Problem Set #7

In the following, \(\cong\) denotes a isomorphism of groups.

**Exercise 1:**
Let \(G\) be a finite abelian group with \(|G| = n\), we will see next week that \(x^n = e\). Let \(k > 0\) be an integer such that \(\gcd(k, n) = 1\). Prove that every \(g \in G\) can be written in the form \(g = x^k\) for some \(x \in G\).

**Exercise 2:**
In \(\text{GL}(n, \mathbb{C})\) and \(\text{SL}(n, \mathbb{C})\) define the subgroups of *scalar* matrices
\[
\mathbb{C}^*I = \{\lambda I : \lambda \neq 0 \text{ in } \mathbb{C}\}, \quad \Omega_n I = \{\lambda I : \lambda \in \Omega_n\}
\]
where \(\Omega_n\) are the complex \(n\)th roots of unity.

(a) Prove that \(\mathbb{C}^*I\) and \(\Omega_n I\) are normal in \(\text{GL}(n, \mathbb{C})\) and \(\text{SL}(n, \mathbb{C})\) respectively.

(b) Prove that \(\text{GL}(n, \mathbb{C})/\mathbb{C}^*I \cong \text{SL}(n, \mathbb{C})/\Omega_n I\)

*Hint:* Use the Second Isomorphism Theorem. If \(N = \mathbb{C}^*I\) show that
\[
N \cdot \text{SL}(n, \mathbb{C}) = \text{GL}(n, \mathbb{C})
\]

**Exercise 3:**
If \(H\) is a subgroup of finite index in a group \(G\), prove that there are only finitely many distinct “conjugate” subgroups \(aHa^{-1}\) for \(a \in G\).

**Exercise 4:**
Let \(G = (\mathbb{R}^*, \cdot)\) be the multiplicative group of nonzero real numbers, and let \(N\) be the subgroup consisting of the numbers \(\pm 1\). Let \(G' = (0, +\infty)\) equipped with multiplication as its group operation. Prove that \(N\) is normal in \(G\) and that \(G/N \cong G' \cong (\mathbb{R}, +)\).

**Exercise 5:**
If \(H\) is a subgroup of \(G\), its *normalizer* is \(N_G(H) = \{g : gHg^{-1} = H\}\). Prove that
(a) $N_G(H)$ is a subgroup.
(b) $H$ is a normal subgroup in $N_G(H)$.
(c) If $H \subseteq K \subseteq G$ are subgroups such that $H$ is a normal subgroup in $K$, prove that $K$ is contained in the normalizer $N_C(H)$.
(d) A subgroup $H$ is normal in $G$ if $N_G(H) = G$.

**Note:** Part (c) shows that $N_G(H)$ is the largest subgroup of $G$ in which $H$ is normal.

**Exercise 6:**
If $x, y \in G$, products of the form $[x, y] = xyx^{-1}y^{-1}$ are called commutators and the subgroup they generate
\[ [G, G] = \langle xyx^{-1}y^{-1} : x, y \in G \rangle \]
is the **commutator subgroup** of $G$. Prove that

(a) The subgroup $[G, G]$ is normal in $G$.
(b) The quotient $G/[G, G]$ is abelian.

**Hint:** In (a) recall that a subgroup $H$ is normal if $gHg^{-1} = H$ for all $g \in G$. What do conjugations $\alpha_g$ do to the generators $[x, y]$ of the commutator subgroup?

**Exercise 7:**
Let $G$ be the group of all real $2 \times 2$ matrices of the form
\[
\begin{pmatrix}
a & b \\
0 & d
\end{pmatrix}
such that \(ad \neq 0\).
\]
Show that the commutator subgroup $[G, G]$ defined in the previous exercise is precisely the subset of matrices in $G$ with 1’s on the diagonal and an arbitrary entry in the upper right corner.

**Exercise 8:**
Consider the group $(\mathbb{Z}/12\mathbb{Z}, +)$.

(a) Identify the set of units $U_{12}$.
(b) What is the order of the multiplicative group $(U_{12}, \cdot)$? Is this abelian group **cyclic**?

**Hint:** What is the maximal order of any element $g \in U_{12}$?

**Exercise 9:**
Let $G$ be any group and let $\text{Int}(G)$ be the set of conjugation operations $\alpha_g(x) = gxg^{-1}$ on $G$. Prove that
(a) Each map $\alpha_g$ is a homomorphism from $G \to G$.
(b) Each map $\alpha_g$ is a bijection, hence an automorphism in $\text{Aut}(G)$.
(c) $\alpha_e = \text{id}_G$, the identity map on $G$.

Exercise 10:
Show that the group $\text{Int}(G)$ of inner automorphisms is a normal subgroup in $\text{Aut}(G)$.

*Note:* The quotient $\text{Aut}(G)/\text{Int}(G)$ is regarded as the group of outer automorphisms $\text{Out}(G)$.

Exercise 11:
The permutation group $G = S_3$ on three objects has $6 = 3!$ elements

$$S_3 = \{e, (12), (23), (13), (123), (132)\}$$

Prove by direct calculation the center of $S_3$ is trivial (Note: you have proven that $G \cong \text{Int}(G)$).

Exercise 12:
For any group $G$ prove that the commutator subgroup $[G, G] = \langle xyx^{-1}y^{-1} | x, y \in G \rangle$ is a characteristic subgroup that is for any $\sigma \in \text{Aut}(G)$, we have $\sigma([G, G]) = [G, G]$.

*Hint:* What does an automorphism do to the generators of $[G, G]$?

*Note:* This example shows that if $G$ is abelian its automorphism gorup may nevertheless be noncommuative (while $\text{Int}(G)$ is trivial).

Exercise 13:
If $G$ is a group, $Z$ is its center, and the quotient group $G/Z$ is cyclic, prove that $G$ must be abelian.