Final

The grader cannot be expected to work his way through a sprawling mess of identities presented without a coherent narrative through line. If he can’t make sense of it in finite time you could lose coherent narrative through line. If he can’t make sense of it in finite time you could lose serious points. Coherent, readable exposition of your work is half the job in mathematics.

Problem 1 :

Prove that if $\phi : G \to G'$ is a group isomorphism, then the inverse function $\phi^{-1}$ is also a group isomorphism.

Problem 2 :

Let $D_n$ be the dihedral group of order $2n$.

1. Give the properties that the generators of a Dihedral group must satisfy which are enough to define it;
2. Express $g = yx^{n-2}yx^5$ in the form $y'x'$ with $i, j \geq 0$.

Problem 3 :

Let $H$ and $N$ be subgroups of a group $G$, and assume that $N$ is normal. We denote their product

$$HN = \{hn | h \in H, n \in N\} \subset G$$

and similarly for $NH$. Prove that $HN = NH$ and this set is the smallest subgroup of $G$ containing both $H$ and $K$. 
Problem 4:
Let \( G \) be the multiplicative group of invertible upper triangular matrices.
\[
\begin{pmatrix}
a & b \\
0 & c
\end{pmatrix}
\]
where \( a, b, c \in \mathbb{R}, \ a, c \neq 0 \).
Let \( S \subset G \) be the subset of all the matrices with \( c = 1 \). Prove that \( S \) is a normal subgroup, by showing that \( S \) is the kernel of a suitable group homomorphism.

Problem 5:
Show that every \( \sigma \in S_n \) is a product of the \( n \)-cycle \( \alpha = (1,2,\ldots,n) \) and the 2-cycle \( \tau = (1,2) \). (Hint: compute \( \alpha^k \tau \alpha^{-k} \)).

Problem 6:

1. Give the properties that the map
\[
\alpha : \ G \times X \rightarrow X \\
(g, x) \mapsto g \cdot x
\]
satisfies so that it defines a group action.

2. Suppose that \( \alpha \) defines a group action. For each \( g \in G \), define the map \( \tau_g : X \rightarrow X \) by \( \tau_g(x) = \alpha(g, x) \), for all \( x \in X \). Show that \( \tau_g \) is a permutation of the set \( X \).

3. Show that the map \( \xi : G \rightarrow S_X \) defined by \( \xi(g) = \tau_g \) is a group homomorphism where \( S_X \) denotes the group of all permutations of \( X \).

4. Conversely, given a group homomorphism \( \phi : G \rightarrow S_X \) from a group \( G \) to the group \( S_X \) of permutation of a set \( X \), show that the map \( \alpha : G \times X \rightarrow X \) defined by
\[
\alpha(g, x) = \phi(g)(x)
\]
is a group action.
Problem 7:

1. Let $G$ be a group and $H$, $N$ be subgroups of $G$. Give two equivalent ways to define an internal semi direct product of $N$ with $H$.

2. Let $G$ be a group of order 6.
   
   (a) Which theorem permits to insure the existence of a subgroup $H_2$ of order 2 and $H_3$ of order 3?
   
   (b) Why $G$ is then a semi-direct product constructed with the group $H_2$ and $H_3$?

   (c) Describe all the possible semi-direct products (write all the details). Identify the dihedral group $D_3$ and the permutation group $S_3$.

   (d) How many non isomorphic groups of order 6 are there? Explain your answer.

Problem 8:

1. Prove that we have a canonical isomorphism

   $$(S^1, \cdot) \cong (\mathbb{R}, +)$$

   where $S^1 = \{ z \in \mathbb{C} | ||z|| = 1 \}$.

2. Prove that there is no cross section for $\mathbb{R}/\mathbb{Z}$. 