Justify all your answers completely (Or with a proof or with a counter example) unless mentioned differently. No step should be a mystery or bring a question. The grader cannot be expected to work his way through a sprawling mess of identities presented without a coherent narrative through line. If he can’t make sense of it in finite time you could lose serious points. Coherent, readable exposition of your work is half the job in mathematics. You will lose serious points if your exposition is messy, incomplete, uses mathematical symbols not adapted... No credit will be given to any unjustified answer.

Problems:

1. (a) Is the the set of all the signals satisfying the equation

\[ y_{k+2} - 4y_{k+1} + 3y_k = 0 \text{ for all } k \]

a subspace of \( \mathbb{S} \) (the vector space of all the signals)? If yes, give a precise proof proving that it is a subspace and if not, explain precisely why.
(b) Is the the set of all the signals satisfying the equation

\[ y_{k+2} - 4y_{k+1} + 3y_k = -4k \quad for \ all \ k \]

a subspace of \( S \)? If yes, give a precise proof proving that it is a subspace and if not, explain precisely why.

2. (a) Verify that the signal \( y_k = k^2 \) satisfies the difference equation

\[ y_{k+2} - 4y_{k+1} + 3y_k = -4k \quad for \ all \ k \ (1) \]
(b) The solutions of a homogeneous difference equation often have the form 
\[ y_k = r^k \] for some \( r \). Find all the solutions of this form for the homogeneous equation \((H)\) associated to (1). What is the dimension of the solution set of \((H)\)? Deduce all the solutions for \((H)\). Express the solution set of \((H)\) as the span of a set of vectors.

(c) Using the previous questions, give explicitly all the solutions for (1).
3. What is the standard basis $S$ of $\mathbb{P}_2$ the vector space of all the polynomials of degree at least 2? What is the dimension of $\mathbb{P}_2$? Give the $S$-coordinate of $a_1 x + a_2 x^2$ in $\mathbb{P}_2$.

4. Let $T : \mathbb{P}_2 \to \mathbb{P}_2$ defined by $T(p) = p + 2$. Is $T$ linear? If yes, give a precise proof that it is a linear and if not, explain precisely why.

5. Let $S : \mathbb{P}_2 \to \mathbb{P}_2$ defined by $S(a_0 + a_1 x + a_2 x^2) = a_1 x + a_2 x^2$. Is $S$ linear? If yes, give a precise proof that it is a linear and if not, explain precisely why.
6. Compute the kernel of $S$, express it as the Span of a set of vectors, deduce a basis for it and its dimension.

7. Compute the range of $S$, express it as the Span of a set of vectors, deduce a basis for it and its dimension.