Quiz #2

Justify all your answers completely (Or with a proof or with a counter example) unless mentioned differently. No step should be a mystery or bring a question. The grader cannot be expected to work his way through a sprawling mess of identities presented without a coherent narrative through line. If he can’t make sense of it in finite time you could lose serious points. Coherent, readable exposition of your work is half the job in mathematics. You will lose serious points if your exposition is messy, incomplete, uses mathematical symbols not adapted...

BE PRECISE!!!!!!

Problems:

   Solution: Observe that $6 \cdot [2] = [0]$ and $k \cdot [2] \neq [0]$ for all $1 \leq k \leq 5$. So that $o([2]) = 6$.


3. Compute the order of $(1,2,3)$ in $S_3$. Justify your answer.
   Solution: Observe that $(1,2,3)^3 = Id$ and $(1,2,3)^k \neq Id$ for all $1 \leq k \leq 2$. So that $o((1,2,3)) = 3$.

4. Let $G$ be a non-trivial group. Let $\tau : G \times G \to G$ be defined as $(g,x) \mapsto \tau(g,x) = \tau_g(x) = gxg^{-1}$.
   (a) What does it mean for $\tau$ to be an action? Prove it is an action.
      Solution: $\tau$ is an action if $\tau_{g_1g_2} = \tau_{g_1} \circ \tau_{g_2}$ and $\tau_e = Id$ for any $g_1, g_2 \in G$.
      For any $g_1, g_2 \in G$, for any $x \in G$, $\tau_{g_1g_2}(x) = (g_1g_2)x(g_1g_2)^{-1} = g_1(g_2xg_2^{-1})g_1^{-1} = \tau_{g_1} \circ \tau_{g_2}(x)$, and $\tau_e(x) = exe^{-1} = x$. So that $\tau$ define an action of $G$ on $G$.

   (b) How is defined the action kernel of $\tau$? Compute it.
      Solution: The action kernel is the kernel of the homomorphism $\phi : G \to Per(G)$ sending $g$ to $\tau_g$. In other words, the action kernel
      $$Ker(\phi) = \{g \in G|\tau_g = Id\} = \{g \in G|\tau_g(x) = x \text{ for all } x \in G\}$$
      Let $g \in Ker(\phi)$, by definition for all $x \in G$, $\tau_g(x) = x$, that is $gxg^{-1} = x$, that is also, $gx = xg$. So that $g \in Z(G)$. This is all equivalence. So that $Z(G) = Ker(\phi)$. So the action kernel is equal to the center.

   (c) Describe the orbit of a point $x \in G$. What is the orbit of $e$ the identity element of $G$? Is the action transitive?
      Solution: The orbit of a point $x \in G$ is $G \cdot x = \{gxg^{-1} : g \in G\}$. So the
orbit of $e$ is $G \cdot e = \{g g^{-1} : g \in G\} = \{e\}$. The action is not transitive since the orbit of $e$ is not equal to $X$. 