Quiz #2

Justify all your answers completely (Or with a proof or with a counter example) unless mentioned differently.

Problems:

1. (10 points) Let $G$ be an abelian group with identity $e$, and let $H$ be the set of all elements $x \in G$ that satisfy the equation $x^3 = e$. Prove that $H$ is a subgroup of $G$.

2. (20 points) Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, viewed as a $2 \times 2$ matrix with entries in $\mathbb{Z}/5\mathbb{Z}$.

   (a) Show that $A$ belongs to $GL_2(\mathbb{Z}/5\mathbb{Z})$.

   (b) Does $A$ belong to $SL_2(\mathbb{Z}/5\mathbb{Z})$? Why, or why not?

   (c) Find all the elements in the cyclic subgroup $\langle A \rangle$ generated by $A$.

   (d) Find the order of $A$ in $GL_2(\mathbb{Z}/5\mathbb{Z})$.

3. (20 pts) Let $(\Omega_n = \{ z \in \mathbb{C} | z^n = 1 \}, \cdot)$ be the group of $n^{th}$ roots of unity and $(\mathbb{Z}/n\mathbb{Z}, +)$ the group of the class of integers modulo $n$. Prove that the map

   $$\phi : (\mathbb{Z}/n\mathbb{Z}, +) \rightarrow (\Omega_n, \cdot)$$

   $$[k] \mapsto e^{2\pi ik/n}$$

   is

   (a) well defined,

   (b) an homomorphism.