The following equations are considered over the reals numbers. All the answers should be justified unless mentioned differently.

**Problem 1 :**
Assume that the matrix $A$ is row equivalent to $B$. Find bases for $\text{Col}(A)$, $\text{Row}(A)$ and $\text{Nul}(A)$.

$$
A = \begin{pmatrix}
1 & 3 & 4 & -1 & 2 \\
2 & 6 & 6 & 0 & -3 \\
3 & 9 & 3 & 6 & -3 \\
3 & 9 & 0 & 9 & 0
\end{pmatrix}
$$

$$
B = \begin{pmatrix}
1 & 3 & 4 & -1 & 2 \\
0 & 0 & 1 & -1 & 1 \\
0 & 0 & 0 & 0 & -5 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

**Solution :** We have proven in class that the $\text{dim}(\text{Col}(A))$ is equal to the number of pivot position, thanks to the echelon form $B$ of $A$ we know that column 1, 3 and 5 are the 3 pivot columns thus $\text{dim}(\text{Col}(A)) = 3$ and we also know that the pivot columns of $A$ form a basis of $\text{Col}(A)$, thus \{ $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 6 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 2 \\ -3 \\ -3 \end{pmatrix}$ \} form a basis for $\text{Col}(A)$.

We have proven that $\text{dim}(\text{Row}(A)) = \text{dim}(\text{Col}(A)) = 3$ and that a basis for $\text{Row}(A)$ is given by the now zero rows of an echelon form of $A$, thus

\{ $\begin{pmatrix} 1 & 3 & 4 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -5 \end{pmatrix}$ \}

form a basis for $\text{Row}(A)$.

We know that if $A \sim B$ then $Ax = 0$ and $Bx = 0$ have same solution set, $Bx = 0$ is equivalent to

$$
\begin{cases}
x_1 + 3x_2 + 3x_4 = 0 \\
x_3 - x_4 = 0 \\
-5x_5 = 0
\end{cases}
$$

with $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$. 

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Thus we have two free \( x_2 \) and \( x_4 \), and we know that \( \text{dim}(\text{Nul}(A)) = 2 \). Also, the form for a general solution is

\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5 \\
\end{pmatrix} = \begin{pmatrix}
  -3x_2 - 3x_4 \\
  x_2 \\
  x_4 \\
  x_4 \\
  0 \\
\end{pmatrix} = x_2 \begin{pmatrix}
  -3 \\
  1 \\
  0 \\
  0 \\
  1 \\
\end{pmatrix} + x_4 \begin{pmatrix}
  0 \\
  0 \\
  0 \\
  1 \\
  0 \\
\end{pmatrix}
\]

with \( x_2, x_4 \) scalars.

Thus \[
\begin{pmatrix}
  -3 \\
  1 \\
  0 \\
  0 \\
  1 \\
\end{pmatrix}, \quad \begin{pmatrix}
  -3 \\
  0 \\
  1 \\
  1 \\
  0 \\
\end{pmatrix}
\]
form a basis for \( \text{Nul}(A) \).

**Problem 2 :**
Let \( u = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \). Find \( v \) in \( \mathbb{R}^3 \) such that

\[
\begin{pmatrix}
  1 & -3 & 4 \\
  2 & -6 & 8 \\
\end{pmatrix} = uv^T
\]

**Solution :** Note that the second row of the matrix is twice the first row. Thus, if \( v = (1, -3, 4) \), then

\[
\begin{pmatrix}
  1 & -3 & 4 \\
  2 & -6 & 8 \\
\end{pmatrix} = uv^T
\]

**Problem 9 :**
Find the change-of-coordinates matrix from \( B \) to \( C \) and the change-of-coordinates matrix from \( C \) to \( B \).

\[
b_1 = \begin{pmatrix} 7 \\ 5 \end{pmatrix}, \quad b_2 = \begin{pmatrix} -3 \\ -1 \end{pmatrix}, \quad c_1 = \begin{pmatrix} 1 \\ -5 \end{pmatrix}, \quad c_2 = \begin{pmatrix} -2 \\ 2 \end{pmatrix}
\]

**Solution :** In order to find the change-of-coordinates matrix from \( B \) to \( C \), \( P_{C \rightarrow B} \), we know that

\[
P_{C \rightarrow B} = [[b_1]_C, [b_2]_C]
\]

By definition, we know that if \([b_1]_C = (x_1, x_2)\) and \([b_2]_C = (y_1, y_2)\) then we have \(b_1 = x_1c_1 + x_2c_2\) and \(b_2 = y_1c_1 + y_2c_2\). So we need to solve these systems in order to solve them simultaneously, we can augment the coefficient matrix by \(b_1\) and \(b_2\) and row reduce this augmented matrix

\[
[c_1, c_2, b_1, b_2] \sim \text{Row reduce} \sim \begin{pmatrix}
  1 & 0 & -3 & 1 \\
  0 & 1 & -5 & 2 \\
\end{pmatrix}
\]

Thus, thanks to the reduce form, we get easily that

\[
[b_1]_C = \begin{pmatrix} -3 \\ -5 \end{pmatrix}, \quad \text{and} \quad [b_2]_C = \begin{pmatrix} 1 \\ 2 \end{pmatrix}
\]
As a consequence,

\[ P_{C \to \mathcal{B}} = \begin{pmatrix} -3 & 1 \\ -5 & 2 \end{pmatrix} \]

and we know that

\[ P_{\mathcal{B} \to C} = P_{C \to \mathcal{B}}^{-1} = \text{(compute...)} = \begin{pmatrix} -2 & 1 \\ -5 & 3 \end{pmatrix} \]

**Problem 3 :**
In \( \mathbb{P}_2 \), find the change-of-coordinates matrix from the basis \( \mathcal{B} = \{1 - 2t + t^2, 3 - 5t + 4t^2, 2t + 3t^2\} \) to the standard basis \( \mathcal{C} = \{1, t, t^2\} \). Then find the \( \mathcal{B} \)-coordinate vector for \( -1 + 2t \).

**Solution :** The \( \mathcal{C} \) coordinate of a polynomial \( p \) are \([p]_C = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\) with \( x_1, x_2, x_3 \) scalar such that \( p = x_11 + x_2t + x_3t^3 \).

Thus, the \( \mathcal{C} \) coordinate of \( b_1, b_2 \) and \( b_3 \) are

\[ [b_1]_C = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, [b_2]_C = \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}, [b_3]_C = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} \]

Thus the the change-of-coordinates matrix from the basis \( \mathcal{B} = \{1 - 2t + t^2, 3 - 5t + 4t^2, 2t + 3t^2\} \) to the standard basis \( \mathcal{C} = \{1, t, t^2\} \),

\[ P_{C \to \mathcal{B}} = [[b_1]_C, [b_2]_C, [b_3]_C] = \begin{pmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{pmatrix} \]

Now let \( x = -1 + 2t \), by definition of \( P_{C \to \mathcal{B}} \) we have;

\[ P_{C \to \mathcal{B}}[x]_\mathcal{B} = [x]_C = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \]

This system may be solved by row reducing its augmented matrix

\[
\begin{pmatrix} 1 & 3 & 0 & -1 \\ -2 & -5 & 2 & 2 \\ 1 & 4 & 3 & 0 \end{pmatrix} \sim \text{Row reduce!} \sim \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{pmatrix}
\]

By the reduced form of the augmented matrix, one can find easily that the coordinate of \( x \)

\[ [x]_\mathcal{B} = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix} \]
Problem 4:
Show that the given signal is a solution of the difference equation. Then find the general solution of that difference equation.

\[ y_k = k^2, \quad y_{k+2} + 3y_{k+1} - 4y_k = 7 + 10k \]

Solution: To show that \( y_k = k^2 \) is a solution of \( y_{k+2} + 3y_{k+1} - 4y_k = 10k + 7 \), substitute \( y_k = k^2, \ y_{k+1} = (k + 1)^2 \) and \( y_{k+2} = (k + 2)^2 \):

\[
\begin{align*}
y_{k+2} + 3y_{k+1} & = (k + 2)^2 + 3(k + 1)^2 - 4k^2 \\
& = (k^2 + 4k + 4) + 3(k^2 + 2k + 1) - 4k^2 \\
& = k^2 + 4k + 4 + 3k^2 + 6k + 3 - 4k^2 \\
& = 10k + 7 \text{ for all } k
\end{align*}
\]

The auxiliary equation for the homogeneous difference equation \( y_{k+2} + 3y_{k+1} - 4y_k = 0 \) is \( r^2 + 3r - 4 = 0 \). By quadratic formula (or factoring), \( r = -4 \) or \( r = 1 \), so two solution of difference equation are \((-4)^k\) or \(1^k\). The signals \((-4)^k\) and \(1^k\) are linearly independent because neither is multiple of the other. The solution space is two-dimensional, so the two linearly independent signals \((-4)^k\) and \(1^k\) form a basis for the solution space of the homogeneous difference equation is thus \(c_1(-4)^k + c_21^k\). Adding the particular solution \(k^2\) of the nonhomogeneous differential equation, we find that the general solution of the difference equation \( y_{k+2} + 3y_{k+1} - 4y_k = 10k + 7 \) is \( y_k = k^2 + c_1(-4)^k + c_2\).

Problem 5:
Let \( y_k = k^2 \) and \( z_k = 2k|k| \). Are the signals \( \{y_k\} \) and \( \{z_k\} \) linearly independent? Evaluate the associated Casorati matrix \( C(k) \) for \( k = 0, k = -1 \) and \( k = -2 \), and discuss your results.

Solution: The Casorati matrix \( C(k) \) is:

\[
C(k) = \begin{pmatrix} y_k & z_k \\ y_{k+1} & z_{k+1} \end{pmatrix} = \begin{pmatrix} k^2 & 2k \cdot |k| \\ (k + 1)^2 & 2(k + 1)|k + 1| \end{pmatrix}
\]

is particular.

\[
C(0) = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}, \quad C(-1) = \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} \text{ and } C(-2) = \begin{pmatrix} 4 & -8 \\ 1 & -2 \end{pmatrix}
\]

none of which are invertible. In fact, \( C(k) \) is not invertible for all \( k \), since

\[
det(C(k)) = 2k^2(k + 1)|k + 1| - 2(k + 1)^2k|k| - 2k(k + 1)(k + 1)|k + 1| - (k + 1)|k|
\]

If \( k = 0 \) or \( k = -1 \), \( det(C(k)) = 0 \). If \( k > 0 \), then \( k + 1 > 0 \) and \( k|k + 1| - (k + 1)|k| = k(k + 1) - (k + 1)k = 0 \), so \( det(C(k)) = 0 \). Thus, \( det(C(k)) = 0 \) for all \( k \), and \( C(k) \) is not invertible for all \( k \). Since \( C(k) \) is not invertible for all \( k \), it provides no information about whether the signals \( \{y_k\} \) and \( \{z_k\} \) are linearly dependent or linearly independent. In fact, neither signal is a multiple of the other, so the signals \( \{y_k\} \) and \( \{z_k\} \) are linearly independent.
Problem 6:
Write the difference equation as first-order systems, \(x_{k+1} = Ax_k\), for all \(k\).

\[y_{k+4} + 3y_{k+3} - 8y_{k+2} + 6y_{k+1} - 2y_k = 0\]

**Solution:** Let \(x_k = \left( \begin{array}{c} y_k \\ y_{k+1} \\ y_{k+2} \\ y_{k+3} \end{array} \right)\). Then

\[x_{k+1} = \left( \begin{array}{c} y_{k+1} \\ y_{k+2} \\ y_{k+3} \\ y_{k+4} \end{array} \right) = \left( \begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & -6 & 8 & -3 \end{array} \right) \left( \begin{array}{c} y_k \\ y_{k+1} \\ y_{k+2} \\ y_{k+3} \end{array} \right) = Ax_k\]

Problem 7:
Find the steady-state vector.

\[\begin{pmatrix} 0.4 & 0.5 & 0.8 \\ 0 & 0.5 & 0.1 \\ 0.6 & 0 & 0.1 \end{pmatrix}\]

**Solution:** Finding the steady-vector is the same as finding \(x\) such that \(Px = x\) that is \(x\) such that \((P - I)x = 0\), where

\[P - I = \left( \begin{array}{ccc} -0.6 & 0.5 & 0.8 \\ 0 & -0.5 & 0.1 \\ 0.6 & 0 & -0.9 \end{array} \right)\]

Row reducing the augmented matrix for the homogeneous system \((P - I)x = 0\) gives

\[\left( \begin{array}{cccc} -0.6 & 0.5 & 0.8 & 0 \\ 0 & -0.5 & 0.1 & 0 \\ 0.6 & 0 & -0.9 & 0 \end{array} \right) = \left( \begin{array}{cccc} 1 & 0 & -3/2 & 0 \\ 0 & 1 & -1/5 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)\]

Thus, the system associated to the reduce form gives

\[x = \left( \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right) = x_3 \left( \begin{array}{c} 3/2 \\ 1/5 \\ 1 \end{array} \right),\]

and one solution is \(\begin{pmatrix} 15 \\ 2 \\ 0 \end{pmatrix}\) sum to 27, multiply by \(1/27\) to obtain the steady-state vector

\[q = \left( \begin{array}{c} 15/27 \\ 2/27 \\ 10/27 \end{array} \right) = \left( \begin{array}{c} 0.556 \\ 0.74 \\ 0.370 \end{array} \right)\]