Problem Set # 2

1 The matrix equation $Ax = b$

The following equations are considered over the reals numbers.

Problem 1 :
Let $u = \begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix}$ and $A = \begin{pmatrix} 3 & -5 \\ -2 & 6 \\ 1 & 1 \end{pmatrix}$. Is $u$ in the plane passing through the origin with directions the columns of $A$?

Solution : The vector $u$ is in the plane spanned by the columns of $A$ if and only if $u$ is a linear combination of the column of $A$. This happens if and only if the equation $Ax = u$ has a solution. To study this equation, reduce the augmented matrix $[A, u]$

$$
\begin{pmatrix}
3 & -5 & 0 \\
-2 & 6 & 4 \\
1 & 1 & 4
\end{pmatrix}
\sim_{R_1 \leftrightarrow R_3}
\begin{pmatrix}
1 & 1 & 4 \\
-2 & 6 & 4 \\
3 & -5 & 0
\end{pmatrix}
\sim_{R_2 \leftrightarrow 2R_1 + R_2} \sim_{R_3 \leftrightarrow 3R_1 - 3R_2}
\begin{pmatrix}
1 & 1 & 4 \\
0 & 8 & 12 \\
0 & -8 & -12
\end{pmatrix}
\sim_{R_3 \leftrightarrow R_3 + R_2}
\begin{pmatrix}
1 & 1 & 4 \\
0 & 8 & 12 \\
0 & 0 & 0
\end{pmatrix}
$$

The equation $Ax = u$ has a solution, so $u$ is in the plane spanned by the columns of $A$. For your information : The unique solution of $Ax = u$ is $(5/2, 3/2)$.

Problem 2 :
Let

$$A = \begin{pmatrix}
4 & -5 & -1 & 8 \\
3 & -7 & -4 & 2 \\
5 & -6 & -1 & 4 \\
9 & 1 & 10 & 7
\end{pmatrix}$$
Determine if the columns of the matrix $A$ span $\mathbb{R}^4$.

**Solutions** : In order to answer to this question, we will row reduce the matrix $A$ :

\[
\begin{pmatrix}
  4 & -5 & -1 & 8 \\
  3 & -7 & -4 & 2 \\
  5 & -6 & -1 & 4 \\
  9 & 1 & 10 & 7
\end{pmatrix}
\]

$\sim R_2 \rightarrow R_2 - 3/4 R_1$ and $R_3 \rightarrow R_3 - 5/4 R_1$ and $R_4 \rightarrow R_4 - 9/4 R_1$

\[
\begin{pmatrix}
  4 & -5 & -1 & 8 \\
  0 & -13/4 & -13/4 & -4 \\
  0 & 1/4 & 1/4 & -6 \\
  0 & 49/4 & 49/4 & -11
\end{pmatrix}
\]

$\sim R_3 \rightarrow R_3 + 1/13 R_2$ and $R_4 \rightarrow R_4 + 49/13 R_2$

\[
\begin{pmatrix}
  4 & -5 & -1 & 8 \\
  0 & -13/4 & -13/4 & -4 \\
  0 & 0 & 0 & -82/13 \\
  0 & 0 & 0 & 0
\end{pmatrix}
\]

The augmented matrix has only pivots in the first three rows, that mean from a theorem of the class that the columns that do not span $\mathbb{R}^4$.

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2 Solution sets of linear system

**Problem 3** :

1. Write the solution set of the given homogeneous system in parametric vector form, express it as a Span and give its geometric description,

\[
\begin{align*}
  x_1 + 2x_2 - 3x_3 & = 0 \\
  2x_1 + x_2 - 3x_3 & = 0 \\
  -x_1 + x_2 & = 0
\end{align*}
\]
Solutions: The augmented matrix associated to the system is:

\begin{pmatrix}
  1 & 2 & -3 & 0 \\
  2 & 1 & -3 & 0 \\
  -1 & 1 & 0 & 0 \\
\end{pmatrix}

\sim R_2 \rightarrow R_2 - 2R_1 \quad \text{and} \quad R_3 \rightarrow R_3 + R_1

\begin{pmatrix}
  1 & 2 & -3 & 0 \\
  0 & -3 & 3 & 0 \\
  0 & 3 & -3 & 0 \\
\end{pmatrix}

\sim R_3 \rightarrow R_3 + R_2

\begin{pmatrix}
  1 & 2 & -3 & 0 \\
  0 & -3 & 3 & 0 \\
  0 & 0 & 0 & 0 \\
\end{pmatrix}

\sim R_3 \rightarrow -1/3R_3

\begin{pmatrix}
  1 & 2 & -3 & 0 \\
  0 & 1 & -1 & 0 \\
  0 & 0 & 0 & 0 \\
\end{pmatrix}

\sim R_1 \rightarrow R_1 - 2R_2

\begin{pmatrix}
  1 & 0 & -1 & 0 \\
  0 & 1 & -1 & 0 \\
  0 & 0 & 0 & 0 \\
\end{pmatrix}

The last column of the augmented matrix is not a pivot column and we have one free variable \(x_3\), so the system is consistent with infinitely many solutions. More precisely, the system corresponding to the last matrix in reduced echelon form is

\[
\begin{cases}
  x_1 - 3x_3 = 0 \\
  x_2 - 3x_3 = 0 \\
  0 = 0
\end{cases}
\]

So, a general solution in parametric form for the system is:

\[
x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}
\]

The solution set is \(\text{Span}\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\}\).

Geometrically, the solution are on the line passing through the origin with direction \(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\).

2. Describe the solutions of the following system in parametric vector form, and provide a geometric comparison with the solution set of the previous question.

\[
\begin{cases}
  x_1 + 2x_2 - 3x_3 = 5 \\
  2x_1 + x_2 - 3x_3 = 13 \\
  -x_1 + x_2 = -8
\end{cases}
\]
Solution : The augmented matrix associated to the system is:
\[
\begin{pmatrix}
1 & 2 & -3 & 5 \\
2 & 1 & -3 & 13 \\
-1 & 1 & 0 & -8
\end{pmatrix}
\]

After performing the following row operations:
\( R_2 \rightarrow R_2 - 2R_1 \) and
\( R_3 \rightarrow R_3 + R_1 \)

\[
\begin{pmatrix}
1 & 2 & -3 & 5 \\
0 & -3 & 3 & 3 \\
0 & 3 & -3 & -3
\end{pmatrix}
\]

\( R_3 \rightarrow R_3 + R_2 \)

\[
\begin{pmatrix}
1 & 2 & -3 & 5 \\
0 & -3 & 3 & 3 \\
0 & 0 & 0 & -3
\end{pmatrix}
\]

\( R_3 \rightarrow -1/3R_3 \)

\[
\begin{pmatrix}
1 & 2 & -3 & 5 \\
0 & 1 & -1 & -1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\( R_1 \rightarrow R_1 - 2R_2 \)

\[
\begin{pmatrix}
1 & 0 & -1 & 7 \\
0 & 1 & -1 & -1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

The last column of the augmented matrix is not a pivot column and we have one free variable \( x_3 \), so the system is consistent with infinitely many solution. More precisely, the system corresponding to the last matrix in reduced echelon form is
\[
\begin{align*}
x_1 & -x_3 = 7 \\
x_2 & -x_3 = -1 \\
0 & = 0
\end{align*}
\]

So, a general solution in parametric form for the system is:
\[
x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 + x_3 \\ -1 + x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ -1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}
\]

The solution set is \( \{ \begin{pmatrix} 7 \\ -1 \\ 0 \end{pmatrix} \} + \text{Span} \{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \} \).

Geometrically, the solution are on the line passing through the point \( \begin{pmatrix} 7 \\ -1 \\ 0 \end{pmatrix} \) with direction \( \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \), this line is parallel to the one found in the previous question.

3 Application of linear system

Problem 4:
Chemical equations describe the quantities of substances consumed and produced by
chemical reactions. Alka-Selter contains sodium bicarbonate \((NaHCO_3)\) and citric acid \((H_3C_6H_5O_7)\). When a tablet is dissolved in water the following reaction produces sodium citrate, water and carbon dioxide (gas):

\[
NaHCO_3 + H_3C_6H_5O_7 \rightarrow Na_3C_6H_5O_7 + H_2O + CO_2
\]

Balance the chemical equation. (For this, a systematic method for balancing chemical equation is to set up a vector equation that describes the numbers of atoms of each type present in a reaction. Since the equation involves 4 atoms: Sodium (Na), Hydrogen (H), carbon (C) and oxygen (O). For instance, \(NaHCO_3\) will be expressed as the vector \(\begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \end{pmatrix}\). Then translate the problem into a vector equation.)

**Solution**: Putting the number of atom of Sodium in first position of the vector, of hydrogen in second, the carbon in third and the oxygen in 4th (the order in which they appear is not important, here we just choose the order suggested by the name of the molecules, you could choose another order but then you have to stick with it until the end of the exercise.)

We have

\[
NaHCO_2 : \begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \end{pmatrix}, \ H_3C_6H_5O_7 : \begin{pmatrix} 0 \\ 8 \\ 6 \\ 7 \end{pmatrix}, \ Na_3C_6H_5O_7 : \begin{pmatrix} 3 \\ 5 \\ 6 \\ 7 \end{pmatrix}, \ H_2O : \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \ CO_2 : \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}
\]

The order of the various atoms is not important. The list here was selected by writing the element in the order in which they first appear in the chemical equation, reading left to right:

\[x_1NaHCO2 + x_3H_3C_6H_5O_7 \leftarrow x_3Na_3C_6H_5O_7 + x_4H_2O + x_5CO_2\]

The coefficients \(x_1, \ldots, x_5\) satisfy the vector equation

\[x_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 8 \\ 6 \\ 7 \end{pmatrix} = x_3 \begin{pmatrix} 3 \\ 5 \\ 6 \\ 7 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} + x_5 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}\]

Move all the terms to the left side (changing the sign of each entry in the third, fourth, and fifth vectors) and reduce the augmented matrix (show your work as usual):

\[
\begin{pmatrix} 1 & 0 & -3 & 0 & 0 & 0 \\ 1 & 8 & -5 & -2 & 0 & 0 \\ 1 & 6 & -6 & 0 & -1 & 0 \\ 3 & 7 & -7 & -1 & -2 & 0 \end{pmatrix} \sim \text{Row reduce!} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1/3 & 0 \\ 0 & 0 & 1 & 0 & -1/3 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}
\]
The last column is not a pivot column and the system corresponding to the reduced echelon form has a free variable $x_5$, so the system is consistent with infinitely many solutions.

More precisely the system corresponding to the reduced echelon form of the augmented matrix is

\[
\begin{align*}
    x_1 - x_5 &= 0 \\
    x_2 - \frac{1}{3}x_5 &= 0 \\
    x_3 - \frac{1}{3}x_5 &= 0 \\
    x_4 - x_5 &= 0 \\
\end{align*}
\]

For instance, one can take $x_5 = 3$. Then $x_1 = x_4 = 3$ and $x_2 = x_3 = 1$. So, a balanced equation is

\[3\text{NaHCO}_2 + \text{H}_3\text{C}_6\text{H}_5\text{O}_7 \leftrightarrow \text{Na}_3\text{C}_6\text{H}_5\text{O}_7 + 3\text{H}_2\text{O} + 3\text{CO}_2\]

**Problem 5:**

A network consist of a set of points called junction or nodes, with lines or arcs, called branches connecting some or all of the junction. The direction of flow of each branch is indicated and the flow amount (or rate) is either shown or is denoted by a variable. The basic assumption of network flow is that the total flow into the network equals the total flow out of the junction. For instance

![Diagram of network flow](image)

Shows that 40 units flowing into a junction through one branch, with $x_1$ and $x_2$ denoting the flows out of the junction through other branches. Since the flow is conserved at each junction, we must have $x_1 + x_2 = 40$. In a similar fashion, the flow at each junction is described by a linear equation. The problem of network analysis is to determine the flow in each branch when partial information (such as the flow into and out the network) is known.

Find the general flow pattern of the network shown in the figure. Assuming the flows are all nonnegative, what is the smallest possible value for $x_4$?
For this complete the table

<table>
<thead>
<tr>
<th>Intersection</th>
<th>Flow in</th>
<th>Flow out</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$x_1 + x_4$</td>
<td>$x_2$</td>
</tr>
<tr>
<td>B</td>
<td>$x_2$</td>
<td>$x_3 + 100$</td>
</tr>
<tr>
<td>C</td>
<td>$x_3 + 80$</td>
<td>$x_4$</td>
</tr>
</tbody>
</table>

and then translate the problem into solving a system of linear equations.

**Solution:** Write the equation for each intersection:

<table>
<thead>
<tr>
<th>Intersection</th>
<th>Flow in</th>
<th>Flow out</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$x_1 + x_4$</td>
<td>$x_2$</td>
</tr>
<tr>
<td>B</td>
<td>$x_2$</td>
<td>$x_3 + 100$</td>
</tr>
<tr>
<td>C</td>
<td>$x_3 + 80$</td>
<td>$x_4$</td>
</tr>
</tbody>
</table>

Since the flow is conserved at each junction, we must have, that $x_1$, $x_2$, $x_3$, $x_4$ satisfy:

\[
\begin{align*}
  x_1 + x_4 &= x_2 \\
  x_2 &= x_3 + 100 \\
  x_3 + 80 &= x_4
\end{align*}
\]

We rearrange the equation in order to have all the unknown in the left hand side and the constants in the right in order to be able to use the method that we know to solve the system:

\[
\begin{align*}
  x_1 - x_2 + x_4 &= 0 \\
  x_2 - x_3 &= 100 \\
  x_3 - x_4 &= -80
\end{align*}
\]

We then reduce as usual the augmented matrix associated to the system (show your work!):

\[
\begin{pmatrix}
  1 & -1 & 0 & 1 & 0 \\
  0 & 1 & -1 & 0 & 100 \\
  0 & 0 & 1 & -1 & -80
\end{pmatrix} \sim \text{Row reduce!} \sim
\begin{pmatrix}
  1 & 0 & 0 & 0 & 20 \\
  0 & 1 & 0 & -1 & 20 \\
  0 & 0 & 1 & -1 & -80
\end{pmatrix}
\]

The system corresponding to the obtained reduced echelon form is

\[
\begin{align*}
  x_1 &= 20 \\
  x_2 - x_4 &= 20 \\
  x_3 - x_4 &= -80 \\
  x_4 &= \text{free}
\end{align*}
\]

So, a general solution is of the form:

\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4
\end{pmatrix} =
\begin{pmatrix}
  20 \\
  20 + x_4 \\
  -80 + x_4 \\
  x_4
\end{pmatrix}
\]
Since $x_3$ cannot be negative, the minimum value of $x_4$ is 80.

**Problem 6:**
Suppose an economy has four sectors: Mining, Lumber, Energy, and Transportation. Mining sells 10% of its output to Lumber, 60% to Energy, and retains the rest. Lumber sells 15% of its output to Mining, 50% to Energy, 20% to Transportation, and retains the rest. Energy sells 20% of its output to Mining, 15% to Lumber, 20% to Transportation and retains the rest. Transportation sells 20% of its output to mining, 10% to lumber, 50% to Energy, and retains the rest.

1. Construct the exchange table for this economy.
2. Find a set of equilibrium prices for this economy.

**Solutions:**

1. Fill in the exchange table one column at a time. The entries in each column must sum to 1.

<table>
<thead>
<tr>
<th>Distribution output from</th>
<th>Mining</th>
<th>Lumber</th>
<th>Energy</th>
<th>Transportation</th>
<th>Purchased by</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.30</td>
<td>0.15</td>
<td>0.20</td>
<td>0.20</td>
<td>Mining</td>
</tr>
<tr>
<td></td>
<td>0.10</td>
<td>0.15</td>
<td>0.15</td>
<td>0.10</td>
<td>Lumber</td>
</tr>
<tr>
<td></td>
<td>0.60</td>
<td>0.50</td>
<td>0.45</td>
<td>0.50</td>
<td>Energy</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0.20</td>
<td>0.20</td>
<td>0.20</td>
<td>Transportation</td>
</tr>
</tbody>
</table>

2. Denote the total annual output of the sectors by $p_M$, $p_L$, $p_E$ and $p_T$, respectively. From the first row of the table, the total input to Agriculture is $0.30p_M + 0.15p_L + 0.20p_E + 0.20p_T$. So the equilibrium prices must satisfy

$$p_M = 0.30p_M + 0.15p_L + 0.20p_E + 0.20p_T$$

Similarly we do the same with the other row and we obtain the system:

$$p_M = 0.30p_M + 0.15p_L + 0.20p_E + 0.20p_T$$
$$p_L = 0.10p_M + 0.15p_L + 0.15p_E + 0.10p_T$$
$$p_E = 0.60p_M + 0.50p_L + 0.45p_E + 0.50p_T$$
$$p_T = 0.20p_L + 0.20p_E + 0.20p_T$$

Move all variables to the left side and combine like terms:

$$0.70p_M - 0.15p_L - 0.20p_E - 0.20p_T = 0$$
$$-0.10p_M + 0.85p_L - 0.15p_E - 0.10p_T = 0$$
$$-0.60p_M - 0.50p_L + 0.55p_E - 0.50p_T = 0$$
$$-0.20p_L - 0.20p_E + 0.80p_T = 0$$

Reduce the augmented matrix to reduced echelon form:

$$\begin{pmatrix}
-0.70 & -0.15 & -0.20 & -0.20 & 0 \\
-0.10 & 0.85  & -0.15 & -0.10 & 0 \\
-0.60 & -0.50 & 0.55  & -0.50 & 0 \\
0      & -0.20  & -0.20 & 0.80  & 0 \\
\end{pmatrix} \xrightarrow{\text{Row reduce!}} \begin{pmatrix}
1 & 0 & 0 & -1.37 & 0 \\
0 & 1 & 0 & -0.84 & 0 \\
0 & 0 & 1 & -3.16 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}$$
Solve for the basic variable in terms of the free variable $p_T$, and obtain

\[
\begin{align*}
p_M &= 1.37p_T \\
p_L &= 0.84p_T \\
p_E &= 3.16p_T
\end{align*}
\]

Take for instance $p_T = 100$, then the other equilibrium prices become $p_M = 137$, $p_L = 84$ and $p_E = 316$.