Quiz #1

Justify all your answers completely (Or with a proof or with a counter example) unless mentioned differently.

Problems:

1. (20pt) Find the intersection between the plane passing through the origin with direction $(2, 1, 0)$ and $(-2, 0, 1)$ and the plane passing through origin with directions $(-2, 1, 0)$ and $(1, 0, 1)$. For this, follow the following steps:

(a) after giving me the general form for an implicit description of an arbitrary plane, give an implicit description for these two planes,

\textbf{Solution:} The general form for an implicit description of an arbitrary plane is

$$ax_1 + bx_2 + cx_3 = d$$

where $a, b, c, d$ are scalars.

Since the plane are passing through the origin we have

$$d = a \times 0 + b \times 0 + c \times 0 = 0$$

Since the plane plane $P_1$ passes through the origin with direction $(2, 1, 0)$ and $(-2, 0, 1)$, it also passes through the points $(2, 1, 0)$ and $(-2, 0, 1)$. So that,

$$a \times 2 + b \times 1 + c \times 0 = 0 \Rightarrow b = -2a$$

And

$$a \times (-2) + b \times 0 + c \times 1 = 0 \Rightarrow c = 2a$$

Now, take $a = 1$ then $b = -2$ and $c = 2$. So an implicit description for $P_1$ is

$$x_1 - 2x_2 + 2x_3 = 0$$

Since the plane $P_2$ passes through origin with directions $(-2, 1, 0)$ and $(1, 0, 1)$, it also passes through the points $(-2, 1, 0)$ and $(1, 0, 1)$.

So that

$$a \times (-2) + b \times 1 + c \times 0 = 0 \Rightarrow b = 2a$$

and

$$a \times 1 + b \times 0 + c \times 1 = 0 \Rightarrow c = -a$$

Then take for instance $a = 1$ then we have $c = -1$ and $b = 2$. So that an implicit description of a plane is

$$x_1 + 2x_2 - x_3 = 0$$
(b) then translate the problem into solving a linear system of equation, solve
the system and answer to the question. Solution: In order to find the
intersection we need to find the points satisfying both implicit equation that
is satisfying the system:

\[
\begin{align*}
    x_1 - 2x_2 + 2x_3 &= 0 \\
    x_1 + 2x_2 - x_3 &= 0
\end{align*}
\]

The augmented matrix associated to the system is

\[
\begin{pmatrix}
    1 & -2 & 2 & 0 \\
    1 & 2 & -1 & 0
\end{pmatrix}
\]

\[\sim R_2 \rightarrow R_2 - R_1 \left( \begin{array}{cccc}
    1 & -2 & 2 & 0 \\
    0 & 4 & -3 & 0
\end{array} \right) \]

\[\sim R_2 \rightarrow 1/4R_2 \left( \begin{array}{cccc}
    1 & -2 & 2 & 0 \\
    0 & 1 & -3/4 & 0
\end{array} \right) \]

\[\sim R_1 \rightarrow R_1 + 2R_2 \left( \begin{array}{cccc}
    1 & 0 & 1/2 & 0 \\
    0 & 1 & -3/4 & 0
\end{array} \right) \]

The intersection is non empty since we have a free variable \( x_3 \). The system
(corresponding to the reduced echelon form of the augmented matrix is

\[
\begin{align*}
    x_1 + 1/2x_3 &= 0 \\
    x_2 - 3/4x_3 &= 0
\end{align*}
\]

So that a parametric form of the solution is

\[
\begin{pmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_3
\end{pmatrix} = \begin{pmatrix}
    -1/2x_3 \\
    3/4x_3 \\
    x_3 \\
    x_3
\end{pmatrix} = x_3 \begin{pmatrix}
    -1/2 \\
    3/4 \\
    1
\end{pmatrix}
\]

So the intersection is the line passing through the origin with direction
\[
\begin{pmatrix}
    -1/2 \\
    3/4 \\
    1
\end{pmatrix} .
\]

2. (20pt) Explicit the solution set of the systems corresponding to the following
augmented matrices, in a parametric vector form and geometrically if necessary:

(a)

\[
\begin{pmatrix}
    1 & 0 & 0 & -1/3 & -47 \\
    0 & 1 & 0 & -2 & 0 \\
    0 & 0 & 1 & -2/3 & 0 \\
    0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Solution: From the augmented matrix, we see that \( x_4 \) is a free variable so
the system corresponding is consistent and have an infinity of solutions. The
system corresponding is

\[
\begin{align*}
\begin{cases}
x_1 - 1/3x_4 &= -47 \\
x_2 - 2x_4 &= 0 \\
x_3 - 2/3x_4 &= 0 \\
0 &= 0
\end{cases}
\end{align*}
\]

That is

\[
\begin{align*}
\begin{cases}
x_1 &= 1/3x_4 - 47 \\
x_2 &= 2x_4 \\
x_3 &= 2/3x_4
\end{cases}
\end{align*}
\]

Thus the vector parametric form, for a solution is

\[
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix}
= 
\begin{pmatrix}
-47 \\
0 \\
0 \\
0
\end{pmatrix} +
\begin{pmatrix}
1/3 \\
2 \\
2/3 \\
1
\end{pmatrix}x_4
\]

This is a line passing through \(\begin{pmatrix}
-47 \\
0 \\
0
\end{pmatrix}\) and with direction \(\begin{pmatrix}
1/3 \\
2 \\
2/3
\end{pmatrix}\).

(b)

\[
\begin{pmatrix}
2 & 4 & 3 & -2 \\
0 & 5 & 5 & -4 \\
0 & 0 & 0 & 7
\end{pmatrix}
\]

**Solution:** Since the last line implies 0 = 7 which is impossible, the corresponding system is inconsistent. So the solution set is empty.

(c)

\[
\begin{pmatrix}
1 & -4 & 0 & 0 & 0 & 5 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & -4 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

**Solution:** The system corresponding has two free variables \(x_2\) and \(x_4\) so it is consistent and has infinitely many solutions. The system corresponding to this augmented matrix is:

\[
\begin{align*}
\begin{cases}
x_1 - 4x_2 + 5x_6 &= 0 \\
x_3 - x_6 &= 0 \\
x_5 - 4x_6 &= 0 \\
x_6 &= 0
\end{cases}
\end{align*}
\]

\[
\Leftrightarrow
\begin{align*}
\begin{cases}
x_1 &= 4x_2 \\
x_3 &= 0 \\
x_5 &= 0 \\
x_6 &= 0
\end{cases}
\end{align*}
\]
The parametric vector form of a solution is

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5 \\
  x_6
\end{bmatrix} = \begin{bmatrix}
  4x_2 \\
  x_2 \\
  0 \\
  x_4 \\
  0 \\
  0
\end{bmatrix} = x_2 \begin{bmatrix}
  4 \\
  1 \\
  0 \\
  0 \\
  0 \\
  0
\end{bmatrix} + x_4 \begin{bmatrix}
  0 \\
  0 \\
  0 \\
  0 \\
  0 \\
  1
\end{bmatrix}
\]

The solution set is then equal to \( \text{Span}\{ \begin{bmatrix}
  4 \\
  0 \\
  1 \\
  0 \\
  0 \\
  0
\end{bmatrix}, \begin{bmatrix}
  0 \\
  0 \\
  0 \\
  0 \\
  0 \\
  1
\end{bmatrix}\} \).

Geometrically, this is the plane passing through the origin with direction

\[
\begin{bmatrix}
  4 \\
  1 \\
  0 \\
  0 \\
  0 \\
  0
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
  0 \\
  0 \\
  0 \\
  0 \\
  0 \\
  1
\end{bmatrix}. 
\]

3. Let

\[
A = \begin{pmatrix}
  2 & 3 & 4 & 5 \\
  0 & 7 & 8 & 9 \\
  0 & 0 & 5 & 1 \\
  0 & 0 & 0 & 12
\end{pmatrix}
\]

Do the columns of \( A \) span \( \mathbb{R}^4 \)?

**Solution:** Yes they do span \( \mathbb{R}^4 \) since we have a pivot position in each row of the \( A \).