Quiz #2

Problems:

1. Let $G$ be the matrix group $GL(n, \mathbb{C})$ of all $n \times n$ matrices $A$ with complex entries and $\det(A) \neq 0$. This is a group under matrix multiplication, and so is the subgroup $N = SL(n, \mathbb{C})$ of matrices with determinant +1.
   (a) Prove that $SL(n, \mathbb{C})$ is a normal subgroup of $GL(n, \mathbb{C})$. What does this imply for the quotient $GL(n, \mathbb{C})/SL(n, \mathbb{C})$?
   (b) Let $p$ be the matrix group $GL(2, \mathbb{C})$ by definition of $n$ the usual determinant is multiplicative, and it is surjective because if $\lambda \neq 0$ in $\mathbb{C}$ the diagonal matrix $D = \text{diag}(\lambda^{1/n}, \ldots, \lambda^{1/n})$ has $\det D = \lambda$. (Here $\lambda^{1/n}$ is any complex $n$th root of $\lambda$; for instance if $\lambda$ has polar form $\lambda = re^{i\theta}$ we can take the principal $n$th root $\lambda^{1/n} = r^{1/n}e^{i\theta/n}$ where $r^{1/n}$ is the usual $n$th root of a non-negative real number.) The kernel of $\phi$ is precisely $N = SL(n, \mathbb{C})$, by definition of $SL(n, \mathbb{C})$. The conditions of the First Isomorphism Theorem are fulfilled. We conclude that $GL(n, \mathbb{C})/SL(n, \mathbb{C}) \cong (\mathbb{C}^\times, \cdot)$ as claimed.

2. (a) Give the center of $S_3$. What you can deduce about $S_3$?

   

   $S_3 = \{\text{Id}, (1, 2), (1, 3), (2, 3), (1, 2, 3), (1, 3, 2)\}$

   (1, 2)(1, 2, 3) = (1, 3), and (1, 2, 3)(1, 2) = (2, 3), so neither (1, 2) nor (1, 2, 3) is in the center. (2, 3)(1, 3, 2) = (1, 3), and (1, 3, 2)(2, 3) = (1, 2), so neither (2, 3) nor (1, 3, 2) is in the center. (1, 2)(1, 3) = (1, 2, 3), and (1, 3)(1, 2) = (1, 3, 2), so (1, 3) isn’t in the center either.

   That leaves the identity permutation (1), which has to commute with everything, so the center is just $\{(1)\}$.

   (b) Give the order of $(1, 3, 2)$ in $S_3$ and the group generated by $(1, 3, 2)$, to which well know group is it isomorphic to?

   $(1, 2, 3)^2 = (1, 3, 2)$, and $(1, 2, 3)^3 = \text{Id}$, so $\langle (1, 2, 3) \rangle = \{\text{Id}, (1, 2, 3), (1, 3, 2)\}$.

   $\alpha(1, 2, 3) = 3$ and $\langle (1, 2, 3) \rangle \cong \mathbb{Z}/3\mathbb{Z}$. 
3. For each the following pair of groups, decide whether they are isomorphic or not. In each case, give a brief reason why.
   (a) \( U_5 \) and \( U_{10} \).
   Yes. They are both cyclic of order 4.
   (b) \( U_8 \) and \( \mathbb{Z}/4\mathbb{Z} \).
   No. \( U_8 \) doesn’t have an element of order 4, but \( \mathbb{Z}/4\mathbb{Z} \) does.
   (c) \( U_{10} \) and \( \mathbb{Z}/4\mathbb{Z} \).
   Yes. They are both cyclic of order 4.
   (d) \( S_3 \) and \( \mathbb{Z}/6\mathbb{Z} \).
   No. \( S_3 \) is not abelian, but \( \mathbb{Z}/6\mathbb{Z} \) is.

4. Give the order of \( r^2 s \) in the group \( \mathbb{Z}/6\mathbb{Z} \), give the subgroup of \( \mathbb{Z}/6\mathbb{Z} \) generated by \( [2] \). To which well-known group is it isomorphic to?
   Be careful we are in additive notation here! \( 2[2] = [4], 3[2] = [0] \), then \( o([2]) = 3 \) and \(< [2] >= \{[0],[2],[4]\} \cong \mathbb{Z}/3\mathbb{Z} \).

5. Give the order of \( [5] \) in \( U_6 \), give the subgroup of \( U_6 \), generated by \( [5] \). To which well-known group is it isomorphic to?
   Be careful we are in multiplicative notation here!
   \( [5]^2 = [25] = [1] \), then \( o([5]) = 2 \) and \( < [5] > = \{[1],[5]\} \cong \mathbb{Z}/2\mathbb{Z} \).