**Exercise 1.** Let \( f \) be a nonnegative measurable function. Show that

\[
\int f = \sup \int \varphi,
\]

where \( \varphi \) is taken over all simple functions with \( \varphi \leq f \).

**Answer:**

For each \( n \in \mathbb{N} \) we divide \([0,n)\) to disjoint intervals

\[
I_k = \left[ \frac{k-1}{2^n}, \frac{k}{2^n} \right),
\]

where \( k = 1, \ldots, n2^n \). We then define \( f_n \) for all \( n \in \mathbb{N} \) so that \( f_n(x) = \frac{k-1}{2^n} \) when \( x \in f^{-1}(I_k) \) and \( f_n(x) = n \) when \( x \in f^{-1}[n,\infty] \). In other words,

\[
f_n(x) = \sum_{k=1}^{2^n} k \cdot \chi_{f^{-1}(I_k)} + n \chi_{f^{-1}[n,\infty]}.
\]

Since \( f \) was measurable then \( f_n \) is a simple function. Now \( f_n \leq f_{n+1} \) by construction and \( f_n(x) \to f(x) \) for all \( x \). Indeed if \( x \) is so that \( f(x) < \infty \) then there exists \( m \in \mathbb{N} \) so that \( f(x) < m \). When \( n \geq m \) then \( \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n} \) for some \( k = 1, \ldots, n2^n \). Thus

\[
f_n(x) = \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n} = f_n(x) + \frac{1}{2^n},
\]

which implies that \( f(x) - \frac{1}{2^n} < f_n(x) \leq f(x) \). Letting \( n \to \infty \) it follows that \( \lim_{n \to \infty} f_n(x) = f(x) \). If on the other hand \( x \) is such that \( f(x) = \infty \) then \( f_n(x) = n \) for all \( n \in \mathbb{N} \) and thus \( f_n(x) \to \infty = f(x) \). We have thus constructed a nondecreasing sequence of bounded simple functions \( \{f_n\}_{n=1}^{\infty} \) so that \( f_n \to f \). To truncate the supports, we define \( \varphi_n = f_n \chi_{B(0,n)} \) for all \( n \in \mathbb{N} \), where \( B(0,n) \) is the open \( n \)-radius ball around 0. Note that \( \{\varphi_n\}_{n=1}^{\infty} \) is still a nondecreasing sequence of bounded simple functions converging pointwise to \( f \). In addition
\[ m(\{x : \varphi_n(x) \neq 0\}) \leq m(B(0, n)) = 2n < \infty \text{ for all } n \in \mathbb{N}. \]

Now using the monotone convergence theorem we conclude that
\[ \sup \int \varphi \geq \lim_{n \to \infty} \int \varphi_n = \int \lim_{n \to \infty} \varphi_n = \int f. \]

And the converse inequality \( \sup \int \varphi \leq \int f \) holds by assumptions since the supremum is taken over all simple functions with \( \varphi \leq f \). So the claim follows.

**Exercise 2.** Let \((f_n)_{n=1}^\infty\) be a sequence of nonnegative measurable functions on \((-\infty, \infty)\) such that \(f_n \to f\) almost everywhere, and suppose \(\int f_n \to \int f < \infty\). Then for each measurable set \(E\) we have \(\int_E f_n \to \int_E f\).

**Answer:**

Let \(E\) be a measurable set. Note that since \(f_n \to f\) almost everywhere then \(f_n \chi_E \to f \chi_E\) almost everywhere. Thus by Fatou’s lemma
\[ \int_E f = \int_{-\infty}^\infty f \chi_E = \int_{-\infty}^\infty \liminf_{n \to \infty} f_n \chi_E \leq \liminf_{n \to \infty} \int_{-\infty}^\infty f_n \chi_E = \liminf_{n \to \infty} \int_{E} f_n. \]

Similarly
\[ \int_{E^c} f \leq \liminf_{n \to \infty} \int_{E^c} f_n. \]

Now by assumption and the above inequalities we have
\[ \int_{-\infty}^\infty f = \int_E f + \int_{E^c} f \leq \liminf_{n \to \infty} \int_E f_n + \liminf_{n \to \infty} \int_{E^c} f_n \leq \lim_{n \to \infty} \left( \int_E f_n + \int_{E^c} f_n \right) \]
\[ = \lim_{n \to \infty} \int_{-\infty}^\infty f_n = \int_{-\infty}^\infty f, \]

so there in fact holds an equality everywhere. Since \(\int_{-\infty}^\infty f < \infty\) then it follows that
\[ \liminf_{n \to \infty} \int_E f_n + \liminf_{n \to \infty} \int_{E^c} f_n - \int_{-\infty}^\infty f = 0. \]

In other words, by writing \(\int_{-\infty}^\infty f = \int_E f + \int_{E^c} f\), we have
\[ \left( \liminf_{n \to \infty} \int_E f_n - \int_E f \right) + \left( \liminf_{n \to \infty} \int_{E^c} f_n - \int_{E^c} f \right) = 0. \]
Since both terms are non-negative and their sum is equal to zero, they must both be zero. In other words
\[
\liminf_{n \to \infty} \int_E f_n - \int_E f = 0
\]
and
\[
\liminf_{n \to \infty} \int_{E^c} f_n - \int_{E^c} f = 0
\]
Now the first equation gives
\[
\liminf_{n \to \infty} \int_E f_n = \int_E f
\]
and using the second equation we obtain
\[
\int_{E^c} f = \liminf_{n \to \infty} \int_{E^c} f_n = \liminf_{n \to \infty} \left( \int_{-\infty}^\infty f_n - \int_E f_n \right) \\
= \liminf_{n \to \infty} \int_{-\infty}^\infty f_n - \limsup_{n \to \infty} \int_E f_n \\
= \int_{-\infty}^\infty f - \limsup_{n \to \infty} \int_E f_n.
\]
Above we used the fact that \(\liminf_{n \to \infty} (a_n + b_n) = a + \liminf_{n \to \infty} b_n\) for sequences \((a_n)_{n=1}^\infty\) and \((b_n)_{n=1}^\infty\) for which \(\lim a_n\) exists and \(a_n \to a\). Thus
\[
\limsup_{n \to \infty} \int_E f_n = \int_{-\infty}^\infty f - \int_{E^c} f = \int_E f.
\]
So we conclude that
\[
\limsup_{n \to \infty} \int_E f_n = \int_E f = \liminf_{n \to \infty} \int_E f_n,
\]
which finally implies that the sequence \((\int_E f_n)_{n=1}^\infty\) converges and its limit is \(\int_E f\). In other words, \(\int_E f_n \to \int_E f\).

**Exercise 3.** Let \((f_n)_{n=1}^\infty\) be a sequence of integrable functions such that \(f_n \to f\) almost everywhere and \(f\) is integrable. Show that \(\int |f_n - f| \to 0\) if and only if \(\int |f_n| \to \int |f|\).
Answer:

"⇒": Assume first that \( \int |f_n - f| \to 0 \). Note that by the reverse triangle inequality we have
\[
\left| \int |f_n| - \int |f| \right| \leq \int ||f_n| - |f|| \leq \int |f_n - f|.
\]
Letting \( n \to \infty \) it follows that \( \int |f_n| \to \int |f| \).

"⇐": Assume then the converse that \( \int |f_n| \to \int |f| \) holds. Define \( g_n = |f| + |f_n| - |f - f_n| \) for all \( n \in \mathbb{N} \), which is non-negative by the triangle inequality. Note that \( g_n \to 2|f| \) almost everywhere. Thus by Fatou's lemma
\[
2 \int |f| = \int 2|f| = \int \liminf_{n \to \infty} g_n \leq \liminf_{n \to \infty} \int g_n = \liminf_{n \to \infty} \int |f| + \liminf_{n \to \infty} \int |f_n| - \limsup_{n \to \infty} \int |f - f_n| = 2 \int |f| - \limsup_{n \to \infty} \int |f - f_n|.
\]
Above we used the fact that \( \liminf_{n \to \infty} (a_n + b_n) = a + \liminf_{n \to \infty} b_n \) for sequences \((a_n)_{n=1}^\infty\) and \((b_n)_{n=1}^\infty\) for which \( \lim a_n \) exists and \( a_n \to a \). Since \( f \) is integrable then this implies that
\[
0 \leq \liminf_{n \to \infty} \int |f - f_n| \leq \limsup_{n \to \infty} \int |f - f_n| \leq 0,
\]
so there in fact holds an equality everywhere and in particular
\[
\lim_{n \to \infty} \int |f - f_n| = 0.
\]

Exercise 4. Prove the Riemann-Lebesgue Theorem: If \( f \) is integrable on \( \mathbb{R} \) then
\[
\int_{-\infty}^{\infty} f(x) \cos(nx) \, dx \to 0 \quad \text{as} \quad n \to \infty.
\]

Answer:

The general idea is to approximate \( f \) by step functions with bounded support so we prove the statement in multiple steps:

(a) Assume first that \( f = \chi_{I_k} \) is an indicator function of some bounded interval \( I_k \) with endpoints \( a_k \) and \( b_k \). Then
\[
\left| \int_{-\infty}^{\infty} f(x) \cos(nx) \, dx \right| = \left| \int_{-\infty}^{\infty} \chi_{I_k} \cos(nx) \, dx \right| = \left| \int_{a_k}^{b_k} \cos(nx) \, dx \right| = \frac{1}{n} |\sin(nb_k) - \sin(na_k)| \leq \frac{2}{n} \to 0
\]
as \( n \to \infty \). So the claim holds for indicator functions of bounded intervals.

(b) Assume then that \( f \) is a step function with bounded support. In other words,

\[
 f = \sum_{i=1}^{k} a_i \chi_{A_i}
\]

for some real \( a_i \) and bounded intervals \( A_i \). By part (a) and linearity of the integral we have

\[
 \int_{-\infty}^\infty f(x) \cos(nx) \, dx = \int_{-\infty}^\infty \left( \sum_{i=1}^{k} a_i \chi_{A_i} \right) \cos(nx) \, dx
 = \sum_{i=1}^{k} a_i \int_{-\infty}^\infty \chi_{A_i} \cos(nx) \, dx \to 0
\]

as \( n \to \infty \), as each of the terms in the sum go to zero. Hence the statement holds for all step functions with bounded support.

(c) We then assume that \( f \) is integrable and nonnegative, and let \( \varepsilon > 0 \) be fixed. By Exercises 1. and 3. of this problem set we find a simple function \( \varphi \leq f \) with bounded support so that \( \int_{-\infty}^\infty |f - \varphi| < \varepsilon \). Let \( p \) be the measure of a compact interval \( K \) containing the support of \( \varphi \). We may assume that \( p \geq 1 \) by bounding the support of \( \varphi \) by a larger interval if necessary. Since \( \varphi \) is a simple function with bounded support then \( M := \sup |\varphi| < \infty \). Again, we can assume that \( M \geq 1 \) by replacing \( M \) by \( \max\{1, \sup |\varphi|\} \) if necessary. We then use Exercise 4 of problem set 1 to find a step function \( g \) so that \( |\varphi - g| < \frac{\varepsilon}{pM} \) except on a set of measure less than \( \frac{\varepsilon}{pM} \), and \( 0 \leq g \leq M \). Call the set where the bound fails by \( E \). Now by triangle inequality, the fact that \( |\cos(nx)| \leq 1 \) for all \( n \in \mathbb{N} \) and \( x \in \mathbb{R} \), and the above estimates,
we have

\[
\left| \int_{-\infty}^{\infty} f(x) \cos(nx) \, dx \right|
\]

\[
= \left| \int_{-\infty}^{\infty} (f(x) + \varphi(x) - \varphi(x) + g(x) - g(x)) \cos(nx) \, dx \right|
\]

\[
\leq \int_{-\infty}^{\infty} |f - \varphi| + \int_{-\infty}^{\infty} |\varphi - g| + \int_{-\infty}^{\infty} g(x) \cos(nx) \, dx
\]

\[
\leq \varepsilon + \frac{\varepsilon}{pM} \int_{E} |\varphi - g| + \int_{-\infty}^{\infty} g(x) \cos(nx) \, dx
\]

\[
\leq \varepsilon + \frac{\varepsilon}{M} + \frac{\varepsilon (\sup |\varphi| + \sup |g|)}{pM} + \int_{-\infty}^{\infty} g(x) \cos(nx) \, dx
\]

\[
\leq \varepsilon + \frac{\varepsilon}{M} + \frac{2M}{pM} + \int_{-\infty}^{\infty} g(x) \cos(nx) \, dx
\]

\[
\leq 4\varepsilon + \int_{-\infty}^{\infty} g(x) \cos(nx) \, dx.
\]

Since the choice of \( \varepsilon > 0 \) was independent of \( n \in \mathbb{N} \), we can let \( n \to \infty \) in the above inequality and note that part (b) implies

\[
\limsup_{n \to \infty} \left| \int_{-\infty}^{\infty} f(x) \cos(nx) \, dx \right| \leq 4\varepsilon
\]

since \( g \) was a step function with bounded support. Since the choice of \( \varepsilon > 0 \) was arbitrary then

\[
\limsup_{n \to \infty} \left| \int_{-\infty}^{\infty} f(x) \cos(nx) \, dx \right| = 0,
\]

which implies that

\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} f(x) \cos(nx) \, dx = 0.
\]

Hence the statement is true for any nonnegative integrable function.

(d) To lastly verify the statement for the most general case, assume that \( f \) is an integrable function. Compose \( f \) to its positive and negative parts, i.e. \( f = f^+ - f^- \), where \( f^+ \) and \( f^- \) are integrable
and nonnegative. Now by part (c) we have

\[
\int_{-\infty}^{\infty} f(x) \cos(nx) \, dx = \int_{-\infty}^{\infty} (f^+(x) - f^-(x)) \cos(nx) \, dx
\]

\[
= \int_{-\infty}^{\infty} f^+(x) \cos(nx) \, dx - \int_{-\infty}^{\infty} f^-(x) \cos(nx) \, dx
\]

\[
\to 0
\]

as \( n \to \infty \) since \( f^+ \) and \( f^- \) were integrable and nonnegative. So

\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} f(x) \cos(nx) \, dx = 0.
\]

Hence the statement is true for all integrable functions \( f \).

Exercise 5. Let \( g \) be a bounded measurable function and let \( f \) be integrable on \( \mathbb{R} \). Show that

\[
\lim_{t \to 0} \int_{-\infty}^{\infty} g(x)(f(x) - f(x + t)) \, dx = 0.
\]

Answer:
We first establish the following lemma:

**Lemma (a)** Let \( f \) be integrable. Then for any \( \varepsilon > 0 \) there exists a continuous function \( h \) with compact support so that \( \int_{-\infty}^{\infty} |f - h| \leq \varepsilon \).

**Proof:**
The proof follows the same pattern as the proof of the previous exercise. Assume first that \( f \) is nonnegative. Then by Exercises 1. and 3. of this problem set we find a simple function \( \phi \leq f \) with bounded support so that \( \int_{-\infty}^{\infty} |f - \phi| < \frac{\varepsilon}{4} \). Let \( p \) be the measure of a compact interval \( K \) containing the support of \( \phi \). We may assume that \( p \geq 1 \) by bounding the support of \( \phi \) by a larger interval if necessary. Since \( \phi \) is a simple function then \( M := \sup |\phi| < \infty \). Again, we can assume that \( M \geq 1 \) by replacing \( M \) by \( \max\{1, \sup |\phi|\} \) if necessary. We then use Exercise 4 of problem set 1 to find a continuous function \( h \) on \( K \) so that \( |\phi - h| < \frac{\varepsilon}{4pM} \) except on a set of measure less than \( \frac{\varepsilon}{4pM} \), and \( 0 \leq h \leq M \). Call the set where the bound fails by \( E \). Technically the given \( h \) is only continuous in \( K \). This does not however pose any problems as we can choose one such \( h \) and extend it continuously to the whole real line by truncating it by \( h\chi_{[a+\delta,b-\delta]} \) for small \( \delta > 0 \), where \( K = [a,b] \), and then
linearly interpolating the endpoints to zero and making it zero outside of $K$. The original $h$ can be chosen so that the new $h$ obtained by this truncation satisfies the bounds that we want. Since $h$ and $\varphi$ vanish outside $K$, then by triangle inequality and the above bounds we have

$$\int_{-\infty}^{\infty} |f - h| \leq \int_{-\infty}^{\infty} |f - \varphi| + \int_{-\infty}^{\infty} |\varphi - h|$$

$$= \int_{-\infty}^{\infty} |f - \varphi| + \int_{K\setminus E} |\varphi - h| + \int_{K \cap E} |\varphi - h|$$

$$\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{2M\varepsilon}{4pM} = \frac{\varepsilon}{4} + \frac{\varepsilon}{4M} + \frac{\varepsilon}{2p}$$

$$\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon.$$

So the claim is true when we assume that $f$ is nonnegative. If $f$ is not nonnegative, then compose $f$ to its positive and negative parts $f = f^+ - f^-$ and apply the previous reasoning to the nonnegative integrable functions $f^+$ and $f^-$, i.e. find continuous $h_1$ and $h_2$ with compact support so that $\int_{-\infty}^{\infty} |f^+ - h_1| \leq \frac{\varepsilon}{2}$ and $\int_{-\infty}^{\infty} |f^- - h_2| \leq \frac{\varepsilon}{2}$. Now $h_1 - h_2$ is a continuous function with compact support and

$$\int_{-\infty}^{\infty} |f - h_1 + h_2| = \int_{-\infty}^{\infty} |f^+ - f^- - h_1 + h_2|$$

$$\leq \int_{-\infty}^{\infty} |f^+ - h_1| + \int_{-\infty}^{\infty} |f^- - h_2|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So the proof of the lemma is done. \qed

We then return to the exercise. Denote $M := \sup |g|$, which is finite since $g$ is bounded. Now since $f$ is integrable then by the above Lemma (a) we find a compactly supported continuous function $h$ so that $\int_{-\infty}^{\infty} |f - h| \leq \frac{\varepsilon}{2M}$. Since $h$ is compactly supported there exists $n \in \mathbb{N}$ so that the support of $h$ is contained in $\overline{B(0, n)}$, where $B(0, n)$ is as usual the open $n$-radius ball around $0$ and the overline denotes closure. Now for all $0 \leq t < 1$ we have

$$h(x) = h(x + t) = 0$$

if $x \in \mathbb{R} \setminus \overline{B(0, n + 1)}$. Thus for all $0 \leq t < 1$ we have by triangle
inequality and the above estimates that
\[
\left| \int_{-\infty}^{\infty} g(x)(f(x) - f(x + t)) \, dx \right| \\
= \left| \int_{-\infty}^{\infty} g(x)(f(x) - f(x + t) - h(x) + h(x) - h(x + t) + h(x + t)) \, dx \right| \\
\leq \int_{-\infty}^{\infty} |g(x)||h(x) - h(x + t)| \, dx + \int_{-\infty}^{\infty} |g(x)||f(x) - h(x)| \, dx \\
+ \int_{-\infty}^{\infty} |g(x)||h(x + t) - f(x + t)| \, dx \\
\leq M \int_{B(0, n+1)} |h(x) - h(x + t)| \, dx + 2M \int_{-\infty}^{\infty} |f - h| \\
\leq M \int_{B(0, n+1)} |h(x) - h(x + t)| \, dx + 2M \frac{\varepsilon}{2M} \\
= M \int_{B(0, n+1)} |h(x) - h(x + t)| \, dx + \varepsilon.
\]

Now as a continuous function, \( h \) is uniformly continuous on the compact set \( B(0, n + 1) \). So it follows that \( |h(x) - h(x + t)| \to 0 \) as \( t \to 0 \) uniformly. Hence by letting \( t \to 0 \) we get that
\[
\lim_{t \to 0} \left| \int_{-\infty}^{\infty} g(x)(f(x) - f(x + t)) \, dx \right| \leq \varepsilon.
\]

Since the choice of \( \varepsilon > 0 \) was arbitrary then
\[
\lim_{t \to 0} \int_{-\infty}^{\infty} g(x)(f(x) - f(x + t)) \, dx = 0.
\]

**Exercise 6.** Let \( f \) be integrable over a measurable set \( E \). Show that for any \( \varepsilon > 0 \) there is a continuous function \( g \) supported on a finite measure set such that \( \int_{E} |f - g| \leq \varepsilon \).

**Answer:**

Apply Lemma (a) from Exercise 5. to the integrable function \( f_{\chi_{E}} \) to find a compactly supported continuous function \( g \) so that \( \int_{-\infty}^{\infty} |f_{\chi_{E}} - g| \leq \varepsilon \). Now
\[
\int_{E} |f - g| = \int_{-\infty}^{\infty} |f - g|_{\chi_{E}} = \int_{-\infty}^{\infty} |f_{\chi_{E}} - g_{\chi_{E}}| \leq \int_{-\infty}^{\infty} |f_{\chi_{E}} - g| \leq \varepsilon.
\]

So \( g \) satisfies the desired properties.