1 Practice Exam

Problem 1. Prove Bolzano-Weierstrass

Problem 2. Let \((a_{nm})_{n,m=1}^{\infty} \in \mathbb{R}\). Show that

\[
\sup_n \sup_m a_{nm} = \sup_m \sup_n a_{nm}
\]

and

\[
\inf_n \inf_m a_{nm} = \inf_m \inf_n a_{nm}
\]

Problem 3. Let \(f : \mathbb{R} \rightarrow \mathbb{R}\). Show that there are sequences \(x_n\) and \(y_n\) such that

\[
f(x_n) \rightarrow \inf_{x \in \mathbb{R}} f(x)
\]

and

\[
f(y_n) \rightarrow \sup_{x \in \mathbb{R}} f(x).
\]

Show by example that there are continuous functions on \(\mathbb{R}\) that are bounded above and below which achieve neither their supremum nor their infimum. (Hint: look at \(\tanh(x)\).)

Problem 4. Suppose that \(f : K \rightarrow \mathbb{R}\) is continuous, where \(K \subset \mathbb{R}\) is compact. Prove that there is a constant \(C\) such that

\[
|f| \leq C.
\]

Prove that there is an \(x_*\) and a \(y_*\) such that

\[
f(x_*) = \inf_{x \in K} f(x)
\]

and

\[
f(y_*) = \sup_{y \in K} f(y).
\]

Why doesn’t this contradict the previous problem?

Problem 5. Prove using the definition of continuity that

\[
f(x) = \sqrt{x}
\]

is continuous as a function \(f : (0, \infty) \rightarrow \mathbb{R}\).

Problem 6. Let \(f : \mathbb{R}^d \rightarrow \mathbb{R}\). If \(|f(x) - f(y)| \leq K||x-y||_2\) for some \(K \in \mathbb{R}_+\), then it is said to be \(K\)-Lipschitz. Show that if a function is \(K\)-lipschitz then it is continuous. Provide an example of a lipschitz function on \(\mathbb{R}\). Give an example a function that is continuous but not Lipschitz.

2 More Problems

Problem 7. Prove the Nested Interval Theorem

Problem 8. Prove Abel’s summation by parts formula

Problem 9. Prove the alternating series test

Problem 10. Prove using the definition of the limit that if \(a_n = 1/n\), then

\[
\lim_{n \rightarrow \infty} \sqrt{\frac{1}{n}} = 0
\]