Large deviations asymptotics for spherical integrals

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Abstract

Consider the spherical integral $I_N^{(\beta)}(D_N, E_N) := \int \exp\{N\mathrm{tr}(UD_NU^*E_N)\}dm_N^\beta(U)$, where m_N^β denote the Haar measure on the orthogonal group $\mathcal{O}_N$ when $\beta = 1$ and on the unitary group $\mathcal{U}_N$ when $\beta = 2$, and D_N, E_N are diagonal real matrices whose spectral measures converge to μ_D, μ_E. In this paper we prove the existence and represent as solution to a variational problem the limit $I^{(\beta)}(\mu_D, \mu_E) := \lim N^{-2} \log I_N^{(\beta)}(D_N, E_N)$. This limit appears in so called "matrix models" but also in the evaluation of large deviations of the spectral measure of generalized Wishart matrices. Our technique is based on stochastic calculus, large deviations, and elements from free probability.

Keywords : Large deviations, random matrices, non-commutative measure, integration.

Mathematics Subject of Classification : 60F10, 15A52, 46L50.

1 Introduction and statement of results

1.1 Asymptotics of spherical integrals

Let m_N^β denote the Haar measure on the orthogonal group $\mathcal{O}_N$ when $\beta = 1$ and on the unitary group $\mathcal{U}_N$ when $\beta = 2$. We shall consider in this paper the following integrals, which we will call spherical integrals, given, for two $N \times N$ matrices D_N and E_N, by

$$I_N^{(\beta)}(D_N, E_N) := \int \exp\{N\mathrm{tr}(UD_NU^*E_N)\}dm_N^\beta(U).$$

We will restrict ourselves to the case where D_N and E_N are symmetric if $\beta = 1$ and Hermitian if $\beta = 2$.

Such integrals appear in physics in the so-called matrix models where one is interested in evaluating integrals of the form

$$Z_N = \int \exp\{\sum_{i=1}^n \mathrm{tr}V_i(M_i) + \sum_{i,j=1}^n \alpha_{ij}\mathrm{tr}(M_iM_j)\} \prod_{i=1}^n dM_i$$

where dM is the Lebesgue measure on the set $\mathcal{H}_N$ of $N \times N$ Hermitian (or symmetric) matrices and tr denotes the usual trace on the set $\mathcal{M}_N$ of $N \times N$ matrices: $\mathrm{tr}(A) = \sum_{i=1}^N A_{ii}$. It turns out that such an

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evaluation, even for the first term in the large N asymptotics, is highly non trivial. We refer to [18], [25], [26] and [10] for research in this direction. As already noticed by [8], [25], when α_{ij} is null if $j \neq i+1$, it is enough to obtain the first order asymptotics of the spherical integrals as the spectral measures of D_N and E_N converge in order to estimate Z_N as N goes to infinity.

Thus, if for a $N \times N$ matrix A with eigenvalues $(a_1, \cdots, a_N)$ we denote by $\hat{\mu}_A^N = (1/N)\sum_{i=1}^N \delta_{a_i}$ the spectral measure of A, one wishes to investigate the large N limit of $N^{-2}\log I_N^{(\beta)}(D_N, E_N)$ when the spectral measures $\hat{\mu}_{D_N}^N$ and $\hat{\mu}_{E_N}^N$ converge weakly towards the probability measures μ_D and μ_E respectively.

The very same question arises when one studies the large deviations properties of the spectral measure of Gaussian sample covariance matrices XTX^* with T a general positive definite matrix whose spectral distribution converges as the size of the sample goes to infinity. The limiting spectral measure for such matrices is well known, see [19], [21], and the evaluation of the large deviations properties for them was actually the original motivation for this paper. The solution to this problem is described in Section 1.2 below.

Recall that by a formula due to Harish-Chandra, and used in this context by Itzykson and Zuber, see [17, Appendix 5], whenever the eigenvalues of D_N and E_N are distinct then

$$I_N^{(2)}(D_N, E_N) = \frac{\det\{\exp N D_N(i) E_N(j)\}}{\Delta(D_N)\Delta(E_N)},$$

where $\Delta(D_N) = \prod_{i<j}(D_N(j) - D_N(i))$ and $\Delta(E_N) = \prod_{i<j}(E_N(j) - E_N(i))$ are the VanderMonde determinants associated with D_N, E_N. Although this formula seems to solve the problem, it is far from doing so, due to the possible cancellations appearing in the determinant. Only particular cases can be handled, notably the (trivial) case where one of D_N or E_N is a multiple of the identity, or the case where $D_N(i) = 1 + i/N$, implying $\mu_D([1, x+1)) = x$ for $x \in (0,1)$, in which case algebraic manipulations involving VanderMonde determinants yield

$$I_N^{(2)}(D_N, E_N) = C_N \exp(2N\mathrm{tr}E_N) \prod_{i<j} \frac{(1 - e^{E_N(j) - E_N(i)})}{(E_N(j) - E_N(i))}.$$

Obviously, the asymptotics of $N^{-2}\log I_N^{(2)}(D_N, E_N)$ may easily be read off the above formula in this case.

Our approach to the evaluation of the above asymptotics is based on the theory of large deviations (we refer to [9] for background on large deviations). It is somewhat related to the formal derivation of the same asymptotics in [16], although both the language and methods differ. The key point is to relate the evaluation of the spherical integrals with the deviations of the law of the spectral measure of a Gaussian Wigner matrix with non degenerate initial data. Namely, let $\mathbb{P}_{D_N}^\beta$ be the law of the spectral measure of $W + D_N$ for a $N \times N$ Gaussian Wigner matrix W with real (resp. complex) entries if $\beta = 1$ (resp. $\beta = 2$) (that is a $N \times N$ symmetric (resp. Hermitian) matrix with centered Gaussian entries of covariance N^{-1}); for any measurable subset A of the set $M_1(\mathbb{R})$ of probability measures on $\mathbb{R}$, $\mathbb{P}_{D_N}^\beta$ is given by

$$\mathbb{P}_{D_N}^\beta(A) \quad = \quad (Z_N^\beta)^{-1} \int_{\hat{\mu}_{X_N}^N \in A} e^{-\frac{N}{2}\mathrm{tr}(X_N - D_N)^2} dX_N$$

with dX_N the Lebesgue measure on the set of Hermitian ($\beta = 2$) or symmetric ($\beta = 1$) matrices.

Let d denote a distance on the space of probability measures on $\mathbb{R}$, compatible with the weak topology. The main outcome of our large deviations analysis, see Corollary 1.6 below, is that if D_N is a sequence of uniformly bounded operators, then for any probability measure μ in $M_1(\mathbb{R})$,

$$\lim_{\delta\downarrow 0}\liminf_{N\uparrow\infty}\frac{1}{N^2}\log \mathbb{P}^{\beta}_{D_N}\left(d(\hat\mu^N_{X_N},\mu)<\delta\right) = \lim_{\delta\downarrow 0}\limsup_{N\uparrow\infty}\frac{1}{N^2}\log \mathbb{P}^{\beta}_{D_N}\left(d(\hat\mu^N_{X_N},\mu)<\delta\right) = -J_\beta(\mu_D,\mu),$$

with a function $J_\beta(\cdot,\cdot)$ given in terms of the solution of an appropriate variational problem, see (1.6). The main result of this paper is then the following consequence:

Theorem 1.1 *Assume the existence of a compact subset $\mathcal{K}$ of $\mathbb{R}$ such that $\operatorname{supp}\hat\mu^N_{D_N}\subset\mathcal{K}$ for all $N\in\mathbb{N}$. Moreover, suppose that $\hat\mu^N_{E_N}(x^2)$ is uniformly bounded (in N). Suppose that $\hat\mu^N_{E_N}$ and $\hat\mu^N_{D_N}$ converge weakly towards μ_E and μ_D respectively. Then,*

$$I^{(\beta)}(\mu_D,\mu_E):=\lim_{N\to\infty}\frac{1}{N^2}\log I^{(\beta)}_N(D_N,E_N) = -J_\beta(\mu_D,\mu_E)+I_\beta(\mu_E)-\inf_{\mu\in M_1(\mathbb{R})}I_\beta(\mu)+\frac{1}{2}\int x^2 d\mu_D(x)$$

where, for any $\mu\in M_1(\mathbb{R})$,

$$I_\beta(\mu)=\frac{1}{2}\int x^2 d\mu(x)-\frac{\beta}{2}\int\log|x-y|d\mu(x)d\mu(y).$$

1.2 Large deviation for the spectral measure of Gaussian sample covariance matrices

Sample covariance matrices (or Wishart matrices) are matrices of the form

$$Y_{N,M}=X_{N,M}T_M X^*_{N,M}\,.$$

Here, $X_{N,M}$ is an $N\times M$ matrix with centered real or complex i.i.d. entries of covariance N^{-1} and T_M is an $M\times M$ Hermitian (or symmetric) matrix. These matrices are often considered in the limit where M/N goes to a constant $\alpha>0$. Let us assume that $M\le N$, and hence $\alpha\in[0,1]$, to fix the notations. Then, $Y_{N,M}$ has $N-M$ null eigenvalues. Let $(\lambda_1,\cdots,\lambda_M)$ be the M non trivial remaining eigenvalues and denote $\hat\mu^M=M^{-1}\sum_{i=1}^M\delta_{\lambda_i}$. In the case where $T_M=I$ and the entries of $X_{N,M}$ are Gaussian, Hiai and Petz [12] proved that the law of $\hat\mu^M$ satisfies a large deviation principle. We generalize this result to positive definite matrices T_M whose spectral measures converge, while keeping the hypothesis of Gaussian entries. In fact, when dealing with Gaussian entries, we have the following formula for the joint law of the eigenvalues

$$\begin{aligned}
d\sigma^\beta_M(\lambda_1,\cdots,\lambda_M) &= \frac{1}{Z^\beta_{T_M}}\prod_{i<j}|\lambda_i-\lambda_j|^\beta\prod\lambda_i^{\frac{\beta}{2}(N-M+1)-1}\int e^{-\frac{N}{2}\mathrm{tr}(UT_M^{-1}U^*D(\lambda))}dm^\beta_N(U)\prod 1_{\lambda_i\ge 0}d\lambda_i\\
&= \frac{1}{Z^\beta_{T_M}}\prod_{i<j}|\lambda_i-\lambda_j|^\beta\prod\lambda_i^{\frac{\beta}{2}(N-M+1)-1}I^{(\beta)}_N(D(\lambda),(2T_M)^{-1})\prod 1_{\lambda_i\ge 0}d\lambda_i
\end{aligned}$$

with $D(\lambda)$ the $M\times M$ diagonal matrix with entries $(\lambda_1,\cdots,\lambda_M)$ and, as before, $\beta=1$ if the entries of $X_{N,M}$ are real, $\beta=2$ if they are complex. $Z^\beta_N(T_M)$ is the normalizing constant such that σ^β_M has mass one. This formula can be found in [13, (58) and (95)].

From this formula, the asymptotics of the spherical integrals found in the previous section and Laplace methods as developed in [1] (or [12]), one can easily obtain the following theorem

Theorem 1.2 *Assume that $(T_M, M \in \mathbb{N})$ is a sequence of matrices with eigenvalues $(t_1^M, \cdots, t_M^M)$ such that*

a) There exist $\rho \geq \eta > 0$, such that for each M and each $1 \leq i \leq M$, $\eta \leq t_i^M \leq \rho$.

b) As M tends to infinity, $\hat{\mu}^M_{T_M} = \frac{1}{M}\sum_{i=1}^M \delta_{t_i^M}$ converges towards a probability measure μ_T.

Then, the law of $\hat{\mu}^M$ under σ_M^β satisfies a large deviation principle with the speed M^2 and the good rate function $W_\beta : M_1(\mathbb{R}^+) \to \mathbb{R}^+$ given by

$$W_\beta(\mu) = \frac{\beta}{2}\int \log|x-y|^{-1} d\mu(x)d\mu(y) - \frac{\beta}{2}(\alpha^{-1}-1)\int \log(x)d\mu(x) - I^{(\beta)}(\mu_T \circ (2x)^{-1}, \mu) - m$$

with

$$m := \inf_{\nu \in M_1(\mathbb{R}^+)} \{\frac{\beta}{2}\int \log|x-y|^{-1} d\nu(x)d\nu(y) - \frac{\beta}{2}(\alpha^{-1}-1)\int \log(x)d\nu(x) - I^{(\beta)}(\mu_T \circ (2x)^{-1}, \nu)\}.$$

Here, $\mu_T \circ (2x)^{-1}$ is the law of $(2x)^{-1}$ under μ_T, that is the law given, for all bounded measurable function f by

$$\mu_T \circ (2x)^{-1}(f) = \int f(\frac{1}{2x})d\mu_T(x).$$

The proof, as we mentioned above, is straightforward. Indeed, one notices that our assumptions on T_M imply that T_M^{-1} has uniformly bounded spectrum with converging spectral measure so that Theorem 1.1 applies and hence that the techniques of [1] yield

$$\liminf_{M\to\infty} \frac{1}{M^2} \log Z^\beta_{T_M} \geq -m.$$

Therefore, since

$$I_N^{(\beta)}(D(\lambda), (2T_M)^{-1}) \leq e^{-\frac{N}{2\rho}\sum_{i=1}^M \lambda_i},$$

and finding, following the techniques of [1], a finite constant C such that

$$\limsup_{M\to\infty} M^{-2} \log Z^\beta_{(2\rho)^{-1}I} \leq C,$$

we conclude by Chebyshev's inequality that for any $L \in \mathbb{R}^+$,

$$\limsup_{M\to\infty} \frac{1}{M^2} \log \sigma_M^\beta(\hat{\mu}^M(x) \geq L) = -(4\rho)^{-1}L + C + m.$$

As a consequence, $\sigma_M^\beta \circ (\hat{\mu}^M)^{-1}$ is exponentially tight and one can apply again Theorem 1.1 and the ideas of [1] to obtain the convergence of $M^{-2}\log Z^\beta_{T_M}$ towards $-m$ and then the weak large deviation principle for $\sigma_M^\beta \circ (\hat{\mu}^M)^{-1}$ with rate function W_β. The fact that W_β is a good rate function is now a direct consequence of the above weak large deviation principle and exponential tightness (see [9, Lemma 1.2.18]).

1.3 Large deviations

We describe in this section our main large deviations results. Our analysis follows and improves the ideas of [6] where large deviations estimates were obtained for the spectral process of $H_N(\cdot) + D_N$ with a Hermitian (resp. symmetric) Brownian motion constructed as Wigner matrices but with Brownian motion entries (see Sections 2 and 3 for details). To understand this point of view, consider the measure valued process given by $\hat{\mu}_t^N = \hat{\mu}^N_{H_N(t)+D_N}$ as an element of the space $\mathcal{C}([0,1], M_1(\mathbb{R}))$ of continuous measure-valued processes furnished with the topology generated by the weak topology on $M_1(\mathbb{R})$ and the uniform topology on $[0,1]$. $\mathcal{C}([0,1],M_1(\mathbb{R}))$ is a Polish space with respect to the distance given by

$$D(\mu,\nu) := \sup_{t\in[0,1]} d(\mu_t,\nu_t)$$

with d the Wasserstein (also called the Monge-Kantorovich-Rubinstein) distance on $M_1(\mathbb{R})$ given by

$$d(\mu,\nu) := \sup_{||f||_{\mathcal{L}}\le 1} |\int f d\mu - \int f d\nu|$$

where

$$||f||_{\mathcal{L}} = \sup_{x\in\mathbb{R}} |f(x)| + \sup_{x,y\in\mathbb{R}} \left|\frac{f(x)-f(y)}{x-y}\right| .$$

We shall establish a large deviation principle for the law of $\hat{\mu}^N_\cdot$ with a good rate function defined as follows. We set, for any $f,g \in \mathcal{C}_b^{2,1}(\mathbb{R}\times[0,1])$, any $s\le t\in[0,1]$, and any $\nu_\cdot \in \mathcal{C}([0,1],M_1(\mathbb{R}))$,

$$\begin{aligned} S^{s,t}(\nu,f) &= \int f(x,t)d\nu_t(x) - \int f(x,s)d\nu_s(x) \\ &- \int_s^t \int \partial_u f(x,u)d\nu_u(x)du - \frac{1}{2}\int_s^t \int\int \frac{\partial_x f(x,u)-\partial_x f(y,u)}{x-y} d\nu_u(x)d\nu_u(y)du, \end{aligned} \quad (1.1)$$

$$< f,g >^{\nu}_{s,t} = \int_s^t \int \partial_x f(x,u)\partial_x g(x,u)d\nu_u(x)du\,, \quad (1.2)$$

and

$$\bar{S}^{s,t}(\nu,f) = S^{s,t}(\nu,f) - \frac{1}{2} < f,f >^{\nu}_{s,t} . \quad (1.3)$$

Set, for any probability measure $\mu\in M_1(\mathbb{R})$,

$$S_\mu(\nu) := \begin{cases} +\infty\,, & \text{if } \nu_0 \ne \mu, \\ S^{0,1}(\nu) := \sup_{f\in\mathcal{C}_b^{2,1}(\mathbb{R}\times[0,1])} \sup_{0\le s\le t\le 1} \bar{S}^{s,t}(\nu,f)\,, & \text{otherwise.} \end{cases}$$

We often make the following assumption on the sequence D_N:

Assumption 1.3

$$\lim_{N\to\infty} \hat{\mu}^N_{D_N} = \mu_D\,, \quad \sup_N \hat{\mu}^N_{D_N}(x^2) < \infty\,.$$

For both $\beta=1$ and $\beta=2$, our main large deviations result is:

Theorem 1.4 *1) For any $\mu\in M_1(\mathbb{R})$, S_μ is a good rate function on $\mathcal{C}([0,1],M_1(\mathbb{R}))$.*

2) Assume Assumption 1.3. Then,

a) For any closed set $F\subset\mathcal{C}([0,1],M_1(\mathbb{R}))$

$$\limsup_{N\to\infty} \frac{1}{N^2}\log \mathbb{P}\left(\hat{\mu}^N_\cdot \in F\right) \le -\frac{\beta}{2}\inf_{\nu\in F} S_{\mu_D}(\nu).$$

b) Denote

$$\mathcal{A} = \{\nu \in \mathcal{C}([0,1], M_1(\mathbb{R}))\,;\ \exists \eta > 0,\ \sup_{t\in[0,1]} \nu_t(|x|^{5+\eta}) < \infty.\}$$

Then, for any open subset $O \in \mathcal{C}([0,1], M_1(\mathbb{R}))$,

$$\liminf_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}\left(\hat{\mu}^N_{\cdot} \in O\right) \geq -\frac{\beta}{2} \inf_{\nu \in O \cap \mathcal{A}} S_{\mu_D}(\nu).$$

The main observation needed in order to relate this theorem with spherical integrals is that $H_N(1)$ is in fact a Gaussian Wigner matrix so that $\mathbb{P}^{\beta}_{D_N}$ can be seen as the law of the spectral measure $\hat{\mu}^N_1$ of $H_N(1) + D_N$. Henceforth, as a consequence of the contraction principle of large deviations theory, see [9, Theorem 4.2.1], we obtain from Theorem 1.4 the

Theorem 1.5 *Assume Assumption 1.3. Then, for any probability measure $\mu \in M_1(\mathbb{R})$,*

$$\begin{aligned} -\lim_{\delta\to 0} \frac{\beta}{2} \inf\{S_{\mu_D}(\nu_\cdot) \quad &: \quad \nu \in \mathcal{A}, d(\nu_1, \mu) < \delta\} \leq \lim_{\delta\to 0} \liminf_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}^{\beta}_{D_N}\left(d(\hat{\mu}^N_{X_N}, \mu) < \delta\right) \\ &\leq \lim_{\delta\to 0} \limsup_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}^{\beta}_{D_N}\left(d(\hat{\mu}^N_{X_N}, \mu) < \delta\right) \leq -\frac{\beta}{2} \inf\{S_{\mu_D}(\nu_\cdot); \nu_1 = \mu\}. \end{aligned} \tag{1.4}$$

Observe here that, because $\mathcal{A}$ is not closed, it is not clear whether the left hand side of (1.4) should be equal to $-\inf\{S_{\mu_D}(\nu_\cdot) : \nu \in \mathcal{A}, \nu_1 = \mu\}$ or not.

Finally, we shall prove in Theorem 4.1 below that, when μ_D is compactly supported,

$$\lim_{\delta\to 0} \inf\{S_{\mu_D}(\nu_\cdot) : \ \nu \in \mathcal{A}, d(\nu_1, \mu) < \delta\} = \inf\{S_{\mu_D}(\nu_\cdot); \nu_1 = \mu\} \tag{1.5}$$

implying together with Theorem 1.5 that

Corollary 1.6 *Assume Assumption 1.3 with a compactly supported μ_D. Then, for any probability measure $\mu \in M_1(\mathbb{R})$, $N^{-2} \log \mathbb{P}^{\beta}_{D_N}\left(d(\hat{\mu}^N_{X_N}, \mu) < \delta\right)$ converges as first N goes to infinity and then δ goes to zero towards the quantity $-J_\beta(\mu_D, \mu)$ given by*

$$J_\beta(\mu_D, \mu) = \frac{\beta}{2} \inf\{S_{\mu_D}(\nu_\cdot); \nu_1 = \mu\}. \tag{1.6}$$

We refer to Section 6 for a discussion of candidates for the minimizing path in (1.6).

We remark that in the context of random matrix theory, it is natural to consider also the symplectic ensemble, where the matrix considered are quaternion matrices, and $\beta = 4$, see [17]. To keep this article within reasonable length, we do not treat this case in details here, except for mentioning that the methods carry over to that case too.

The organization of this paper is as follows; in the next section we tackle the heart of the paper, namely the proof of Theorem 1.4 where we consider Hermitian Brownian motions. The generalization to symmetric Brownian motions is done in Section 3. We prove Corollary 1.6 in section 4. Equipped with these preliminaries, we then relate in Section 5 the deviations of Wigner matrices with the asymptotics of spherical integrals, and prove Theorem 1.1. In Section 6, we discuss the relation between our variational problem which gives the value of J_β to the one appearing in [16]. The discussion of matrix models will be the subject of another research.

2 Large deviation for the law of the spectral process of the Hermitian Brownian motion

The Hermitian Brownian motion H^N starting from the origin is defined as the Markov process $(H_N(t))_{t\in\mathbb{R}^+}$ with values in the space $\mathcal{H}_N$ of Hermitian matrices of dimension N and complex Brownian motions entries so that

$$E[H_N^{i,j}(t)H_N^{k,l}(s)] = \frac{t\wedge s}{N}\delta_i^l\delta_k^j$$

Explicitly, we can construct the entries $\{H_N^{i,j}(t), t \geq 0, (i,j) \in \{1,..,N\}\}$ via independent real valued Brownian motions $(\beta_{i,j}, \widetilde{\beta}_{k,l})_{1\leq i\leq j\leq N}^{1\leq k<l\leq N}$ by

$$H_N^{k,l} = \begin{cases} \frac{1}{\sqrt{2N}}\left(\beta_{k,l} + i\widetilde{\beta}_{k,l}\right), & \text{if } k<l \\ \frac{1}{\sqrt{2N}}\left(\beta_{l,k} - i\widetilde{\beta}_{l,k}\right), & \text{if } k>l \\ \frac{1}{\sqrt{N}}\beta_{l,l}, & \text{if } k=l. \end{cases}$$

Let D_N be a matrix in $\mathcal{H}_N$ with eigenvalues $(d_i)_{1\leq i\leq N} \in \mathbb{R}^N$, and set $X_N(t) = D_N + H_N(t)$. Let $(\lambda_i^N(t))_{1\leq i\leq N}$ be the (real-valued) eigenvalues of $X_N(t)$ and define the spectral empirical process by

$$\begin{array}{rccl} \hat{\mu}^N: & [0,1] & \longrightarrow & M_1(\mathbb{R}) \\ & t & \longrightarrow & \hat{\mu}_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i^N(t)}\,. \end{array}$$

We shall prove in this section Theorem 1.4 in the case $\beta = 2$. This theorem is rather close to [6, Theorem 1.1] where the authors considered the case $\mu_D = \delta_0$. However, the lower bound is here much sharper than in [6] and actually this highly non trivial sharpening is the main object of this section. We shall first present the key to our approach: Itô's calculus. Then, we shall obtain the large deviation upper bound and study the rate function S_{μ_D}. Finally, we investigate the large deviation lower bound.

2.1 Itô's calculus

It was proved in [6, Section 2.1] that $\hat{\mu}_\cdot^N$ satisfies an Itô's formula when $\hat{\mu}_{D_N}^N = \delta_0$. This assumption is in fact clearly irrelevant and one can apply exactly the same arguments to check it for general $\hat{\mu}_{D_N}^N$. Then, with the notations of (1.1) and (1.2), we have

Theorem 2.1 ([6, Lemma 1.1]) *For any $N \in \mathbb{N}$, any $f \in \mathcal{C}_b^{2,1}(\mathbb{R}\times[0,1])$ and any $s \in [0,1)$, $\left(S^{s,t}(\hat{\mu}^N, f), s \leq t \leq 1\right)$ is a bounded martingale with quadratic variation*

$$< S^{s,\cdot}(\hat{\mu}^N, f) >_t = \frac{1}{N^2} < f, f >_{s,t}^{\hat{\mu}^N}\,.$$

The restriction to functions $f \in \mathcal{C}_b^{2,1}(\mathbb{R}\times[0,1])$ was not due to technical reasons but only motivated by the goal to achieve in [6]. For our needs, we slightly generalize this formula to functions of the form

$$f(x,t) = \sum_{k=0}^{n} 1\!\!1_{t\in[t_k,t_{k+1}[} f_k(x) \tag{2.1}$$

with times $0 = t_0 < t_1, \cdots, t_n < t_{n+1} = 1$ and twice continuously differentiable functions f_k with uniformly bounded first and second derivatives (note that we do not want to impose a boundedness restriction on the f_k!).

For $\mu_. \in \mathcal{C}([0,1], M_1(\mathbb{R}))$ such that $\sup_{t\in[0,1]} \mu_t(x^2) < \infty$, $\mu_t(f_k)$ is well defined for any $k \in \{0, \cdots, n\}$ and $t \in [0,1]$. Hence, we can extend $S^{s,t}$ for such processes as follows

$$S^{s,t}(\mu, f) = \sum_{k=0}^{n} S^{t_k \vee s \wedge t, t_{k+1} \vee s \wedge t}(\mu, f_k).$$

Observe that for a given $N \in \mathbb{N}$, $\sup_{t\in[0,1]} \hat{\mu}_t^N(x^2) < \infty$ almost surely so that $S^{s,t}(\hat{\mu}^N, f)$ is well defined. We claim that

Corollary 2.2 *For any $N \in \mathbb{N}$, any f of the form (2.1) and any $s \in [0,1)$, $\left(S^{s,t}(\hat{\mu}^N, f), s \le t \le 1\right)$ is a bounded martingale with quadratic variation*

$$< S^{s,\cdot}(\hat{\mu}^N, f) >_t = \frac{1}{N^2} < f, f >_{s,t}^{\hat{\mu}^N} .$$

Further, for any $g \in \mathcal{C}_b^{2,1}(\mathbb{R} \times [0,1])$ and any $0 \le s \le t \le 1$,

$$< S^{s,\cdot}(\hat{\mu}^N, f), S^{s,\cdot}(\hat{\mu}^N, g) >_t = \frac{1}{N^2} < f, g >_{s,t}^{\hat{\mu}^N} .$$

Proof. Note first that we can approximate any function of type (2.1) by a sequence $f^p(x,t) = \sum_{k=0}^{n} \mathbb{1}_{t\in[t_k,t_{k+1}[} f_k^p(x)$ with functions $f_k^p \in \mathcal{C}_b^2(\mathbb{R})$ such that

- On $|x| \le p$, for $k \in \{0, \cdots, n\}$, $f_k^p(x,t) = f_k(x)$,
- On $|x| \ge p+1$ and $k \in \{0, \cdots, n\}$, $f_k^p(x,t) = f_k((p+1)\mathrm{sgn}(x))$.
- On $p \le |x| \le p+1$ and $k \in \{0, \cdots, n\}$, $f_k^p(x,t)$ is smooth and pointwise bounded by $c|x| + d$ with c, d such that $f_k(x) \le c|x| + d$ for all $k \in \{1, \cdots, n\}$. Further, its first and second space derivatives are uniformly bounded by $M := 2\sup_{k\in\{0,\cdots n\}}(||\partial_x f_k||_\infty + ||\partial_x^2 f_k(x)||_\infty)$.

With such a choice of approximation, it is not difficult to verify that, for any $0 \le s \le t \le 1$

$$\begin{aligned} \mathbb{E}\left[|S^{s,t}(\hat{\mu}^N, f) - S^{s,t}(\hat{\mu}^N, f^p)|^2\right] &\le 16 \sup_{u\in[0,1]} \mathbb{E}\left[\hat{\mu}_u^N[(c|x|+d)\mathbb{1}_{|x|\ge p}]^2\right] \\ &+ 4M^2 \sup_{u\in[0,1]} \mathbb{E}\left[\hat{\mu}_u^N[\mathbb{1}_{|x|\ge p}]\right] . \end{aligned}$$

Hence, since for all given $N \in \mathbb{N}$, $\sup_{u\in[0,1]} \mathbb{E}\left[\hat{\mu}_u^N[|x|^4]\right] < \infty$, we conclude that for any $s, t \in [0,1]$,

$$\lim_{p\to\infty} \mathbb{E}\left[|S^{s,t}(\hat{\mu}^N, f) - S^{s,t}(\hat{\mu}^N, f^p)|^2\right] = 0.$$

Further,

$$\mathbb{E}\left[< S^{s,\cdot}(\hat{\mu}^N, f) - S^{s,\cdot}(\hat{\mu}^N, f^p) >_t\right] \le M^2 \sup_{u\in[0,1]} \mathbb{E}\left[\hat{\mu}_u^N[\mathbb{1}_{|x|\ge p}]\right]$$

goes as well to zero as p goes to infinity. Thus, since for any $p \in \mathbb{N}$ and $k \in \{1, \cdots, n\}$, $(S^{s\vee t_k \wedge t_{k+1}, t\vee t_k\wedge t_{k+1}}(\hat{\mu}^N, f_k^p), t \geq s)$ are martingales such that

$$< S^{s\vee t_k\wedge t_{k+1}, .\vee t_k\wedge t_{k+1}}(\hat{\mu}^N, f_k^p), S^{s\vee t_l\wedge t_{l+1}, .\vee t_l\wedge t_{l+1}}(\hat{\mu}^N, f_l^p) >_t = 0, \qquad \text{if } k \neq l,$$

the proof of the lemma is complete.

□

Remark 2.3: Note here that the condition $\sup_{t\in[0,1]} \mu_t(x^2) < \infty$ is not in fact necessary to define $S^{s,t}(\mu, f)$ with f of the form described in (2.1). For instance, assume that

$$f(x, u) = \sum_{k=0}^{n} 1\!\!1_{u\in[t_k, t_{k+1}[} c_k x + g(x, u)$$

with finite constants c_k and $g \in \mathcal{C}^{2,1}(\mathbb{R} \times [t_k, t_{k+1}[)$ for all $k \in \{0, \cdots, n\}$ such that $\sup_{t\in[0,1]} \mu_t(g(\cdot, t)^2) < \infty$ and $g(\cdot, t)$ has bounded spatial derivatives. Then, under the additional assumption that $\mu_t = P * \nu_t$ for P a Cauchy law and a process $\nu_.$ satisfying $\sup_{t\in[0,1]} \nu_t(x^2) < \infty$, we can set for any $s, t \in [t_k, t_{k+1}[$, $k \in \{0, \cdots n\}$,

$$S^{s,t}(\mu, f) := S^{s,t}(\mu, g) + c_k (\nu_t(x) - \nu_s(x)) \tag{2.2}$$

and $S^{s,t}(\mu, g)$ is well defined. Further, Corollary 2.2 holds for such functions whatever is the initial condition D_N since its entries are finite.

2.2 Large deviation upper bound

From the previous Itô's formula, one can deduce as in [6] a large deviation upper bound for the measure valued process $\hat{\mu}^N_. \in \mathcal{C}([0,1], M_1(\mathbb{R}))$. To this end, we shall make the following assumption on the initial condition D_N ;

(H)

$$C_D := \sup_{N\in\mathbb{N}} \hat{\mu}^N_{D_N}(\log(x^2 + 1)) < \infty,$$

implying that $(\hat{\mu}^N_{D_N}, N \in \mathbb{N})$ is tight. Moreover, $\hat{\mu}^N_{D_N}$ converges weakly, as N goes to infinity, towards a probability measure μ_D.

Then, we shall prove, with the notations of (1.1)-(1.3), the following

Theorem 2.4 *Assume (H). Then*
(1) S_{μ_D} is a good rate function on $\mathcal{C}([0,1], M_1(\mathbb{R}))$.
(2) For any closed set F of $\mathcal{C}([0,1], M_1(\mathbb{R}))$,

$$\limsup_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}\left(\hat{\mu}^N_. \in F\right) \leq - \inf_{\nu\in F} S_{\mu_D}(\nu).$$

The proof is very similar to that given in [6, Sections 2.2, 2.3, 2.4]. However, some arguments have to be changed since we do not assume that $\sup_{N\in\mathbb{N}} \hat{\mu}^N_{D_N}(x^2) < \infty$. Since these arguments shall be important in our derivation of the lower bound, we shall detail them below. The parts of the proof which are identical to that given in [6] will be either merely sketched or omitted.

We first prove that S_{μ_D} is a good rate function. Then, we show that exponential tightness hold and then obtain a weak large deviation upper bound, these two arguments yielding (2).

2.2.1 Study of the rate function

Let us first observe that $S_{\mu_D}(\nu)$ is also given, when $\nu_0 = \mu_D$, by

$$S_{\mu_D}(\nu) = \frac{1}{2} \sup_{f \in \mathcal{C}_b^{2,1}(I\!\!R \times [0,1])} \sup_{0 \le s \le t \le 1} \frac{S^{s,t}(\nu, f)^2}{< f, f >_{s,t}^{\nu}}. \tag{2.3}$$

Consequently, S_{μ_D} is non negative. Moreover, S_{μ_D} is obviously lower semi-continuous as a supremum of continuous functions. Hence, we merely need to check that its level sets are contained in relatively compact sets. For K_M and C_n compact subsets of $M_1(I\!\!R)$ and $\mathcal{C}([0,1], I\!\!R)$, respectively, set

$$\mathcal{K}(K) = \{\nu \in \mathcal{C}([0,1], M_1(I\!\!R)), \nu_t \in K \ \forall t \in [0,1]\}$$

and

$$\mathcal{C}(C, f) = \{\nu \in \mathcal{C}([0,1], M_1(I\!\!R)), (t \to \nu_t(f)) \in C\}.$$

With $(f_n)_{n \in I\!\!N}$ a family of bounded continuous functions dense in the set $\mathcal{C}_c(I\!\!R)$ of compactly supported continuous functions, and K_M and C_n compact subsets of $M_1(I\!\!R)$ and $\mathcal{C}([0,1], I\!\!R)$, respectively, recall (see [6, Section 2.2]) that the sets

$$\mathcal{K} = \mathcal{K}(K_M) \bigcap \Big(\bigcap_{n \in I\!\!N} \mathcal{C}(C_n, f_n) \Big)$$

are relatively compact subsets of $\mathcal{C}([0,1], M_1(I\!\!R))$. Indeed, the elements of $\bigcap_{n \in I\!\!N} \mathcal{C}(C_n, f_n)$ can easily be seen to be tight by a standard diagonalization procedure with limit points in $\mathcal{C}([0,1], \mathcal{C}_c(I\!\!R)')$, where $\mathcal{C}_c(I\!\!R)'$ denotes the algebraic dual of $\mathcal{C}_c(I\!\!R)$. If they also belong to $\mathcal{K}(K_M)$, their limit points can be seen to belong to $\mathcal{C}([0,1], M_1(I\!\!R))$.

Following the above description of relatively compact subsets of $\mathcal{C}([0,1], M_1(I\!\!R))$, and the well known characterizations of compact subsets of $M_1(I\!\!R)$ and $\mathcal{C}([0,1], I\!\!R)$, to achieve our proof, it is enough to show that, for any $M > 0$,

•1) For any integer m, there is a positive real number L_m^M so that for any $\nu \in \{S_{\mu_D} \le M\}$,

$$\sup_{0 \le s \le 1} \nu_s(|x| \ge L_m^M) \le \frac{1}{m}. \tag{2.4}$$

•2) For any integer m and $f \in \mathcal{C}_b^2(I\!\!R)$, there exists a positive real number δ_m^M so that for any $\nu \in \{S_{\mu_D} \le M\}$,

$$\sup_{|t-s| \le \delta_m^M} |\nu_t(f) - \nu_s(f)| \le \frac{1}{m}. \tag{2.5}$$

To prove (2.4), we consider, for $\delta > 0$, $f_\delta(x) = \log\left(x^2(1+\delta x^2)^{-1} + 1\right) \in \mathcal{C}_b^{2,1}(I\!\!R \times [0,1])$. We observe that

$$C := \sup_{0 < \delta \le 1} ||\partial_x f_\delta||_\infty + \sup_{0 < \delta \le 1} ||\partial_x^2 f_\delta||_\infty$$

is finite and, for $\delta \in (0,1]$,

$$\left| \frac{\partial_x f_\delta(x) - \partial_x f_\delta(y)}{x - y} \right| \le C.$$

Hence, (2.3) implies, by taking $f = f_\delta$ in the supremum, that for any $\delta \in (0,1]$, any $t \in [0,1]$, any $\mu_. \in \{S_{\mu_D} \le M\}$,

$$\mu_t(f_\delta) \le \mu_0(f_\delta) + 2Ct + 2C\sqrt{Mt}.$$

Consequently, we deduce by the monotone convergence theorem and letting δ decrease to zero that for any $\mu_. \in \{S_{\mu_D} \le M\}$,

$$\sup_{t\in[0,1]} \mu_t(\log(x^2+1)) \le \mu_D(\log(x^2+1)) + 2C(1+\sqrt{M}).$$

Chebycheff's inequality and hypothesis (H) thus imply that for any $\mu_. \in \{S_{\mu_D} \le M\}$ and any $K \in \mathbb{R}^+$,

$$\sup_{t\in[0,1]} \mu_t(|x| \ge K) \le \frac{C_D + 2C(1+\sqrt{M})}{\log(K^2+1)}$$

which finishes the proof of (2.4).

The proof of (2.5) again relies on (2.3) which implies that for any $f \in \mathcal{C}_b^2(\mathbb{R})$, any $\mu_. \in \{S_{\mu_D} \le M\}$ and any $0 \le s \le t \le 1$,

$$|\mu_t(f) - \mu_s(f)| \le ||\partial_x^2 f||_\infty |t-s| + 2||\partial_x f||_\infty \sqrt{M}\sqrt{|t-s|}. \tag{2.6}$$

□

2.2.2 Exponential tightness

Here, we shall prove that

Lemma 2.5 *For any integer number L, there exists a finite integer number $N_0 \in \mathbb{N}$ and a compact set $\mathcal{K}_L$ in $\mathcal{C}([0,1], M_1(\mathbb{R}))$ such that $\forall N \ge N_0$,*

$$\mathbb{P}(\hat{\mu}^N \in \mathcal{K}_L^c) \le \exp\{-LN^2\}.$$

Proof. In view of the previous description of the relatively compact subsets of $\mathcal{C}([0,1], M_1(\mathbb{R}))$, we need to show that

- a) For every positive real numbers L and m, there is an $N_0 \in \mathbb{N}$ and a positive real number $M_{L,m}$ so that $\forall N \ge N_0$
$$\mathbb{P}\Big(\sup_{0\le t\le 1} \hat{\mu}_t^N(|x| \ge M_{L,m}) \ge \frac{1}{m}\Big) \le \exp(-LN^2)$$

- b) For any $f \in \mathcal{C}_b^2(\mathbb{R})$, for any positive real numbers L and m, there exists an $N_0 \in \mathbb{N}$ and a positive real number $\delta_{L,m,f}$ such that $\forall N \ge N_0$
$$\mathbb{P}\Big(\sup_{|t-s|\le\delta_{L,m,f}} |\hat{\mu}_t^N(f) - \hat{\mu}_s^N(f)| \ge \frac{1}{m}\Big) \le \exp(-LN^2)$$

The proof of the second point is exactly the same as that given in [6, Lemma 2.16]; we shall omit it here. (a) is slightly different since the initial data here plays a role and we describe its proof below.

Let us first note that Chebycheff's inequality implies that

$$\sup_{0\le s\le 1}\hat{\mu}_s^N(|x|\ge M)\le \frac{1}{\log(M^2+1)}\sup_{0\le s\le 1}\hat{\mu}_s^N(\log(x^2+1)). \tag{2.7}$$

Denote by tr_N the normalized trace; $\mathrm{tr}_N(A)=\frac{1}{N}\sum_{i=1}^N A_{ii}$. Then, by the definition of $\hat{\mu}_\cdot^N$, for any $s\in[0,1]$,

$$\hat{\mu}_s^N(\log(x^2+1))=\mathrm{tr}_N\log\left((H_N(s)+D_N)^2+1\right). \tag{2.8}$$

Remark that if $A,B\in\mathcal{H}_N$ are such that $0\le A\le B$ (in the sense that for any $u\in\mathbb{C}^N$, $0\le<u,Au>\le<u,Bu>$),

$$\mathrm{tr}_N(\log(A))\le \mathrm{tr}_N(\log(B)).$$

Indeed, if A,B are two self-adjoint matrices with eigenvalues $\lambda_A^1\le\lambda_A^2\le\ldots\le\lambda_A^N$ (resp. $\lambda_B^1\le\lambda_B^2\le\ldots\le\lambda_B^N$) such that $A\le B$, then, for any $i\in\{1,\cdots,N\}$, $\lambda_A^i\le\lambda_B^i$, and the monotonicity of $\log x$ proves the claim. Therefore, since $(H_N(s)+D_N)^2+I_N\le 2H_N(s)^2+2D_N^2+I_N$, (2.8) implies that for $s\in[0,1]$,

$$\hat{\mu}_s^N(\log(x^2+1))\quad\le\quad \log 2+\mathrm{tr}_N\log(H_N(s)^2+D_N^2+1). \tag{2.9}$$

Now,

$$\begin{aligned}
\mathrm{tr}_N\log(H_N(s)^2+D_N^2+1) &= \mathrm{tr}_N\log(D_N^2+1)+\int_0^1\partial_\alpha\mathrm{tr}_N\log(\alpha H_N(s)^2+D_N^2+1)d\alpha\\
&= \mathrm{tr}_N\log(D_N^2+1)+\int_0^1\mathrm{tr}_N\left(H_N(s)(\alpha H_N(s)^2+D_N^2+1)^{-1}H_N(s)\right)d\alpha\\
&\le \mathrm{tr}_N\log(D_N^2+1)+\int_0^1\mathrm{tr}_N\left(H_N(s)(\alpha H_N(s)^2+1)^{-1}H_N(s)\right)d\alpha\\
&= \mathrm{tr}_N\log(D_N^2+1)+\mathrm{tr}_N\log(H_N(s)^2+1).
\end{aligned}$$

As a consequence, (2.9) gives, with hypothesis (H) and the concavity of $x\to\log x$,

$$\sup_{s\in[0,1]}\hat{\mu}_s^N(\log(x^2+1))\quad\le\quad\log 2+C_D+\sup_{s\in[0,1]}\log\left(1+\mathrm{tr}_N((H_N(s))^2)\right). \tag{2.10}$$

Now, note that Chebycheff's inequality yields for any $K\in\mathbb{R}^+$,

$$\begin{aligned}
\mathbb{P}\left(\sup_{s\in[0,1]}\mathrm{tr}_N((H_N(s))^2)\ge K\right) &\le e^{-\frac{1}{8}N^2K}E[e^{\frac{1}{4}\sum_{i\le j}\sup_{0\le s\le 1}\beta_{ij}^2(s)+\frac{1}{4}\sum_{i<j}\sup_{0\le s\le 1}\tilde{\beta}_{ij}^2(s))}]\\
&= 2^{\frac{3N^2}{2}}e^{-\frac{1}{8}N^2K}
\end{aligned}$$

where we have used that for $\eta<1/2$, by Désirè André's reflection principle,

$$\mathbb{E}(\exp(\eta\sup_{0\le s\le 1}\beta_{ij}^2(s))\le 2\ \mathbb{E}(\exp(\eta\beta_{ij}^2(1)))=2(1-2\eta)^{-\frac{1}{2}}.$$

Using this estimate with (2.7) and (2.10) shows that with $L_0 := \log 2 + C_D$, any $\delta > 0$,

$$\begin{aligned}\mathbb{P}\left(\sup_{s\in[0,1]} \hat{\mu}_s^N(|x| \geq L) \geq \delta\right) &\leq \mathbb{P}\left(\sup_{s\in[0,1]} \mathrm{tr}_N((H_N(s))^2) \geq (L^2+1)^\delta e^{-L_0} - 1\right)\\ &\leq 2^{\frac{3N^2}{2}} e^{-\frac{1}{8}N^2\{(L^2+1)^\delta e^{-L_0}-1\}}\end{aligned}$$

which completes the proof of (a). □

2.2.3 Weak large deviation upper bound

We here summarize the main arguments giving the weak large deviation upper bound.

Lemma 2.6 *For every process ν in $\mathcal{C}([0,1], M_1(\mathbb{R}))$, if $B_\delta(\nu)$ denotes the open ball with center ν and radius δ for the distance D, then*

$$\lim_{\delta\to 0}\limsup_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}\big(\hat{\mu}^N \in B_\delta(\nu)\big) \leq -S_{\mu_D}(\nu)$$

Proof. Note first that, since $\hat{\mu}^N_{D_N}$ converges weakly towards μ_D, for any $\delta > 0$, any N large enough, $d(\hat{\mu}^N_{D_N}, \mu) < \delta$ ensuring that

$$\lim_{\delta\to 0}\limsup_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}(\hat{\mu}^N \in B_\delta(\nu)) = -\infty$$

if $\nu_0 \neq \mu_D$. Hence, we shall assume hereafter that $\nu_0 = \mu_D$. We shall follow the ideas developed in [15] and [6]. To this end, we define a family of positive super-martingales $\{\zeta_f^{N,s}, f \in \mathcal{C}_b^{2,1}(\mathbb{R} \times [0,1])\}$, equal to 1 at $t = s$, thanks to Lemma 2.1: we set, for $t \geq s \geq 0$,

$$\begin{aligned}\zeta_f^{N,s}(t) &= \exp\left(N^2(S^{s,t}(\hat{\mu}^N, f) - \frac{1}{2} < f, f >_{\hat{\mu}^N}^{s,t})\right)\\ &= \exp\left(N^2 \bar{S}^{s,t}(\hat{\mu}^N, f)\right).\end{aligned}$$

Let $\nu \in \mathcal{C}([0,1], M_1(\mathbb{R}))$ and $f \in \mathcal{C}_b^{2,1}(\mathbb{R} \times [0,1])$; then Chebycheff's inequality implies that for any $0 \leq s \leq t \leq 1$

$$\begin{aligned}\mathbb{P}\big(\hat{\mu}^N \in B(\nu,\delta)\big) &= \mathbb{E}\Big[1_{\hat{\mu}^N\in B(\nu,\delta)} \frac{\zeta_f^{N,s}(t)}{\zeta_f^{N,s}(t)}\Big]\\ &\leq \sup_{\nu'\in B(\nu,\delta)} \exp\Big(-N^2 \bar{S}^{s,t}(\nu', f)\Big)\\ &= \exp\Big(-N^2 \inf_{\nu'\in B(\nu,\delta)} \big(\bar{S}^{s,t}(\nu', f)\big)\Big).\end{aligned}$$

Notice that if f belongs to $\mathcal{C}_b^{2,1}(\mathbb{R}\times[0,1])$, the function $\nu' \to \bar{S}^{s,t}(\nu', f)$ is continuous. Thus, for any function $f \in \mathcal{C}_b^{2,1}(\mathbb{R} \times [0,1])$, we deduce

$$\lim_{\delta\to 0}\limsup_{N\to\infty} \frac{1}{N^2} \ln \mathbb{P}(\hat{\mu}^N \in B_\delta(\nu)) \leq -S^{s,t}(\nu, f).$$

We conclude by taking the supremum over the functions f and the times $0 \leq s \leq t \leq 1$. □

2.3 Large deviation lower bound

In this section we shall prove a large deviation lower bound estimate in the case where $\mu_.$ satisfies,

(A) for some $\eta > 0$,

$$\sup_{t\in[0,1]} \mu_t(|x|^{5+\eta}) < \infty.$$

We shall further strengthen (H) by assuming

(H') (H) holds and

$$\sup_{N\in\mathbb{N}} \mathrm{tr}_N(D_N^2) < \infty.$$

Theorem 2.7 *Assume that (H') holds. Then, for any $\mu_. \in \mathcal{C}([0,1], M_1(\mathbb{R}))$ satisfying (A),*

$$\lim_{\delta\to 0}\liminf_{N\to\infty} \frac{1}{N^2}\log \mathbb{P}\left(D(\hat{\mu}_.^N, \mu_.) \le \delta\right) \ge -S_{\mu_D}(\mu_.) .$$

A lower bound was already obtained in [6, Section 2.4] but for processes $\mu_.$ satisfying a much less transparent condition, and further possessing fixed initial conditions at 0. Here, we shall generalize this result by using several approximations. The proof is hence rather technical and actually far from straightforward. It is the most difficult part of this paper.

The key to the theorem is to approximate the measure valued process $\hat{\mu}_.^N$. To this end, we introduce a matrix C_N in $\mathcal{H}_N$ with real-valued eigenvalues $(c_i)_{1\le i\le N}$ such that the spectral measure $\hat{\mu}_C^N = (1/N)\sum_{i=1}^N \delta_{c_i}$ converges towards the Cauchy law $P(dx) = \pi^{-1}(x^2+1)^{-1}dx$. Further, we choose the entries $\{c_i\}$ such that

$$\limsup_{N\to\infty} \hat{\mu}_C^N((\log(x^2+1))^2) < \infty. \tag{2.11}$$

Moreover, we assume that for any $\epsilon > 0$, the limit distribution of $D_N + \epsilon C_N$ is the free convolution $P_\epsilon \boxed{+} \mu_D$ (see [2]) where we have denoted by $P_\epsilon(dx) = \epsilon\pi^{-1}(x^2+\epsilon^2)^{-1}dx$ the Cauchy law with parameter ϵ. Note that, because P_ϵ is the Cauchy law, it has been remarked (see [2, Section 7]) that $P_\epsilon \boxed{+} \mu_D$ is just the usual convolution $P_\epsilon * \mu_D$. The couple (C_N, D_N) can be constructed as follows, once the eigenvalues $(c_i)_{1\le i\le N}$ and $(d_i)_{1\le i\le N}$ of C_N and D_N satisfying (2.11) and (H') have been given. Let $\phi_M(x) = x \vee (-M) \wedge M$ and denote $(\widetilde{C}_{N,M}, \widetilde{D}_{N,M})$ the $N\times N$ diagonal matrices with entries $(\phi_M(c_i))_{1\le i\le N}$ and $(\phi_M(d_i))_{1\le i\le N}$ respectively (here, $(d_i)_{1\le i\le N}$ denotes the spectrum of D_N). Let U_N be a $N\times N$ unitary matrix following the Haar measure m_N^2 on the unitary group and define $(C_{N,M}, D_{N,M}) = (U_N^*\widetilde{C}_{N,M}U_N, \widetilde{D}_{N,M})$. Since $(\widetilde{C}_{N,M}, \widetilde{D}_{N,M})$ are uniformly bounded operators, [23, Pg. 328-330] insures that $(C_{N,M}, D_{N,M})$ are asymptotically free. Therefore, the limit distribution of $D_{N,M} + \epsilon C_{N,M}$ is $P_\epsilon \circ \phi_{\epsilon M}^{-1} \boxed{+} \mu_D \circ \phi_M^{-1}$. Further, with d denoting the Wasserstein distance,

$$d(\hat{\mu}^N_{C_{N,M}}, \hat{\mu}^N_{C_N}) \le \frac{2}{N}\sum_{i=1}^N 1_{|c_i|\ge M} \le \frac{2}{(\log(1+M^2))^2}\hat{\mu}^N_{C_N}[(\log(1+x^2))^2]$$

so that (2.11) results with

$$\lim_{M\to\infty}\sup_{N\in\mathbb{N}} d(\hat{\mu}^N_{C_{N,M}}, \hat{\mu}^N_{C_N}) = 0.$$

Similarly, (H) implies that $\lim_{M\to\infty}\sup_{N\in\mathbb{N}} d(\hat{\mu}^N_{D_{N,M}}, \hat{\mu}^N_{D_N}) = 0$, and both (H) and (2.11) imply that

$\lim_{M\to\infty}\sup_{N\in\mathbb{N}} d(\hat{\mu}^N_{D_{N,M}+\epsilon C_{N,M}}, \hat{\mu}^N_{D_{N,\infty}+\epsilon C_{N,\infty}}) = 0$. Consequently, by uniformity on $N \in \mathbb{N}$, $P_\epsilon \circ \phi_{\epsilon M}^{-1} \boxplus \mu_D \circ \phi_M^{-1}$ converges as M goes to infinity towards $P_\epsilon \boxplus \mu_D$ and we can choose $M = M_N$ so that the limit distribution of $D_{N,M_N} + \epsilon C_{N,M_N}$ is $P_\epsilon \boxplus \mu_D$. We then set $(C_N, D_N) := (C_{N,M_N}, D_{N,M_N})$. Clearly, this new choice still verifies (2.11) and (H'). We assume in the following that such a construction has been made, independently of the Hermitian Brownian motion to come next, and work with such a given realization of the U_N's (hence with quenched (C_N, D_N)).

We then introduce, for $\epsilon > 0$, the following approximation X_N^ϵ of the matrix-valued process X_N

$$X_N^\epsilon(t) := X_N(t) + \epsilon C_N = H_N(t) + D_N + \epsilon C_N.$$

We denote by $\hat{\mu}_.^{N,\epsilon}$ the empirical process of the eigenvalues of X_N^ϵ. The central lemma in the proof of Theorem 2.7 is the following

Lemma 2.8 *For any $\epsilon > 0$, for any $\mu_. \in \mathcal{C}([0,1], M_1(\mathbb{R}))$ satisfying Assumption (A) and such that $S_{\mu_D}(\mu) < \infty$, for any $\delta > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}\left(\hat{\mu}_.^{N,\epsilon} \in B(P_\epsilon * \mu_., \delta)\right) \geq -S_{P_\epsilon * \mu_D}(P_\epsilon * \mu).$$

Note that Lemma 2.8 and Theorem 2.4 yield already the following

Corollary 2.9 *For any $\epsilon > 0$, for any closed subset F of $\mathcal{C}([0,1], M_1(\mathbb{R}))$,*

$$\limsup_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}\left(\hat{\mu}_.^{N,\epsilon} \in F\right) \leq -\inf\{S_{P_\epsilon * \mu_D}(\nu), \nu \in F\}.$$

Further, for any open set O of $\mathcal{C}([0,1], M_1(\mathbb{R}))$,

$$\liminf_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}\left(\hat{\mu}_.^{N,\epsilon} \in O\right) \geq -\inf\{S_{P_\epsilon * \mu_D}(\nu), \nu \in O, \nu = P_\epsilon * \mu, \mu \text{ satisfies } (A) \text{ and } S_{\mu_D}(\mu) < \infty\}.$$

To deduce Theorem 2.7 from Lemma 2.8, we shall need the following two auxiliary lemmas. First,

Lemma 2.10 *For any $\epsilon > 0$, any $\mu_. \in \mathcal{C}([0,1], M_1(\mathbb{R}))$ such that $S_{\mu_D}(\mu) < \infty$,*

$$S_{P_\epsilon * \mu_D}(P_\epsilon * \mu) \leq S_{\mu_D}(\mu).$$

And secondly

Lemma 2.11 *Consider, for $L \in \mathbb{R}^+$, the compact set K_L of $M_1(\mathbb{R})$ given by*

$$K_L = \{\mu \in M_1(\mathbb{R}); \mu(\log(x^2+1)) \leq L\}.$$

Then, on $\mathcal{K}_\epsilon^N(K_L) := \bigcap_{t\in[0,1]} \left\{\{\hat{\mu}_t^{N,\epsilon} \in K_L\} \cap \{\hat{\mu}_t^N \in K_L\}\right\}$,

$$D(\hat{\mu}_.^{N,\epsilon}, \hat{\mu}_.^N) \leq f(N,\epsilon)$$

where

$$\limsup_{\epsilon\to 0}\limsup_{N\to\infty} f(N,\epsilon) = 0.$$

In the next paragraph, we shall deduce Theorem 2.7 from Lemmas 2.8, 2.10 and 2.11. Lemma 2.8 will be proved in the next subsection. Lemmas 2.10 and 2.11 will be the subject of the two last paragraphs.

2.3.1 Proof of Theorem 2.7

Following Lemma 2.5 and its proof, we deduce from hypothesis (H') and (2.11) that for any $M \in I\!R^+$, we can find $L_M \in I\!R^+$ such that for any $L \geq L_M$,

$$\sup_{0 \leq \epsilon \leq 1} I\!P(\mathcal{K}_\epsilon^N(K_L)^c) \leq e^{-MN^2}. \tag{2.12}$$

Fix $M > S_{\mu_D}(\mu) + 1$ and $L \geq L_M$. Let $\delta > 0$ be given. Next, observe that $P_\epsilon * \mu_.$ converges weakly towards $\mu_.$ as ϵ goes to zero and choose consequently ϵ small enough so that $D(P_\epsilon * \mu_., \mu_.) < \frac{\delta}{3}$. Then, write

$$\begin{aligned}
I\!P\left(\hat{\mu}_.^N \in B(\mu_., \delta)\right) &\geq I\!P\left(D(\hat{\mu}_.^N, \mu_.) < \frac{\delta}{3}, \hat{\mu}_.^{N,\epsilon} \in B(P_\epsilon * \mu_., \frac{\delta}{3}), \mathcal{K}_\epsilon^N(K_L)\right) \\
&\geq I\!P\left(\hat{\mu}_.^{N,\epsilon} \in B(P_\epsilon * \mu_., \frac{\delta}{3})\right) - I\!P(\mathcal{K}_\epsilon^N(K_L)^c) - I\!P\left(D(\hat{\mu}_.^{N,\epsilon}, \hat{\mu}_.^N) \geq \frac{\delta}{3}, \mathcal{K}_\epsilon^N(K_L)\right) \\
&= I - II - III.
\end{aligned}$$

(2.12) implies, up to terms of smaller order, that

$$II \leq e^{-N^2(S_{\mu_D}(\mu)+1)}.$$

Lemma 2.11 shows that $III = 0$ for ϵ small enough and N large, while Lemmas 2.8 and 2.10 imply for any $\eta > 0$, N large and $\epsilon > 0$

$$I \geq e^{-N^2 S_{P_\epsilon * \mu_D}(P_\epsilon * \mu) - N^2\eta} \geq e^{-N^2 S_{\mu_D}(\mu) - N^2\eta}.$$

Theorem 2.7 is proved. $\square$

2.3.2 Proof of Lemma 2.8

In this section, we are given a process $\mu_. \in \{S_{\mu_D} < \infty\}$ and a threshold $\epsilon > 0$. We set $\mu_t^\epsilon := P_\epsilon * \mu_t$ for $t \in [0,1]$. By Lemma 2.10, $S_{P_\epsilon * \mu_D}(\mu^\epsilon) < \infty$ and thus $S^{0,1}(\mu^\epsilon) = S_{P_\epsilon * \mu_D}(\mu^\epsilon) < \infty$. The main advantage of $\mu_.^\epsilon$ (in comparison with $\mu_.$) is that its marginals possess bounded densities. However, we shall later need also some additional time regularity of this process. Hence, we shall in a first step approximate $\mu_.^\epsilon$ by smooth polygonal approximations $\mu_.^{\epsilon,\Delta}$. Once this is done, we shall study the processes $\mu_.^{\epsilon,\Delta}$ and show that the lower bound obtained in [6, Section 2.4], applies in small neighborhood of $\mu_.^{\epsilon,\Delta}$, hence proving the lemma.

<u>*Step 1 : Time approximations*</u> We shall see that if $0 = t_1 < t_2 < \ldots < t_n = 1$ with $t_i = (i-1)\Delta$ and if we set, for $t \in [t_k, t_{k+1}[$,

$$\mu_t^{\epsilon,\Delta} = \mu_{t_k}^\epsilon + \frac{(t - t_k)}{\Delta}[\mu_{t_{k+1}}^\epsilon - \mu_{t_k}^\epsilon],$$

then,

$$\lim_{\Delta \downarrow 0} S^{0,1}(\mu^{\epsilon,\Delta}) = S^{0,1}(\mu^\epsilon). \tag{2.13}$$

Observe first that since $\mu^{\epsilon,\Delta}$ converges weakly to μ^ϵ as Δ goes to zero, the lower semi-continuity of $S^{0,1}$ implies that

$$\liminf_{\Delta \downarrow 0} S^{0,1}(\mu^{\epsilon,\Delta}) \geq S^{0,1}(\mu^\epsilon).$$

Now, recall that, since $S^{0,1}(\mu^\epsilon)$ is finite, Riesz's theorem implies that there exists h^ϵ with

$$\int_0^1 \int \left(\partial_x h_u^\epsilon(x)\right)^2 d\mu_u^\epsilon(x) du < \infty$$

so that

$$\inf_{f \in \mathcal{C}_b^{2,1}(\mathbb{R}\times[0,1])} \int_0^1 \int (\partial_x f(x,u) - \partial_x h_u^\epsilon(x))^2 d\mu_u^\epsilon(x) du = 0 \tag{2.14}$$

and for any $f \in \mathcal{C}_b^{2,1}(\mathbb{R} \times [0,1])$,

$$S^{0,1}(\mu^\epsilon, f) = \int_0^1 \int \partial_x h_u^\epsilon(x) \partial_x f(x,u) d\mu_u^\epsilon(x) du. \tag{2.15}$$

Further,

$$S^{0,1}(\mu^\epsilon) = \frac{1}{2}\int_0^1 \int \left(\partial_x h_u^\epsilon(x)\right)^2 d\mu_u^\epsilon du. \tag{2.16}$$

It will be convenient to also recall that, using the continuity of $u \mapsto \mu_u^\epsilon$ and (2.15), one has that for any time independent smooth function $g(\cdot)$,

$$\mu_{t_{k+1}}^\epsilon(g) - \mu_{t_k}^\epsilon(g) = \int_{t_k}^{t_{k+1}} \int \partial_x h_s^\epsilon(x) \partial_x g(x) d\mu_s^\epsilon(x) ds + \frac{1}{2}\int_{t_k}^{t_{k+1}} \int\int \frac{g'(x) - g'(y)}{x-y} d\mu_s^\epsilon(x) d\mu_s^\epsilon(y) ds\,. \tag{2.17}$$

Let $H_{\mu_u^\epsilon}(x) = PV \int (x-y)^{-1} d\mu_u^\epsilon(y)$ (the occurrence of the principal value is due to the fact that while $(\partial_x f_t(x) - \partial_y f_t(y))/(x-y)$ is bounded continuous, $\int d\mu_s^\epsilon(x) \partial_x f_t(y)/(x-y)$ is not defined except as principal value). Then, separating the double integral in (2.17) using the definition of principal value and Fubini's theorem,

$$\mu_{t_{k+1}}^\epsilon(g) - \mu_{t_k}^\epsilon(g) = \int_{t_k}^{t_{k+1}} \int (\partial_x h_s^\epsilon(x) + H_{\mu_s^\epsilon}(x)) \partial_x g(x) d\mu_s^\epsilon(x) ds\,. \tag{2.18}$$

Consequently, for $t \in [t_k, t_{k+1}[$, writing $\mu_t^\epsilon(dx) = \rho_t^\epsilon(x)dx$ and using $g(x) = f(t,x)$ to obtain the first equality,

$$\begin{aligned}
\partial_t \mu_t^{\epsilon,\Delta}(f(t,\cdot)) &= \mu_t^{\epsilon,\Delta}(\partial_t f(t,\cdot)) + \Delta^{-1} \int_{t_k}^{t_{k+1}} \mu_s^\epsilon[(H_{\mu_s^\epsilon} + \partial_x h_s^\epsilon)\partial_x f(t,\cdot)] ds \\
&= \mu_t^{\epsilon,\Delta}(\partial_t f(t,\cdot)) + \mu_t^{\epsilon,\Delta}(\partial_x f(t,\cdot)(H_{\mu_t^{\epsilon,\Delta}} + \partial_x h_t^{\epsilon,\Delta}))
\end{aligned}$$

where

$$\partial_x h_t^{\epsilon,\Delta}(x) \equiv \frac{\int_{t_k}^{t_{k+1}} (H_{\mu_u^\epsilon}(x) + \partial_x h_u^\epsilon(x)) \rho_u^\epsilon(x) du}{\Delta(\rho_{t_k}^\epsilon + \frac{(t-t_k)}{\Delta}[\rho_{t_{k+1}}^\epsilon - \rho_{t_k}^\epsilon])} - H_{\mu_{t_k}^\epsilon}(x) - \frac{(t-t_k)}{\Delta}(H_{\mu_{t_{k+1}}^\epsilon}(x) - H_{\mu_{t_k}^\epsilon}(x)). \tag{2.19}$$

Then, by definition of $S^{0,1}(\cdot)$,

$$S^{0,1}(\mu^{\epsilon,\Delta}) \le \frac{1}{2}\int_0^1 \int (\partial_x h_u^{\epsilon,\Delta}(x))^2 d\mu_u^{\epsilon,\Delta}(x) du \tag{2.20}$$

with equality if

$$\inf_{f \in \mathcal{C}_b^{2,1}(\mathbb{R}\times[0,1])} \int_0^1 \int (\partial_x f(x,u) - \partial_x h_u^{\epsilon,\Delta}(x))^2 d\mu_u^{\epsilon,\Delta}(x) du = 0.$$

Note that, since $\mu_t^\epsilon = P_\epsilon * \mu_t$,

$$|H_{\mu_u^\epsilon}(x)| = |\int \frac{(x-y)}{(x-y)^2+\epsilon^2} d\mu_u(y)| \le \frac{1}{2\epsilon}$$

is uniformly bounded, as well as

$$\rho_u^\epsilon(x) = \int \frac{\epsilon}{\pi} \frac{1}{(x-y)^2+\epsilon^2} d\mu_u(y) \le \frac{1}{\epsilon\pi}.$$

Also, for a given x, $\rho_u^\epsilon(x) \neq 0$. Finally, by continuity of $u \to \mu_u$, for any given x, $u \to \rho_u^\epsilon(x)$ and $u \to H_{\mu_u^\epsilon}(x)$ are continuous. We can precise these continuity statement as follows. Note that since $S^{0,1}(\mu)$ is finite, we can use (2.6) with $f(y) = \frac{(x-y)}{(x-y)^2+\epsilon^2}$ and $f(y) = \frac{\epsilon}{\pi}\frac{1}{(x-y)^2+\epsilon^2}$ for fixed x, to obtain, as both have uniformly bounded derivatives (independently of the given x but depending on $\epsilon \neq 0$), that

$$\sup_{|u-v|\le\Delta} \sup_x |\rho_u^\epsilon(x) - \rho_v^\epsilon(x)| \le c(\epsilon)\sqrt{\Delta}, \qquad \sup_{|u-v|\le\Delta} \sup_x |H_{\mu_u^\epsilon}(x) - H_{\mu_v^\epsilon}(x)| \le c(\epsilon)\sqrt{\Delta} \tag{2.21}$$

for a finite constant $c(\epsilon)$. Further, since $\mu_. \in \mathcal{C}([0,1], M_1(I\!\!R))$, we know that for M large enough, all $u \in [0,1]$,

$$\mu_u([-M,M]) \ge (1/2).$$

Consequently,

$$\rho_u^\epsilon(x) \ge \frac{\epsilon}{2\pi\left(\sup_{y\in[-M,M]}(x-y)^2+\epsilon^2\right)} = \frac{\epsilon}{2\pi\left((|x|+M)^2+\epsilon^2\right)}. \tag{2.22}$$

Thus, with (2.21), we find a finite constant $c'(\epsilon)$ such that for any $f = f_u(\cdot)$ and any $k \in \{1, \cdots, n\}$,

$$\left|\frac{\int_{t_k}^{t_{k+1}} f_u(x)\rho_u^\epsilon(x)du}{\Delta(\rho_{t_k}^\epsilon + \frac{(t-t_k)}{\Delta}[\rho_{t_{k+1}}^\epsilon - \rho_{t_k}^\epsilon])} - \frac{1}{\Delta}\int_{t_k}^{t_{k+1}} f_u(x)du\right| \le c'(\epsilon)(1+x^2)\sqrt{\Delta}\frac{1}{\Delta}\int_{t_k}^{t_{k+1}} |f_u(x)|du.$$

Applying the above estimate with $f_u(x) = \partial_x h_u^\epsilon(x)$ or $f_u = H_{\mu_u^\epsilon}(x)$, we deduce first that uniformly on k,

$$\begin{aligned}\frac{\int_{t_k}^{t_{k+1}} (H_{\mu_u^\epsilon}(x) + \partial_x h_u^\epsilon(x))\rho_u^\epsilon(x)du}{\Delta(\rho_{t_k}^\epsilon + \frac{(t-t_k)}{\Delta}[\rho_{t_{k+1}}^\epsilon - \rho_{t_k}^\epsilon])} &= \left(\frac{1}{\Delta}\int_{t_k}^{t_{k+1}} (\partial_x h_u^\epsilon(x) + H_{\mu_u^\epsilon}(x))du\right) \\ &+0((x^2+1)\sqrt{\Delta}))\left(\frac{1}{\Delta}\int_{t_k}^{t_{k+1}} (|\partial_x h_u^\epsilon(x)| + |H_{\mu_u^\epsilon}(x)|)du\right).\end{aligned}$$

Then, using (2.21) to estimate the second term in the definition of $\partial_x h^{\epsilon,\Delta}$ and the uniform bound on $H_{\mu_u^\epsilon}$, we obtain

$$\begin{aligned}&\left|\partial_x h_t^{\epsilon,\Delta}(x) - \Delta^{-1}\sum_k 1_{t\in[t_k,t_k+1]}\int_{t_k}^{t_k+1} \partial_x h_u^\epsilon(x)du\right| \\ &\le 0((1+x^2)\sqrt{\Delta})\left[\Delta^{-1}\sum_k 1_{t\in[t_k,t_k+1]}\int_{t_k}^{t_{k+1}} |\partial_x h_u^\epsilon(x)|du + 1\right].\end{aligned} \tag{2.23}$$

Consequently, for any $\eta > 0$,

$$\begin{aligned}\int_0^1 \int_{[-\Delta^{-\frac{1}{4}+\eta}, \Delta^{-\frac{1}{4}+\eta}]} (\partial_x h_u^{\epsilon,\Delta})^2 d\mu_u^{\epsilon,\Delta} du &\le (1+O(\Delta^{2\eta}))\int_0^1 \int_{[-\Delta^{-\frac{1}{4}+\eta}, \Delta^{-\frac{1}{4}+\eta}]} (\partial_x h_u^\epsilon)^2 d\mu_u^\epsilon du + O(\Delta^{2\eta}) \\ &\le 2(1+O(\Delta^{2\eta}))S^{0,1}(\mu^\epsilon) + O(\Delta^{2\eta})\end{aligned} \tag{2.24}$$

To finish the proof of the result, we shall prove that $h^{\epsilon,\Delta}_{\cdot}(x)$ is well controlled for large x.

To see this, note that, since $S_{\mu_D}(\mu)$ is finite and μ_D has finite second moment, we can prove as in [6, Lemma 2.9] that

$$\sup_{u\in[0,1]} \mu_u(x^2) < \infty.$$

We claim that it implies that there exists a finite constant $C = C(\epsilon)$ so that

$$\sup_{u\in[0,1]} \rho_u^\epsilon(x) \le \frac{C}{x^2}. \tag{2.25}$$

Indeed,

$$x^2\rho_u^\epsilon(x) = \frac{\epsilon}{\pi}\int \frac{x^2}{(x-y)^2+\epsilon^2} d\mu_u(y)$$

and

$$f(x,y) = \frac{\epsilon}{\pi}\frac{x^2}{((x-y)^2+\epsilon^2)} \le \frac{\epsilon}{\pi}\frac{x^2}{((x-y)^2+\epsilon^2)}\Big|_{x=y+\epsilon^2/y} \le C(\epsilon)(1+y^2),$$

implying (2.25). As a consequence of (2.22) and (2.25), we deduce that there exists a finite constant $C'(\epsilon)$ so that for x big enough,

$$\sup_{u,v\in[0,1]} \frac{\rho_u^\epsilon}{\rho_v^\epsilon}(x) \le C'(\epsilon).$$

Therefore, for x big enough, for any test function f,

$$\left|\frac{\int_{t_k}^{t_{k+1}} f_u\rho_u^\epsilon(x)du}{\Delta(\rho_{t_k}^\epsilon + \frac{(t-t_k)}{\Delta}[\rho_{t_{k+1}}^\epsilon - \rho_{t_k}^\epsilon])}\right| \le C'(\epsilon)\frac{1}{\Delta}\int_{t_k}^{t_{k+1}} |f_u|du.$$

Thus, since $H_{\mu_u^\epsilon}$ is uniformly bounded, we find a finite constant C so that for x big enough, for $t\in[t_k,t_{k+1})$,

$$|\partial_x h_t^{\epsilon,\Delta}(x)| \le \frac{C}{\Delta}\int_{t_k}^{t_{k+1}} |\partial_x h_u^\epsilon(x)|du + C.$$

Hence,

$$\int_0^1\int_{[-\Delta^{-\frac{1}{4}+\eta},\Delta^{-\frac{1}{4}+\eta}]^c} (\partial_x h_u^{\epsilon,\Delta}(x))^2 d\mu_u^{\epsilon,\Delta}(x)du$$

$$\le C'\left(\int_0^1\int_{[-\Delta^{-\frac{1}{4}+\eta},\Delta^{-\frac{1}{4}+\eta}]^c} (\partial_x h_u^\epsilon(x))^2 d\mu_u^\epsilon(x)du + \int_0^1 \mu_u^{\epsilon,\Delta}([-\Delta^{-\frac{1}{4}+\eta},\Delta^{-\frac{1}{4}+\eta}]^c)du\right)$$

This, together with (2.19) and (2.24), yields

$$\lim_{\Delta\to 0} S^{0,1}(\mu^{\epsilon,\Delta}) \le \lim_{\Delta\to 0} S^{0,1}(\mu^{\epsilon,\Delta},h^{\epsilon,\Delta}) \le S^{0,1}(\mu^\epsilon). \tag{2.26}$$

Step 2 : Study of the field $h^{\epsilon,\Delta}$

Let us recall that the condition of [6, Section 2.4], under which a lower bound was obtained for $\mu^{\epsilon,\Delta}$ is that $h^{\epsilon,\Delta}$ belongs to $\mathcal{C}_b^{2,1}(I\!R\times[0,1])$ and that $\partial_x h^{\epsilon,\Delta}$ has a Fourier transform decreasing exponentially fast at infinity. We shall in this paragraph study $h^{\epsilon,\Delta}$ and show that it nearly satisfies these hypotheses. In the next step, we shall precise the arguments of [6] to show that the properties of $h^{\epsilon,\Delta}$, even though slightly different from that assumed in [6, Section 2.4], still guarantee the lower bound.

We summarize the properties of $h^{\epsilon,\Delta}$ in the following

Lemma 2.12 *Assume that for any $t \in [0,1]$, μ_t satisfies assumption (A). Then, for any $\epsilon, \Delta > 0$, any $u \in [0,1]$,*

$$h_u^{\epsilon,\Delta}(x) = \sum_{k=1}^{n-1} 1\!\!I_{u\in[t_k,t_{k+1}[} h_k^{\epsilon,\Delta}(x,u)$$

with functions $h_k^{\epsilon,\Delta}(x,u)$ of the form

$$h_k^{\epsilon,\Delta}(x,u) = a_k^\Delta x + b_k^\Delta(u)\log(x^2+1) + g_k^{\epsilon,\Delta}(x,u)$$

with finite constants a_k^Δ, continuously differentiable functions $b_k^\Delta(\cdot)$ and functions $g_k^{\epsilon,\Delta} \in \mathcal{C}_b^{2,1}(I\!R \times [t_k, t_{k+1}[)$. Further, for all $k \in \{1, \cdots, n-1\}$, all $x \in I\!R$ and all $u \in [t_k, t_{k+1}[$, there exist functions $l_k^{\epsilon,\Delta}$ and a finite constant C such that

$$\partial_x h_k^{\epsilon,\Delta}(x,u) = a_k^\Delta + \int e^{i\xi x} l_k^{\epsilon,\Delta}(\xi,u)d\xi$$

for any $u \in [t_k, t_{k+1}[$ and any $k \in \{1, \cdots, n-1\}$,

$$|l_k^{\epsilon,\Delta}(\xi,u)| \le Ce^{-\frac{\epsilon}{2}|\xi|}. \tag{2.27}$$

Proof. To study $h^{\epsilon,\Delta}$, we shall first obtain an alternative formula (compare with (2.19)). More precisely, observe that $h^{\epsilon,\Delta}$ is also given by the weak equation

$$S^{0,1}(\mu^{\epsilon,\Delta}, f) = \int_0^1 \int \partial_x h_u^{\epsilon,\Delta}(x) \partial_x f(x,u) d\mu_u^{\epsilon,\Delta}(x) du \tag{2.28}$$

for any $f \in \mathcal{C}_b^{2,1}(I\!R \times [0,1])$. Observe that $d\mu_u^{\epsilon,\Delta}(x) = \rho_u^{\epsilon,\Delta}(x)dx$ with, when $u \in [t_k, t_{k+1}[$,

$$\rho_u^{\epsilon,\Delta}(x) = \frac{\epsilon}{\pi} \int \frac{1}{(x-y)^2+\epsilon^2} \Big(d\mu_{t_k}(y) + \frac{u-t_k}{\Delta} d(\mu_{t_{k+1}} - \mu_{t_k})(y) \Big).$$

By (2.28), reducing again to integration over $u \in [t_k, t_{k+1}[$, it holds that for almost every such u, and every time independent smooth function f,

$$\partial_u \int f(x) \rho_u^{\epsilon,\Delta}(x) dx = \int \partial_x f(x) \rho_u^{\epsilon,\Delta}(x) [H_{\mu_u^{\epsilon,\Delta}}(x) + \partial_x h_u^{\epsilon,\Delta}(x)] dx\,.$$

Hence, for almost every such u,

$$\int f(x) \frac{\rho_{t_{k+1}}^\epsilon(x) - \rho_{t_k}^\epsilon(x)}{\Delta} dx = \int \partial_x f(x) \rho_u^{\epsilon,\Delta}(x) [H_{\mu_u^{\epsilon,\Delta}}(x) + \partial_x h_u^{\epsilon,\Delta}(x)] dx\,,$$

and the last equality extends by continuity to every $u \in [t_k, t_{k+1})$. But then, integrating by parts, we obtain

$$\partial_x h_u^{\epsilon,\Delta}(x) = -(\Delta \rho_u^{\epsilon,\Delta}(x))^{-1} \int_x^\infty (\rho_{t_{k+1}}^\epsilon - \rho_{t_k}^\epsilon)(y) dy - H_{\mu_u^{\epsilon,\Delta}}(x) \tag{2.29}$$

almost everywhere, and then everywhere by continuity.

From (2.29), we first note that

(1) $u \to \partial_x h_u^{\epsilon,\Delta}(x)$ is differentiable for any x (note that for any given x, $u \to \rho_u^{\epsilon,\Delta}(x)$ is continuously differentiable and bounded below by a positive constant) except at the t_k's.

(2) $x \to \partial_x h_u^{\epsilon,\Delta}(x)$ is $\mathcal{C}^\infty$ for any u due to the regularization by the Cauchy kernel.

(3) Let us show that $\partial_x h_u^{\epsilon,\Delta}$ and $\partial_x^2 h_u^{\epsilon,\Delta}$ are uniformly bounded when μ_u satisfies (A) for any u. By continuity of $x, u \to \partial_x h_u^{\epsilon,\Delta}(x)$ and $x, u \to \partial_x^2 h_u^{\epsilon,\Delta}(x)$, it is enough to bound $\sup_u |\partial_x^2 h_u^{\epsilon,\Delta}|$ and $\sup_u |\partial_x^2 h_u^{\epsilon,\Delta}|$ outside of a fixed compact set, chosen below equal to $[-1,1]$.

Remark first that

$$\int_x^\infty (\rho_{t_{k+1}}^{\epsilon,\Delta} - \rho_{t_k}^{\epsilon,\Delta})(x')dx' = \frac{\epsilon}{\pi}\int_x^\infty \int\int \frac{(2x'-y-y')(y-y')}{((x'-y)^2+\epsilon^2)((x'-y')^2+\epsilon^2)} d\mu_{t_k}(y)d\mu_{t_{k+1}}(y')dx' \,. \tag{2.30}$$

Since $\int_{-\infty}^\infty \rho_{t_{k+1}}^{\epsilon,\Delta}(x)dx = \int_{-\infty}^\infty \rho_{t_k}^{\epsilon,\Delta}(x)dx = 1$, the integration over x' can be taken on $(-\infty, x)$ when $x \le 0$ so that we can always assume that $|x'| \ge |x|$. Moreover , when $x' \neq 0$, and for any $y, y' \in \mathbb{R}$,

$$\begin{aligned} & \frac{(2x'-y-y')(y-y')}{((x'-y)^2+\epsilon^2)((x'-y')^2+\epsilon^2)} \\ = \quad & 2(x')^{-3}\left(\frac{(1-(2x')^{-1}(y+y'))(y-y')}{(1-2(y/x')+(y^2+\epsilon^2)/(x')^2)(1-2(y'/x')+((y')^2+\epsilon^2)/(x')^2)}\right) \\ = \quad & 2(x')^{-3}\left(1+3(2x')^{-1}(y+y')+f(x',y,y')\right)(y-y') \end{aligned}$$

with

$$f(x',y,y') = \frac{(1-(2x')^{-1}(y+y'))}{(1-2(y/x')+(y^2+\epsilon^2)/(x')^2)(1-2(y'/x')+((y')^2+\epsilon^2)/(x')^2)} - 1 - 3(2x')^{-1}(y+y').$$

It is not difficult to see that for δ small enough, we can find a finite constant $C(\delta,\epsilon)$ such that for any triple (x', y, y') such that $|y/x'| \le \delta$, $|y'/x'| \le \delta$, $|x'| \ge 1$,

$$|f(x',y,y')| \le C(\delta,\epsilon)\left((y/x')^{-2} + (y'/x')^{-2}\right). \tag{2.31}$$

Further, for any triple (x', y, y'),

$$|f(x',y,y')| \le \epsilon^{-4}(x')^4(1+|\frac{y}{x'}|+|\frac{y'}{x'}|) + 1 + \frac{3}{2}(|\frac{y}{x'}|+|\frac{y'}{x'}|)$$

so that for any $\eta > 0$, any (x', y, y') such that $|y| \ge \delta|x'|$

$$\begin{aligned} |x'|^{1+\eta}|f(x',y,y')| &\le C(\epsilon)\left(|x'|^{5+\eta} + |x'|^\eta(|x'|^4+1)(|y|+|y'|) + |x'|^{1+\eta}\right) \\ &\le C(\epsilon,\delta,\eta)\left(|y|^{5+\eta} + |y|^\eta(|y|^4+1)(|y|+|y'|) + |y|^{1+\eta}\right) \\ &\le C'(\epsilon,\delta,\eta)\left(1+|y|\right)^{5+\eta}\left(1+|y'|\right)^{5+\eta} \end{aligned} \tag{2.32}$$

with finite constants $C(\epsilon), C(\epsilon,\delta,\eta)$ and $C'(\epsilon,\delta,\eta)$. By symmetry, the same estimate holds when $|y'| \ge \delta|x'|$ and therefore we conclude with (2.31) that for any $\eta \in (0,1]$, there exists a finite constant $c(\epsilon,\eta)$ such that for any $|x'| \ge 1$ and any y, y',

$$|x'|^{1+\eta}|f(x',y,y')| \le c(\epsilon,\eta)\left(1+|y|\right)^{5+\eta}\left(1+|y'|\right)^{5+\eta}. \tag{2.33}$$

In the sequel, we always assume that μ satisfies (A), take $\eta \in (0,1]$, write $c_i(\epsilon,\eta)$ for finite constants which depend on ϵ and η only. Then, for any $|x| \geq 1$, any $u \in [t_k, t_{k+1}[$, any $k \in \{1, \cdots, n\}$,

$$\begin{aligned} \varepsilon_u^{(1)}(x) &:= \Big|\frac{\epsilon}{\pi} \int_x^\infty \int\int (x')^{-3} f(x', y, y') d\mu_{t_k}(y) d\mu_{t_{k+1}}(y') dx'\Big| \\ &\leq \frac{\epsilon}{\pi(3+\eta)} c_0(\epsilon,\eta) \left(\sup_{u\in[0,1]} \int (1+|y|)^{5+\eta} \, d\mu_u(y) \right)^2 |x|^{-3-\eta} \end{aligned}$$

and

$$\int_x^\infty (\rho^\epsilon_{t_{k+1}} - \rho^\epsilon_{t_k})(y) dy = \frac{\epsilon}{\pi} \frac{1}{x^2} \left(\mu_{t_k}(y) - \mu_{t_{k+1}}(y) \right) + \frac{3\epsilon}{\pi x^3} (\mu_{t_k}(y^2) - \mu_{t_{k+1}}(y^2)) + \varepsilon^{(1)}_{t_k}(x) . \quad (2.34)$$

Similarly, one gets that for any $|x| \geq 1$, any $u \in [t_k, t_{k+1}[$,

$$\rho^{\epsilon,\Delta}_u(x) = \frac{\epsilon}{\pi x^2} + \frac{2\epsilon}{\pi x^3} \Big(\mu_{t_k}(y) + \frac{u - t_k}{\Delta} (\mu_{t_{k+1}}(y) - \mu_{t_k}(y)) \Big) + \varepsilon^{(2)}_u(x) \quad (2.35)$$

with a function $\varepsilon^{(2)}$ satisfying $\sup_{|x|\geq 1, u\in[0,1]} |\varepsilon^{(2)}_u(x)| \leq c_2(\epsilon,\eta)|x|^{-3-\eta}$. From (2.34) and (2.35), we conclude that for $|x| \geq 1$,

$$\begin{aligned} (\Delta \rho^{\epsilon,\Delta}_u(x))^{-1} \int_x^\infty (\rho^\epsilon_{t_{k+1}} - \rho^\epsilon_{t_k})(y) dy &= \frac{\mu_{t_k}(y) - \mu_{t_{k+1}}(y)}{\Delta} \quad (2.36) \\ +\Delta^{-1} \Big(3(\mu_{t_k}(y^2) - \mu_{t_{k+1}}(y^2)) - 2(\mu_{t_k}(y) - \mu_{t_{k+1}}(y))(\mu_{t_k}(y) &+ \frac{u-t_k}{\Delta}(\mu_{t_{k+1}}(y) - \mu_{t_k}(y)) \Big) x^{-1} \\ +\varepsilon^{(3)}_u(x) \end{aligned}$$

where, as above, $\sup_{|x|\geq 1, u\in[0,1]} |\varepsilon^{(3)}_u(x)| \leq c_3(\epsilon,\eta)|x|^{-1-\eta}$. Similarly, we find that

$$H_{\mu^{\epsilon,\Delta}_u}(x) = x^{-1} + \varepsilon^{(4)}_u(x)$$

with a function $\varepsilon^{(4)}$ such that $\sup_{|x|\geq 1, u\in[0,1]} |\varepsilon^{(4)}_u(x)| \leq c_4(\epsilon,\eta)|x|^{-2}$. Thus, (2.29) implies that for all $|x| \geq 1$,

$$\partial_x h^{\epsilon,\Delta}_u(x) = -\partial_u \mu^{0,\Delta}_u(y) + \left(2\mu^{0,\Delta}_u(y) \partial_u \mu^{0,\Delta}_u(y) - 3\partial_u \mu^{0,\Delta}_u(y^2) - 1 \right) x^{-1} + \varepsilon^{(5)}_u(x) \quad (2.37)$$

with a function $\varepsilon^{(5)}$ such that

$$\sup_{|x|\geq 1, u\in[0,1]} |\varepsilon^{(5)}_u(x)| \leq c_5(\epsilon,\eta)|x|^{-1-\eta}. \quad (2.38)$$

Consequently, $\sup_{x,u\in[0,1]} |\partial_x h^{\epsilon,\Delta}_u| < \infty$, and, by a similar (and easier!) argument, $\sup_{x,u\in[0,1]} |\partial^2_x h^{\epsilon,\Delta}_u| < \infty$.

Letting

$$\begin{aligned} a^\Delta_k &= -\partial_u \mu^{0,\Delta}_u(y) = \frac{\mu_{t_k}(y) - \mu_{t_{k+1}}(y)}{\Delta} \\ b^\Delta_k(u) &= 2\mu^{0,\Delta}_u(y) \partial_u \mu^{0,\Delta}_u(y) - 3\partial_u \mu^{0,\Delta}_u(y^2) - 1 \end{aligned}$$

we find that for some finite constants C_k, $h_u^{\epsilon,\Delta}$ is of the form

$$h_u^{\epsilon,\Delta}(x) = C_k + a_k^\Delta x + b_k^\Delta(u)\log(x^2+1) + g_u^{\epsilon,\Delta}(x)$$

with a function $g_u^{\epsilon,\Delta}(x) \in \mathcal{C}_b^{2,1}(\mathbb{R}\times[t_k,t_{k+1}[)$ going to zero at infinity faster than $|x|^{-\eta}$, uniformly with respect to the time variable. Observe that the C_k's are irrelevant here since $S^{s,t}(\mu^{\epsilon,\Delta},f)$ does not depend on them for any $f\in\mathcal{C}_b^{2,1}(\mathbb{R}\times[0,1])$. Hence, we choose them equal to zero. It remains to prove the last part of the lemma. From the above,

$$\partial_x h_u^{\epsilon,\Delta}(x) - a_k^\Delta = 2b_k^\Delta(u)\frac{x}{x^2+1} + \partial_x g_u^{\epsilon,\Delta}(x).$$

Note that

$$\frac{x}{x^2+\epsilon^2} = \Re(x+i\epsilon)^{-1}$$

can be written for $\epsilon>0$ as

$$\frac{x}{x^2+\epsilon^2} = \Re\left(-i\int_0^\infty e^{i\xi(x+i\epsilon)}d\xi\right) = \int e^{i\xi x}\left(\frac{-i\mathrm{sgn}(\xi)}{2}e^{-\epsilon|\xi|}\right)d\xi. \tag{2.39}$$

Further, observe that $\partial_x h_u^{\epsilon,\Delta}$ can be extended as an analytic function on $\Omega=\{z:\epsilon>\Im(z)>-\epsilon\}$. Indeed, the definitions of $\rho_u^\epsilon(x)$ and $H_{\mu_u^\epsilon}(x)$ extend immediately as analytic functions on Ω, and further $\rho_u^\epsilon(\cdot)$ does not vanish on Ω. Consequently, $\partial_x g_u^{\epsilon,\Delta}(x)$ is well defined and extends as an analytic function on Ω. Thus, for any $\xi\in\mathbb{R}$,

$$\begin{aligned}\widehat{\partial_x g_u^{\epsilon,\Delta}}(\xi) &= \frac{1}{2\pi}\int e^{i\xi x}\partial_x g_u^{\epsilon,\Delta}(x)dx\\ &= \frac{1}{2\pi}\int e^{i\xi x-\epsilon'\xi}\partial_x g_u^{\epsilon,\Delta}(x+i\epsilon')dx.\end{aligned}$$

Observe that we can extend the study of the asymptotics of $\partial_x g_u^{\epsilon,\Delta}$ to the complex line $\{x+i\epsilon', x\in\mathbb{R}\}$ to see that $\partial_x g_u^{\epsilon,\Delta}(x+i\epsilon')$ decreases as $|x+i\epsilon'|^{-1-\eta}$ at infinity. Hence, $\partial_x g_u^{\epsilon,\Delta}(x+i\epsilon')\in L^1(dx)$ and the uniformity in time of our estimate of the function $\varepsilon^{(5)}$ extends to the complex line $\{x+i\epsilon', x\in\mathbb{R}\}$. This shows that, when $\epsilon'<\epsilon$,

$$C(\epsilon',\epsilon,\Delta) := \sup_{u\in[0,1]}\int|\partial_x g_u^{\epsilon,\Delta}(x+i\epsilon')|dx<\infty.$$

Therefore,

$$|\widehat{\partial_x g_u^{\epsilon,\Delta}}(\xi)| \le \int|\partial_x g_u^{\epsilon,\Delta}(x+i\epsilon')|dx e^{-\epsilon'\xi} \le C(\epsilon',\epsilon,\Delta)e^{-\epsilon'\xi}.$$

Finally, $\partial_x g_u^{\epsilon,\Delta}$ decreases at infinity like $|x|^{-1-\eta}$ so that it belongs to $L^1(\mathbb{R})\cap L^2(\mathbb{R})$. Thus, the representation theorem yields

$$\partial_x g_u^{\epsilon,\Delta}(x) = \int e^{i\xi x}\widehat{\partial_x g_u^{\epsilon,\Delta}}(\xi)d\xi$$

which, with the above estimate of $\widehat{\partial_x g_u^{\epsilon,\Delta}}$ completes the proof of the lemma. $\square$

Step 3: Proof of Lemma 2.8

We shall here present the main steps of the proof, leaving only a few technical points for the next (and last) step.

Let

$$A_N^M := \{X_N \in \mathcal{C}([0,1], \mathcal{H}_N); X_N(t) = \epsilon C_N + W_N(t), \quad \sup_{u\in[0,1]} \mathrm{tr}_N W_N^2(u) \le M\}.$$

Write now, with $\delta' < (\delta/2)$, and for Δ small enough,

$$\mathbb{P}\left(\hat{\mu}^{N,\epsilon} \in B(\mu^\epsilon, \delta)\right) \ge \\ \mathbb{P}\left(\{\hat{\mu}^{N,\epsilon} \in B(\mu^{\epsilon,\Delta}, \delta')\} \cap \{X_N^\epsilon \in A_N^M\}; e^{N^2 \bar{S}^{0,1}(\hat{\mu}^{N,\epsilon}, h^{\epsilon,\Delta})} e^{-N^2 \bar{S}^{0,1}(\hat{\mu}^{N,\epsilon}, h^{\epsilon,\Delta})}\right). \quad (2.40)$$

Here, $\bar{S}^{0,1}(\hat{\mu}^{N,\epsilon}, h^{\epsilon,\Delta})$ is defined as in Corollary 2.2. Defining $\bar{S}^{0,1}(\mu^\epsilon, h^{\epsilon,\Delta})$ as in remark 2.3, we shall prove in the next step that, if $D(N, \mu^{\epsilon,\Delta}, \delta', M) = \{\omega : \{X_N^\epsilon(\omega) \in A_N^M\} \cap \{\hat{\mu}^{N,\epsilon} \in B(\mu^{\epsilon,\Delta}, \delta')\}\}$,

$$\limsup_{\delta'\to 0} \sup_{N\to\infty} \sup_{\omega \in D(N,\mu^{\epsilon,\Delta},\delta',M)} \left|\bar{S}^{0,1}(\hat{\mu}^{N,\epsilon}, h^{\epsilon,\Delta}) - \bar{S}^{0,1}(\mu^\epsilon, h^{\epsilon,\Delta})\right| = 0. \quad (2.41)$$

As a consequence, (2.40) shows that for any $\eta > 0$, for N large enough and δ' small enough,

$$\mathbb{P}\left(\hat{\mu}^{N,\epsilon} \in B(\mu^\epsilon, \delta)\right) \ \ge \ e^{-N^2(\bar{S}^{0,1}(\mu^\epsilon, h^{\epsilon,\Delta})+\eta)} \mathbb{P}\left(D(N, \mu^{\epsilon,\Delta}, \delta', M); e^{N^2 \bar{S}^{0,1}(\hat{\mu}^{N,\epsilon}, h^{\epsilon,\Delta})}\right). \quad (2.42)$$

Denote $\mathbb{P}_{h^{\epsilon,\Delta}}$ the probability measure on $\mathcal{H}_N$ given by

$$\mathbb{P}_{h^{\epsilon,\Delta}}(dX_N^\epsilon) = e^{N^2 \bar{S}^{0,1}(\hat{\mu}^{N,\epsilon}, h^{\epsilon,\Delta})} \mathbb{P}(dX_N^\epsilon).$$

Then, (2.42) gives for N large enough

$$\mathbb{P}\left(\{\hat{\mu}^{N,\epsilon} \in B(\mu^\epsilon, \delta)\right) \ge e^{-N^2(\bar{S}^{0,1}(\mu^\epsilon, h^{\epsilon,\Delta})+\eta)} \left(\mathbb{P}_{h^{\epsilon,\Delta}}\left(\hat{\mu}^{N,\epsilon} \in B(\mu^{\epsilon,\Delta}, \delta')\right) - \mathbb{P}_{h^{\epsilon,\Delta}}\left(X_N^\epsilon \in (A_N^M)^c\right)\right). \quad (2.43)$$

We first show that $\mathbb{P}_{h^{\epsilon,\Delta}}\left(X_N^\epsilon \in (A_N^M)^c\right)$ is negligible when M is large enough. In fact, because $\partial_x h^{\epsilon,\Delta}$ is uniformly bounded according to Lemma 2.12, we find a finite constant C such that

$$\int \left(\frac{d\mathbb{P}_{h^{\epsilon,\Delta}}}{d\mathbb{P}}\right)^2 d\mathbb{P} \le C e^{CN^2} \quad (2.44)$$

whereas, by exponential tightness of Lemma 2.5 (see its proof and (H')), for any $L > 0$, there exists $M(L)$ so that for $M \ge M(L)$,

$$\mathbb{P}\left((A_N^M)^c\right) \le e^{-2LN^2}.$$

Hence, Cauchy-Schwarz's inequality gives for $M \ge M(L+C)$,

$$\mathbb{P}_{h^{\epsilon,\Delta}}\left(X_N^\epsilon \in (A_N^M)^c\right) \le \sqrt{C} e^{-LN^2}. \quad (2.45)$$

We shall now prove that for any $\delta' > 0$,

$$\lim_{N\to\infty} \mathbb{P}_{h^{\epsilon,\Delta}}\left(\hat{\mu}^{N,\epsilon} \in B(\mu^{\epsilon,\Delta}, \delta')\right) = 1. \quad (2.46)$$

By (2.44), assumptions (A) and (H'), and Lemma 2.5, we first observe that the law of $\hat{\mu}^{N,\epsilon}$ under $\mathbb{P}_{h^{\epsilon,\Delta}}$ is exponentially tight and hence tight. Let us show that it admits a unique limit point. Girsanov's theorem shows that $\mathbb{P}_{h^{\epsilon,\Delta}}$ is the weak solution of

$$dZ_N(t) = dH_N(t) + (\partial_x h_t^{\epsilon,\Delta})(Z_N(t))dt \quad (2.47)$$

with initial data $Z_N(0) = D_N + \epsilon C_N$. In fact, eventhough not needed here, one can remark (see the proof of Theorem 4.1) that for any given N, since $\partial_x h_t^{\epsilon,\Delta}$ is a Lipschitz operator function according to Lemma 2.12, (2.47) admits a unique solution.

Consequently, we can show as in [6] (see Lemma 2.1) that for any $f \in \mathcal{C}_b^{2,1}(I\!R \times [0,1])$, for any $s \in [0,1)$,

$$M_{f,s}^N(t) = S^{s,t}(\hat{\mu}^{N,\epsilon}, f) - \int_s^t \int \partial_x f(x,u) \partial_x h_u^{\epsilon,\Delta}(x) d\hat{\mu}_u^{N,\epsilon}(x) du$$

is a martingale for $t \in [s,1]$ with quadratic variation $N^{-2} < f,f >_{s,t}^{\hat{\mu}^{N,\epsilon}}$ going to zero as N goes to infinity. Henceforth, any limit point ν of $\hat{\mu}^{N,\epsilon}$ under $I\!P_{h^{\epsilon,\Delta}}$ satisfies the equation

$$S^{s,t}(\nu, f) =< f, h^{\epsilon,\Delta} >_{s,t}^{\nu} . \tag{2.48}$$

Further, according to the construction of C_N and D_N, $\nu_0 = P_\epsilon * \mu_0$. One can prove that this equation admits a unique solution following the proof of [6, Lemma 2.9]. Indeed, take $f(x,t) := e^{i\lambda x}$ in (2.48) and denote by $\mathcal{L}_t(\lambda) = \int e^{i\lambda x} d\nu_t(x)$ the Fourier transform of ν_t. Then, we find, with the notations of Lemma 2.12, that for $t \in [0 = t_1, t_2]$,

$$\mathcal{L}_t(\lambda) = \mathcal{L}_0(\lambda) - \frac{\lambda^2}{2} \int_0^t \int_0^1 \mathcal{L}_s(\alpha\lambda)\mathcal{L}_s((1-\alpha)\lambda) d\alpha ds + i\lambda \int_0^t \left(a_1^\Delta \mathcal{L}_s(\lambda) + \int \mathcal{L}_s(\lambda + \lambda') l_1^{\epsilon,\Delta}(\lambda', s) d\lambda' \right) ds. \tag{2.49}$$

Multiplying both sides of this equality by $e^{-\frac{\epsilon}{4}|\lambda|}$ gives, with $\mathcal{L}_t^\epsilon(\lambda) = e^{-\frac{\epsilon}{4}|\lambda|}\mathcal{L}_t(\lambda)$,

$$\begin{aligned} \mathcal{L}_t^\epsilon(\lambda) &= \mathcal{L}_0^\epsilon(\lambda) - \frac{\lambda^2}{2} \int_0^t \int_0^1 \mathcal{L}_s^\epsilon(\alpha\lambda)\mathcal{L}_s^\epsilon((1-\alpha)\lambda) d\alpha ds \\ &+ i\lambda \int_0^t \left(a_1^\Delta \mathcal{L}_s^\epsilon(\lambda) + \int \mathcal{L}_s^\epsilon(\lambda + \lambda') e^{\frac{\epsilon}{4}|\lambda+\lambda'| - \frac{\epsilon}{4}|\lambda|} l_1^{\epsilon,\Delta}(\lambda', s) d\lambda' \right) ds. \end{aligned} \tag{2.50}$$

Therefore, if $\nu, \widetilde{\nu}'$ are two solutions with Fourier transforms $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ respectively and if we set $\Delta_t^\epsilon(R) = \sup_{|\lambda| \le R} |\mathcal{L}_t^\epsilon(\lambda) - \widetilde{\mathcal{L}}_t^\epsilon(\lambda)|$, we deduce from (2.27) and (2.50) (see [6], proof of Lemma 2.6, for details) that there exists a finite constant C such that

$$\Delta_t^\epsilon(R) \le \frac{CR}{\epsilon} \int_0^t \bar{\Delta}_s^\epsilon(R) ds + 2tRe^{-\frac{\epsilon}{4}R}.$$

By Gronwall's lemma, we deduce that

$$\Delta_t^\epsilon(R) \le 2Re^{-\frac{\epsilon}{4}R} e^{\frac{CR}{\epsilon}t}$$

and thus that $\Delta_t^\epsilon(\infty) = 0$ for $t < \tau \equiv (\epsilon^2/4C) \wedge t_2$. By induction over the time, we conclude that $\Delta_t^\epsilon(\infty) = 0$ for any time $t \le t_2$, and then any time $t \le 1$, and therefore that $\nu = \widetilde{\nu}$. Thus, since $\mu^{\epsilon,\Delta}$ already satisfies this equation by the definition of $h^{\epsilon,\Delta}$, we conclude that $\hat{\mu}^{N,\epsilon}$ converges towards $\mu^{\epsilon,\Delta}$ under $I\!P_{h^{\epsilon,\Delta}}$. Hence, (2.46) is proved.

(2.43), (2.45) and (2.46) shows that for any $\eta > 0$, any $\Delta > 0$ small enough,

$$\liminf_{N\to\infty} \frac{1}{N^2} \log I\!P\left(\hat{\mu}^{N,\epsilon} \in B(\mu^\epsilon, \delta)\right) \ge -\bar{S}^{0,1}(\mu^{\epsilon,\Delta}, h^{\epsilon,\Delta}) - \eta.$$

Letting $\Delta \downarrow 0$, and then $\eta \downarrow 0$, we get, with (2.13), Lemma 2.8.

Step 4 : Proof of the technical result (2.41) In this last part, to prove (2.41), we have to deal with the fact that $h^{\epsilon,\Delta}$ is unbounded and actually growing with x at infinity, which is not integrable with respect to the Cauchy law. This appears to be a real problem when dealing with the convergence of $\hat{\mu}_t^{N,\epsilon}(h_t^{\epsilon,\Delta})$. However, one can observe that our rate function only depends on differences $\hat{\mu}_t^{N,\epsilon}(h_t^{\epsilon,\Delta}) - \hat{\mu}_s^{N,\epsilon}(h_s^{\epsilon,\Delta})$ where these singular terms cancel, and hope that $S^{0,1}(\hat{\mu}^{N,\epsilon}, h^{\epsilon,\Delta})$ will converge towards $S^{0,1}(\mu^\epsilon, h^{\epsilon,\Delta})$ defined as in Remark 2.3.

Since $h^{\epsilon,\Delta}$ is of the type of functions described in Remark 2.3, we consider more generally such functions here. Hence, let

$$f(x,u) = \sum_{k=0}^{n} 1\!\!I_{u\in[t_k,t_{k+1}[} c_k x + g(x,u)$$

with finite constants c_k and $g \in \mathcal{C}^{2,1}(I\!\!R \times [t_k, t_{k+1}[)$ for all $k \in \{0,\cdots,n\}$ such that $\sup_{t\in[0,1]} \mu_t(g_t^2) < \infty$ and g_t has bounded spatial derivatives. For $K \in I\!\!R^+$, set

$$\widetilde{f}_K(x) = \begin{cases} x & \text{if } |x| \le K-1, \\ K - 1/2 & \text{if } |x| \ge K, \\ \widetilde{f}_K & \text{is smooth in between with first and second derivatives bounded by one.} \end{cases}$$

Let

$$f_K(x,u) = \sum_{k=0}^{n} 1\!\!I_{u\in[t_k,t_{k+1}[} c_k \widetilde{f}_K(x) + g(x,u).$$

We first prove that for any $\mu_. = p * \nu_.$ with a probability measure p such that $\sup_t \mu_t(g_t^2) < \infty$ and a probability measure valued process $\nu_.$ such that $\sup_{t\in[0,1]} \nu_t(x^2) < \infty$,

$$\lim_{K\to\infty} S^{0,1}(\mu_., f_K) = \widetilde{S}^{0,1}(\mu_., f) \tag{2.51}$$

where we have used the notations of Remark 2.3 and $S^{0,1}$ is defined as in Corollary 2.2. Note first that for all $s,t \in [t_k, t_{k+1}[$,

$$S^{s,t}(\mu_., f_K) = S^{s,t}(\mu_., g) - \frac{c_k}{2}\int_s^t \mu_u \otimes \mu_u\left(\frac{\partial_x \widetilde{f}_K(x) - \partial_x \widetilde{f}_K(y)}{x-y}\right)du + c_k\left(\mu_t(\widetilde{f}_K) - \mu_s(\widetilde{f}_K)\right) = I - II + III.$$

Observe that since $\mu_. \in \mathcal{C}([0,1], M_1(I\!\!R))$ has tight marginals and $\partial_x \widetilde{f}_K$ is null except for $|x| \in [K-1, K]$ where it can be chosen bounded by one, II goes to zero as K goes to infinity.

Now, let us consider III.

$$c_k^{-1} III = \mu_t(\widetilde{f}_K) - \mu_s(\widetilde{f}_K) = \int (x-y) 1\!\!I_{|x+z|\le K-1} 1\!\!I_{|y+z|\le K-1} d\nu_t(x) d\nu_s(y) dp(z) + IV \tag{2.52}$$

Clearly,

$$\lim_{K\to\infty} \int (x-y) 1\!\!I_{|x+z|\le K-1} 1\!\!I_{|y+z|\le K-1} d\nu_t(x) d\nu_s(y) dp(z) = \nu_t(x) - \nu_s(x).$$

Moreover, IV is bounded by

$$\begin{aligned} V &:= \int |\widetilde{f}_K(x+z) - \widetilde{f}_K(y+z)| 1\!\!I_{\{\{|y+z|\le K-1\}\cap\{|x+z|\le K-1\}\}^c} d\nu_t(x) d\nu_s(y) dp(z) \\ &\le \int |x-y| 1\!\!I_{\{\{|y+z|\le K-1\}\cap\{|x+z|\le K-1\}\}^c} d\nu_t(x) d\nu_s(y) dp(z) \end{aligned}$$

where we have used that $\widetilde{f}_K$ is Lipschitz with Lipschitz constant bounded by one. Hence, Cauchy-Schwarz's inequality gives

$$|IV|^2 \leq \int |x-y|^2 d\nu_t(x) d\nu_s(y) \int 1\!\!I_{\{\{|y+z|\leq K-1\}\cap\{|x+z|\leq K-1\}\}^c} d\nu_t(x) d\nu_s(y) dp(z) \tag{2.53}$$

where the first term is uniformly bounded according to our assumption and the second goes to zero as K goes to infinity by tightness of $(p * \nu_t, t \in [0,1])$. Hence, (2.52) shows that

$$\lim_{K\to\infty} \left(\mu_t(\widetilde{f}_K) - \mu_s(\widetilde{f}_K)\right) = (\nu_t(x) - \nu_s(x))$$

which completes the proof of (2.51). Note here that this convergence is uniform on processes $\nu_.$ such that $\sup_{t\in[0,1]} \nu_t(x^2) \leq M$ for $M \in I\!R^+$. Hence, we see similarly that, uniformly on N,

$$\lim_{K\to\infty} \sup_{\omega\in A_N^M} \left|S^{0,1}(\hat{\mu}^{N,\epsilon}(\omega), f_K) - S^{0,1}(\hat{\mu}^{N,\epsilon}(\omega), f)\right| = 0$$

if $\sup_N \sup_{\omega\in A_N^M} \sup_{t\in[0,1]} \int g_t^2 d\hat{\mu}_t^{N,\epsilon}(\omega)$ is finite. We assume this last property below. But for any $K \in I\!R^+$,

$$\lim_{\delta\to 0} \sup_{d(\mu,\mu^{\epsilon,\Delta})<\delta} \left|S^{0,1}(\mu, f_K) - S^{0,1}(\mu^{\epsilon,\Delta}, f_K)\right| = 0$$

and therefore we can conclude that

$$\begin{aligned}
\sup_{\omega\in D(N,\mu^{\epsilon,\Delta},\delta',M)} \left|S^{0,1}(\hat{\mu}^{N,\epsilon}(\omega), f) - S^{0,1}(\mu^{\epsilon,\Delta}, f)\right| \leq & \sup_{\omega\in A_N^M} \left|S^{0,1}(\hat{\mu}^{N,\epsilon}(\omega), f) - S^{0,1}(\hat{\mu}^{N,\epsilon}(\omega), f_K)\right| \\
& + \left|S^{0,1}(\mu^{\epsilon,\Delta}, f_K) - S^{0,1}(\mu^{\epsilon,\Delta}, f)\right| \\
& + \sup_{d(\mu,\mu^{\epsilon,\Delta})<\delta'} \left|S^{0,1}(\mu, f_K) - S^{0,1}(\mu^{\epsilon,\Delta}, f_K)\right|
\end{aligned}$$

goes to zero as δ' goes to zero.

To complete the argument and apply this result to $f = h^{\epsilon,\Delta}$ one only need to notice that $\log(x^2+1)$ is such that $\sup_N \sup_{\omega\in A_N^M} \sup_{t\in[0,1]} \int (\log(x^2+1))^2 d\hat{\mu}_t^{N,\epsilon}(\omega)$ is finite. This is easily obtained following the lines of the proof of Lemma 2.5 under assumption (2.11).

Hence, we proved

$$\limsup_{\delta'\to 0} \sup_{N\to\infty} \sup_{\omega\in D(N,\mu^{\epsilon,\Delta},\delta',M)} \left|S^{0,1}(\hat{\mu}^{N,\epsilon}, h^{\epsilon,\Delta}) - S^{0,1}(\mu^{\epsilon}, h^{\epsilon,\Delta})\right| = 0. \tag{2.54}$$

Noting that $\nu \to < h^{\epsilon,\Delta}, h^{\epsilon,\Delta} >_{s,t}^{\nu}$ is continuous since $\partial_x h^{\epsilon,\Delta}$ was proved to be bounded piecewise continuous, we extend (2.54) to $\bar{S}^{0,1}$, which completes the proof of (2.41).

2.3.3 Proof of Lemma 2.10

This result is part of a more general theorem of [7] showing that free convolution (which is equivalent to usual convolution in the case of a Cauchy variable) always reduces the entropy. However, the proof is here easier so that we describe it completely. It is easy to check (see [7] or the remark in the proof of Lemma (2.6) after (2.24) in [6]), that for any $\epsilon > 0$ and any $f \in \mathcal{C}_b^1(I\!R)$,

$$\int\int \frac{f(x)-f(y)}{x-y} dP_\epsilon * \mu(x) dP_\epsilon * \mu(y) = \int\int \frac{P_\epsilon * f(x) - P_\epsilon * f(y)}{x-y} d\mu(x) d\mu(y) \tag{2.55}$$

Consequently,

$$\begin{aligned} S^{0,1}(P_\epsilon * \nu) &= \sup_{f\in\mathcal{C}_b^{2,1}(\mathbb{R}\times[0,1])} \Big(S^{0,1}(P_\epsilon * \nu, f) - \frac{1}{2}\int_0^1 \int (\partial_x f(x,u))^2 dP_\epsilon * \nu_u(x) du\Big) \\ &= \sup_{f\in\mathcal{C}_b^{2,1}(\mathbb{R}\times[0,1])} \Big(S^{0,1}(\nu, P_\epsilon * f) - \frac{1}{2}\int dP_\epsilon(z) \int_0^1 \int (\partial_x f(x-z,u))^2 d\nu_u(x) du\Big) \\ &\leq \int dP_\epsilon(z) \sup_{f\in\mathcal{C}_b^{2,1}(\mathbb{R}\times[0,1])} \Big(S^{0,1}(\nu, t_z\circ f) - \frac{1}{2}\int_0^1 \int (\partial_x t_z \circ f(x,u))^2 d\nu_u(x) du\Big) \end{aligned}$$

where $t_z \circ f(x) = f(x-z)$ and we have used the fact that $S^{0,1}(\nu, f)$ is a linear functional of f. Noting that

$$\begin{aligned} &\sup_{f\in\mathcal{C}_b^{2,1}(\mathbb{R}\times[0,1])} \Big(S^{0,1}(\nu, t_z\circ f) - \frac{1}{2}\int_0^1 \int (\partial_x t_z \circ f(x,u))^2 d\nu_u(x) du\Big) \\ &= \sup_{f\in\mathcal{C}_b^{2,1}(\mathbb{R}\times[0,1])} \Big(S^{0,1}(\nu, f) - \frac{1}{2}\int_0^1 \int (\partial_x f(x,u))^2 d\nu_u(x) du\Big) \end{aligned}$$

for any $z \in \mathbb{R}$, we conclude

$$S^{0,1}(P_\epsilon * \nu) \leq S^{0,1}(\nu)\,. \tag{2.56}$$

so that, since $S^{0,1}(\nu)$ is also lower semi continuous, for ν so that $S(\nu) < \infty$,

$$\lim_{\epsilon\downarrow 0} S^{0,1}(P_\epsilon * \nu) = S^{0,1}(\nu) \tag{2.57}$$

Note here that this is not true for S itself since $S(P_\epsilon * \nu) = +\infty$ for $\epsilon \neq 0$ as $P_\epsilon * \nu_0 \neq \nu_0 = \mu_0$.

2.3.4 Proof of Lemma 2.11

Step 1 : compactly supported measure approximation Recall that $(c_i)_{1\leq i\leq N}$ denotes the eigenvalues of C_N. For $M > 0$, we set

$$B_M := \{i : |c_i| > M\} := \{j_1, \cdots, j_{|B_M|}\}\,.$$

Define

$$\widetilde{C}_{N,M}(i,i) = \begin{cases} c_i & \text{if } i \notin B_M \\ 0 & \text{otherwise.} \end{cases}$$

Construct $C_{N,M}$ as C_N, replacing only $\widetilde{C}_N$ by $\widetilde{C}_{N,M}$ (see the discussion at the beginning of the section). Let

$$X_N^{\epsilon,M}(t) = H_N(t) + D_N + \epsilon C_{N,M}$$

and denote $\hat{\mu}_t^{N,\epsilon,M}$ its spectral measure. Then,

$$D(\hat{\mu}_.^{N,\epsilon,M}, \hat{\mu}_.^N) \leq \epsilon M.$$

In fact, for any continuously differentiable function f, any $t \in [0,1]$,

$$\begin{aligned} |\hat{\mu}_t^{N,\epsilon,M}(f) - \hat{\mu}_t^N(f)| &= \epsilon \left| \int_0^1 \operatorname{tr}_N \left(f'(X_N(t) + \alpha\epsilon C_{N,M}) C_{N,M} \right) d\alpha \right| \\ &\le \frac{\epsilon}{N} \sum_{i \in B_M} |c_i| \int_0^1 |< e_i, f'(X_N(t) + \alpha\epsilon C_{N,M}) e_i >| \, d\alpha \\ &\le \epsilon M \int_0^1 \left(\operatorname{tr}_N (f'(X_N(t) + \alpha\epsilon C_{N,M})^2) \right)^{\frac{1}{2}} d\alpha. \end{aligned}$$

Extending this inequality to Lipschitz functions, we deduce that

$$|\hat{\mu}_t^{N,\epsilon,M}(f) - \hat{\mu}_t^N(f)| \le ||f||_{\mathcal{L}} \epsilon M$$

which gives the desired estimate on $D(\hat{\mu}_.^{N,\epsilon,M}, \hat{\mu}_.^N)$.

<u>*Step 2 : Tightness*</u> Observe that , under assumption (2.11), for any $M \in \mathbb{R}^+$,

$$\sup_{N \in \mathbb{N}} \operatorname{tr}_N ((\log(C_{N,M}^2 + 1))^2) \le \sup_{N \in \mathbb{N}} \operatorname{tr}_N ((\log(C_N^2 + 1))^2) < \infty$$

which insures, in view of the proof of Lemma 2.5, that, on $\mathcal{K}_\epsilon^N(K_L)$, the family of the marginals $\{\hat{\mu}_t^{N,\epsilon,M}, M \in \mathbb{R} \cup \{\infty\}\}_{t \in [0,1]}$ is tight. In particular, we find a compact set K_1 of $M_1(\mathbb{R})$, independent of ϵ, such that for all $t \in [0,1]$ and $M \in \mathbb{R}^+$,

$$\hat{\mu}_t^{N,\epsilon,M} \in K_1, \qquad \hat{\mu}_t^{N,\epsilon} \in K_1.$$

<u>*Step 3*</u> On the compact set K_1, the Wasserstein distance is equivalent to the distance

$$d_1(\mu,\nu) = \sup_{||f||_{\mathcal{L}} \le 1, f \uparrow} \left| \int f d\nu - \int f d\mu \right|.$$

Write

$$X_N^\epsilon(t) = X_N^{\epsilon,M}(t) + \epsilon \sum_{i=1}^{|B_M|} c_{j_i} e_{j_i} e_{j_i}^T.$$

Now, let $\lambda_1, \ldots, \lambda_N$ denote the eigenvalues of $X_N^{\epsilon,M}(t)$ and $\lambda_1^1, \ldots, \lambda_N^1$ that of $X_N^{\epsilon,M}(t) + \epsilon c_{j_1} e_{j_1} e_{j_1}^T$. Then, by Lidskii's theorem (see [14, Theorem 6.10]),

$$\lambda_1 \le \lambda_1^1 \le \lambda_2 \cdots \le \lambda_N \le \lambda_N^1.$$

Thus, for any increasing Lipschitz function f, we get, with $\hat{\mu}_t^{N,\epsilon,M,1}$ the spectral measure of $X_N^{\epsilon,M}(t) + \epsilon c_{j_1} e_{j_1} e_{j_1}^T$,

$$\left| \int f d\hat{\mu}_t^{N,\epsilon,M} - \int f d\hat{\mu}_t^{N,\epsilon,M,1} \right| \le \frac{2}{N} ||f||_{\mathcal{L}} + \frac{f(\lambda_{max}) - f(\lambda_{min})}{N} \le \frac{4}{N} ||f||_{\mathcal{L}}$$

ensuring that

$$d_1(\hat{\mu}_t^{N,\epsilon,M}, \hat{\mu}_t^{N,\epsilon,M,1}) \le \frac{4}{N}.$$

Repeating this operation $|B_M|$ times we conclude

$$d_1(\hat{\mu}_t^{N,\epsilon,M}, \hat{\mu}_t^{N,\epsilon}) \le \frac{4|B_M|}{N}.$$

But Chebycheff's inequality yields

$$\begin{aligned}\frac{|B_M|}{N} &= \int 1\!\!I_{|x|\geq M} d\hat{\mu}_C^N(x) \\ &\leq \frac{1}{(\log(M^2+1))^2} \sup_N \int (\log(x^2+1))^2 d\hat{\mu}_C^N(x)\end{aligned}$$

giving finally, according to condition (2.11), a finite constant C such that

$$d_1(\hat{\mu}_t^{N,\epsilon,M}, \hat{\mu}_t^{N,\epsilon}) \leq \frac{C}{(\log(M^2+1))^2}.$$

Steps 1-3 give the lemma by taking first $\epsilon \downarrow 0$ and then $M \uparrow \infty$. □

3 Large deviation for the law of the spectral process of the symmetric Brownian motion

The symmetric Brownian motion S^N is defined as the Markov process $(H_N(t))_{t\in I\!R}$ with values in the space $\mathcal{S}_N$ of symmetric matrices of dimension N and real Brownian motions entries. We can construct the entries $\{S_N^{i,j}(t), t \geq 0, (i,j) \in \{1,..,N\}\}$ via independent real valued Brownian motions $(\beta_{i,j})_{1\leq i\leq j\leq n}$ by

$$S_N^{k,l} = \frac{\sqrt{1+\delta_{k=l}}}{\sqrt{N}} \beta_{k\wedge l, k\vee l}.$$

We let $(\widetilde{\lambda}_i^{(N)}(t), 1 \leq i \leq N)$ be the eigenvalues of $S_N(t) + D_N$ and $\widetilde{\mu}_\cdot^N$ be their empirical process. We shall prove now Theorem 1.4 for $\beta = 1$.

In fact, the proof of this theorem for $\beta = 1$ follows the case $\beta = 2$ once one obtains the following Itô's formula for $\widetilde{\mu}_\cdot^N$;

Lemma 3.1 ([6, Lemma 3.1]) *For every $f \in \mathcal{C}_b^{2,1}(I\!R \times [0,1])$,*

$$\begin{aligned}\int f(x,t) d\widetilde{\mu}_t^N &= \int f(x,0) d\widetilde{\mu}_0^N + \int_0^t \mathrm{tr}_N(f'(S_N(s);s) dS_N(s)) + \int_0^t \int \partial_s f(x,s) d\widetilde{\mu}_s^N ds \\ &+ \int_0^t \int \frac{\partial_x f(x,s) - \partial_x f(y,s)}{2(x-y)} d\widetilde{\mu}_s^N(x) d\widetilde{\mu}_s^N(y) ds + \frac{1}{2N} \int_0^t \int \partial_{xx} f(x,s) d\widetilde{\mu}_s^N(x) ds\end{aligned}$$

Furthermore, the martingale bracket for $\int_0^t \mathrm{tr}_N(f'(S_N(s);s) dS_N(s))$ is given by

$$\langle \int_0^\cdot \mathrm{tr}_N(f'(S_N(s);s) dS_N(s)) \rangle_t = \frac{2}{N^2} \int_0^t \int (\partial_x f(S_N(s),s))^2 d\widetilde{\mu}_s^N(x) ds.$$

The above Itô's formula is very similar to that obtained in Theorem 2.1 for the Hermitian Brownian motion. The only differences are an error term (see the last term in Itô's formula) which will not change either the analysis of the large deviation or the result, and the quadratic variation of the martingale which is twice what it was for the Hermitian Brownian motion. From this last fact, the analysis of the previous section shows that the rate function governing the deviations of $\widetilde{\mu}^N_.$ is given, when $\widetilde{\mu}^N_0$ converges towards μ_D, for any $\nu \in \mathcal{C}([0,1], M_1(\mathbb{R}))$ so that $\nu_0 = \mu_D$,

$$\begin{aligned} S^s_{\mu_D}(\nu) &= \sup_{0\le s\le t\le 1} \sup_{f\in\mathcal{C}^{2,1}(\mathbb{R}\times[0,1])} \{S^{s,t}(\nu,f) - < f,f >^{\nu}_{s,t}\} \\ &= \frac{1}{2} \sup_{0\le s\le t\le 1} \sup_{f\in\mathcal{C}^{2,1}(\mathbb{R}\times[0,1])} \{S^{s,t}(\nu,f) - \frac{1}{2} < f,f >^{\nu}_{s,t}\} = \frac{1}{2} S_{\mu_D}(\nu) \end{aligned}$$

where in the second line we have only performed the homothety $f \to (1/2)f$. Hence, Theorem 1.4 for $\beta = 1$ is a direct consequence of the proof of Theorem 1.4 for $\beta = 2$ and Lemma 3.1. □

4 Study of the minimizer: proof of (1.5) and Corollary 1.6

In this section, we prove the following theorem, yielding (1.5) and hence Corollary 1.6:

Theorem 4.1 *Assume that μ_D is compactly supported. Then,*

$$\lim_{\vartheta\to 0} \inf\{S_{\mu_D}(\nu_.);\ \nu \in \mathcal{A},\ d(\nu_1,\mu) < \vartheta\} = \inf\{S_{\mu_D}(\nu_.); \nu_1 = \mu\}.$$

Proof. Let us first observe that of course, since S_{μ_D} is a good rate function,

$$\lim_{\vartheta\to 0} \inf\{S_{\mu_D}(\nu_.);\ \nu \in \mathcal{A}, d(\nu_1,\mu) < \vartheta\} \ge \lim_{\vartheta\to 0} \inf\{S_{\mu_D}(\nu_.);\ d(\nu_1,\mu) < \vartheta\} = \inf\{S_{\mu_D}(\nu_.);\ \nu_1 = \mu\}$$

so that we only need to prove the opposite inequality. Now, let $\nu^\delta \in \mathcal{C}([0,1], M_1(\mathbb{R}))$ be such that $\nu^\delta_1 = \mu$ and

$$\inf\{S_{\mu_D}(\nu_.); \nu_1 = \mu\} \ge S_{\mu_D}(\nu^\delta_.) - \delta = S^{0,1}(\nu^\delta_.) - \delta$$

for some $\delta > 0$. Of course, $\nu^\delta_0 = \mu_D$. We shall prove that, up to a small error, $S^{0,1}(\nu^\delta_.)$ can be bounded below by $S^{0,1}(\widetilde{\nu}^{\delta,\epsilon,\Delta}_.)$ with $\widetilde{\nu}^{\delta,\epsilon,\Delta}_.$ a compactly supported process with initial law equal to μ_D and $\widetilde{\nu}^{\delta,\epsilon,\Delta}_1$ as close as wished to μ. To construct the process $\widetilde{\nu}^{\delta,\epsilon,\Delta}_.$, recall that we saw in Lemma 2.10 that, for any $\epsilon > 0$,

$$S^{0,1}(\nu^\delta_.) \ge S^{0,1}(P_\epsilon * \nu^\delta_.). \tag{4.1}$$

Moreover, we can also perform the time regularization of the first step of Section 2.3.2 to find a $\nu^{\delta,\epsilon,\Delta}$ such that

$$\inf\{S_{\mu_D}(\nu_.); \nu_1 = \mu\} \ge S^{0,1}(\nu^{\delta,\epsilon,\Delta}_.) - \delta - \Delta \tag{4.2}$$

and $\nu^{\delta,\epsilon,\Delta}_0 = P_\epsilon * \mu_D, \nu^{\delta,\epsilon,\Delta}_1 = P_\epsilon * \mu$. Further, we saw in the second step of Section 2.3.2 (see (2.28) that $\nu^{\delta,\epsilon,\Delta}$ is described as the unique solution of the so-called free Fokker-Planck equation given by

$$S^{0,t}(f,\mu_.) = \int_0^t \int \partial_x f(x,s) \partial_x h^{\delta,\epsilon,\Delta}(x,s) d\mu_s ds \tag{4.3}$$

with a field $\partial_x h^{\delta,\epsilon,\Delta}(x,s)$ described in Lemma 2.12. Now, let us consider a free probability space $(\mathcal{A},\tau)$ and the algebra $\widetilde{\mathcal{A}}_{sa}$ of self adjoint possibly unbounded operators affiliated to $\mathcal{A}$. Consider a bounded operator

D with law μ_D and a free Brownian motion S, free with D, in $(\mathcal{A},\tau)$. Let C_ϵ be an unbounded operator of $\widetilde{\mathcal{A}}_{sa}$ with Cauchy law P_ϵ, C_ϵ free with S and D. Then, we consider the free equation

$$dY_t^{\delta,\epsilon,\Delta} = dS_t + \partial_x h^{\delta,\epsilon,\Delta}(Y_t^{\delta,\epsilon,\Delta} + C_\epsilon, t)dt$$

with bounded initial data $Y_0^{\delta,\epsilon,\Delta} = D$. Observe that, by Lemma 2.12, the field $\partial_x h^{\delta,\epsilon,\Delta}$ is of the form

$$\partial_x h^{\delta,\epsilon,\Delta}(x,s) = a_k^\Delta + \int e^{i\xi x} l_k^{\delta,\epsilon,\Delta}(\xi,s)d\xi, \qquad s\in[t_k,t_{k+1}[$$

with, for any $s\in[t_k,t_{k+1}[$ and any $k\in\{0,\cdots,n\}$,

$$|l_k^{\delta,\epsilon,\Delta}(\xi,s)| \le Ce^{-\frac{\epsilon}{2}|\xi|}$$

for a finite constant C. In particular, thanks to Duhamel's formula, for $t\in[t_k,t_{k+1}]$, for any $X,Y\in\mathcal{A}$,

$$\begin{aligned}||\partial_x h^{\delta,\epsilon,\Delta}(X+C_\epsilon,t) - \partial_x h^{\delta,\epsilon,\Delta}(Y+C_\epsilon,t)|| &\le \int l_k^{\delta,\epsilon,\Delta}(\xi,u)||\xi\int_0^1 e^{i\alpha\xi(X+C_\epsilon)}(X-Y)e^{i(1-\alpha)\xi(Y+C_\epsilon)}d\alpha||d\xi \\ &\le K||X-Y||,\end{aligned}$$

with a finite constant K. As a consequence, we can check that $Y^{\delta,\epsilon,\Delta}$ exists and is uniquely defined. Indeed, considering the sequence $Y_.^0 = D$, $Y_0^n = D$,

$$dY_t^{n+1} = dS_t + \partial_x h^{\delta,\epsilon,\Delta}(Y_t^n + C_\epsilon, t)dt,$$

we see first that for all $n\in\mathbb{N}$,

$$\sup_{t\in[0,1]}||Y_t^n|| \le ||D|| + \sup_{t\in[0,1]}||S_t|| + \sup_{t\in[0,1]}||\partial_x h^{\delta,\epsilon,\Delta}(\cdot,t)||_\infty < \infty$$

ensuring that for all $n\in\mathbb{N}$, Y^n is uniformly bounded, and then that

$$||Y_t^{n+1} - Y_t^n|| \le K\int_0^t ||Y_s^{n-1} - Y_s^n||ds$$

ensuring that Y^n is a Cauchy sequence in $\mathcal{A}$, which is strongly closed. Its limit $Y\in\mathcal{A}$ satisfies the desired equation. Observe also that

$$\sup_{t\in[0,1]}||Y_t^{\delta,\epsilon,\Delta}|| \le ||D|| + 2 + \sup_{t\in[0,1]}||\partial_x h^{\delta,\epsilon,\Delta}(\cdot,t)||_\infty < \infty \tag{4.4}$$

Further, uniqueness can be obtained by similar Picard arguments. We refer to [4] for similar results.

We claim that $X_.^{\delta,\epsilon,\Delta} = Y_.^{\delta,\epsilon,\Delta} + C_\epsilon$ has time marginals $X_t^{\delta,\epsilon,\Delta}$ with law $\nu_t^{\delta,\epsilon,\Delta}$, $t\in[0,1]$. In fact, it is not hard to see that it satisfies the same free Fokker-Planck equation (4.3) with same initial data. Hence, since it was proved below (2.48) that the solution to (4.3) is unique and that $\nu^{\delta,\epsilon,\Delta}$ satisfies the very same equation, we conclude that $X_t^{\delta,\epsilon,\Delta}$ has distribution $\nu_t^{\delta,\epsilon,\Delta}$.

We proceed using the free Itô calculus as developed in [3], whose notations we borrow. For any $f(x) = \int e^{i\xi x}d\mu(\xi)$ with $\int|\xi|^2 d|\mu|(\xi)<\infty$, we can follow [3, Proposition 4.3.4], to see that

$$f(Y_t^{\delta,\epsilon,\Delta}) = f(Y_0^{\delta,\epsilon,\Delta}) + \int_0^t D_0 f(Y_s^{\delta,\epsilon,\Delta})\sharp dS_s + \int_0^t D_0 f(Y_s^{\delta,\epsilon,\Delta})\sharp\partial_x h^{\delta,\epsilon,\Delta}(X_s^{\delta,\epsilon,\Delta},s)ds + \int_0^t (\tau\otimes I)Lf(Y_s^{\delta,\epsilon,\Delta})ds$$

with, for any $A,B,C\in\mathcal{A}$, $A\otimes B\sharp C=ACB$, and, if we identify $\mathcal{C}_b^0(\mathbb{R}^2)$ with $\mathcal{C}_b^0(\mathbb{R})\otimes\mathcal{C}_b^0(\mathbb{R})$,

$$D_0 f(x,y)=\frac{f(x)-f(y)}{x-y}\text{ and }Lf(x,y)=\partial_x\circ D_0 f(x,y).$$

Taking the trace on both sides of this equality, we see that the law $\tilde{\nu}_t^{\delta,\epsilon,\Delta}$ of $Y_t^{\delta,\epsilon,\Delta}$ satisfies the free Fokker-Planck equation

$$S^{0,t}(\tilde{\nu}_.^{\delta,\epsilon,\Delta},f)=\int_0^t\int\partial_x f(x,s)K_s^{\delta,\epsilon,\Delta}(x)d\tilde{\nu}_s^{\delta,\epsilon,\Delta}(x)ds$$

with $K_s^{\delta,\epsilon,\Delta}=\tau(\partial_x h^{\delta,\epsilon,\Delta}(X_s^{\delta,\epsilon,\Delta},s)|Y_s^{\delta,\epsilon,\Delta})$ the $L^2(\tau)$ projection of $\partial_x h^{\delta,\epsilon,\Delta}(X_s^{\delta,\epsilon,\Delta},s)$ on the algebra generated by $Y_s^{\delta,\epsilon,\Delta}$. Consequently, we find that

$$\begin{aligned}
S^{0,1}(\nu^{\delta,\epsilon,\Delta}) &= \frac{1}{2}\int_0^1 \nu_s^{\delta,\epsilon,\Delta}((\partial_x h^{\delta,\epsilon,\Delta}(.,s))^2)ds\\
&= \frac{1}{2}\int_0^1 \tau((\partial_x h^{\delta,\epsilon,\Delta}(X_s^{\delta,\epsilon,\Delta},s))^2)ds\\
&\geq \frac{1}{2}\int_0^1 \tau((\tau(\partial_x h^{\delta,\epsilon,\Delta}(X_s^{\delta,\epsilon,\Delta},s)|Y_s^{\delta,\epsilon,\Delta})^2)ds\\
&= \frac{1}{2}\int_0^1 \tilde{\nu}_s^{\delta,\epsilon,\Delta}((K_s^{\delta,\epsilon,\Delta})^2)ds\\
&\geq S^{0,1}(\tilde{\nu}^{\delta,\epsilon,\Delta}).
\end{aligned}\tag{4.5}$$

Thus, we have constructed a law $\tilde{\nu}^{\delta,\epsilon,\Delta}$ such that

- $\tilde{\nu}_0^{\delta,\epsilon,\Delta}=\mu_D$.
- $\tilde{\nu}_.^{\delta,\epsilon,\Delta}$ is uniformly compactly supported (see (4.4)), and therefore belongs to $\mathcal{A}$.
- $S^{0,1}(\nu^{\delta,\epsilon,\Delta})\geq S^{0,1}(\tilde{\nu}^{\delta,\epsilon,\Delta})$.
- For any $\vartheta>0$, we can choose $\epsilon>0$ small enough so that $D(\nu^{\delta,\epsilon,\Delta},\tilde{\nu}^{\delta,\epsilon,\Delta})\leq\vartheta/2$. Indeed, this is a direct consequence from the previous observation that $X^{\delta,\epsilon,\Delta}=Y^{\delta,\epsilon,\Delta}+C_\epsilon$ has distribution $\nu^{\delta,\epsilon,\Delta}$ (see Lemma 4.2 at the end of the section).

From (4.2) we thus deduce that for any $\epsilon>0$,

$$\begin{aligned}
\inf\{S_{\mu_D}(\nu_.);\nu_1=\mu\} &\geq S^{0,1}(\nu^{\delta,\epsilon,\Delta})-\delta-\Delta\\
&\geq S^{0,1}(\tilde{\nu}^{\delta,\epsilon,\Delta})-\delta-\Delta\\
&\geq \inf\{S_{\mu_D}(\nu_.);\ \nu\in\mathcal{A},\ d(\nu_1,\mu)<\vartheta\}-\delta-\Delta
\end{aligned}\tag{4.6}$$

where we have chosen above $\epsilon>0$ small enough so that also $d(\nu_1^{\delta,\epsilon,\Delta},\mu)=d(P_\epsilon*\mu,\mu)<(\vartheta/2)$.

We can finally let δ,Δ and ϑ (and therefore ϵ) going to zero to conclude that

$$\inf\{S_{\mu_D}(\nu_.);\nu_1=\mu\}\geq\limsup_{\vartheta\to0}\inf\{S_{\mu_D}(\nu_.);\ \nu\in\mathcal{A},\ d(\nu_1,\mu)<\vartheta\}.$$

The proof is complete, except for the:

Lemma 4.2 *Let $(\mathcal{A},\tau)$ be a non-commutative probability space and $C_\epsilon \in \widetilde{\mathcal{A}}_{sa}$ be an operator with Cauchy distribution P_ϵ. Then, there exists functions $g_\gamma : \mathbb{R}^+ \mapsto \mathbb{R}^+$, such that $g_\gamma(\epsilon) \to_{\epsilon\to 0} 0$ and, for all Lipschitz function f, $||f||_{\mathcal{L}} \le 1$,*

$$|\tau(f(Y+C_\epsilon)) - \tau(f(Y))| \le g_{||Y||}(\epsilon).$$

Further, if $\gamma > \gamma'$ then $g_\gamma \ge g_{\gamma'}$.

Proof. Note first that by density it is enough to prove the result for functions of $\mathcal{C}_b^1(\mathbb{R})$. Let $\eta \in (0,1)$ and $f \in \mathcal{C}_b^1(\mathbb{R})$ be fixed. Throughout, we let K_η denote a constant, which depends on η only, and whose value may change from line to line. We consider, for $\kappa > 0$

$$f_\kappa(x) := \frac{f(x)}{(1+\kappa x^2)^{\frac{\eta}{2}}}.$$

Observe that

$$|f_\kappa(x) - f(x)| \le K_\eta ||f||_\infty \left(\kappa^{1/2}|x|\right)^\eta .$$

Thus, for all $Y \in (\mathcal{A},\tau)$, $||Y|| \le M$,

$$|\tau\left(f(Y+C_\epsilon)\right) - \tau\left(f_\kappa(Y+C_\epsilon)\right)| \le K_\eta ||f||_\infty \kappa^{\eta/2} \tau\left((M+|C_\epsilon|)^\eta\right) ,$$

so that for any $M \in \mathbb{R}^+$,

$$\sup_{Y\in(\mathcal{A},\tau),||Y||\le M}\ \sup_{\epsilon\in[0,1]} |\tau\left(f(Y+C_\epsilon)\right) - \tau\left(f_\kappa(Y+C_\epsilon)\right)| \le K_{\eta,M}||f||_\infty \kappa^{\eta/2} \tag{4.7}$$

where $K_{\eta,M}$ is a constant depending on η, M only, monotone nondecreasing in M, and whose value may again change from line to line.

Further, since $f \in \mathcal{C}_b^1(\mathbb{R})$, $f_\kappa \in \mathcal{C}_b^1(\mathbb{R})$ and we have

$$\tau\left(f_\kappa(Y+C_\epsilon) - f_\kappa(Y)\right) = \int_0^1 \tau\left(f_\kappa'(Y+\alpha C_\epsilon)C_\epsilon\right) d\alpha. \tag{4.8}$$

Now,

$$f_\kappa'(x) = \frac{f'(x)}{(1+\kappa x^2)^{\eta/2}} - \frac{\eta\kappa x f(x)}{(1+\kappa x^2)^{\frac{\eta+2}{2}}}$$

ensuring

$$|f_\kappa'(x)| \le \left(||f'||_\infty + \frac{\eta\sqrt{\kappa}}{2}||f||_\infty\right)(1+\kappa x^2)^{-\eta/2}.$$

Consequently, for any $\alpha > 0$, since for any self adjoint operators A, B, $|\tau(AB)| \le \tau(|A||B|)$,

$$|\tau\left(f_\kappa'(Y+\alpha C_\epsilon)C_\epsilon\right)| \le \left(||f'||_\infty + \frac{\eta\sqrt{\kappa}}{2}||f||_\infty\right)\tau\left((1+\kappa(Y+\alpha C_\epsilon)^2)^{-\eta/2}|C_\epsilon|\right). \tag{4.9}$$

Now, using that for $||Y|| \le M$,

$$(Y+\alpha C_\epsilon)^2 \ge ((\alpha|C_\epsilon| - M)^+)^2,$$

we arrive at

$$\begin{aligned}
\tau\left((1+\kappa(Y+\alpha C_\epsilon)^2)^{-\eta/2}|C_\epsilon|\right) &\le \tau\left((1+\kappa((\alpha|C_\epsilon|-M)^+)^2)^{-\eta/2}|C_\epsilon|\right)\\
&= \frac{\epsilon}{\pi}\int \frac{1}{1+x^2}\frac{|x|}{(1+\kappa(\epsilon\alpha|x|-M)^+)^2)^{\eta/2}}dx\\
&\le \frac{\epsilon}{\pi}\int_{|x|\le\frac{2M}{\epsilon\alpha}}\frac{|x|}{1+x^2}dx+\frac{\epsilon}{\pi\kappa^{\eta/2}}\int_{|x|\ge\frac{2M}{\epsilon\alpha}}\frac{2^{\eta/2}}{|x|(\epsilon\alpha|x|)^{\eta/2}}dx\\
&= \frac{2\epsilon}{\pi}\log(1+(\frac{2M}{\epsilon\alpha})^2)+\frac{2\epsilon}{\pi\eta(\sqrt{\kappa}M)^\eta}\,.
\end{aligned}$$

Hence, using this control in (4.8) and (4.9), we find that

$$|\tau\left(f_\kappa(Y+C_\epsilon)\right)-\tau\left(f_\kappa(Y)\right)|\le K_{\eta,||Y||}\left(||f'||_\infty+\frac{\eta\sqrt{\kappa}}{2}||f||_\infty\right)(1+\frac{1}{\kappa^{\eta/2}}-\log\epsilon)\epsilon \tag{4.10}$$

With (4.7), we conclude that

$$|\tau\left(f(Y+C_\epsilon)\right)-\tau\left(f(Y)\right)|\le K_{\eta,||Y||}||f||_\infty\kappa^{\eta/2}+K_{\eta,||Y||}\left(||f'||_\infty+\frac{\eta\sqrt{\kappa}}{2}||f||_\infty\right)(1+\frac{1}{\kappa^{\eta/2}}-\log\epsilon)\epsilon \tag{4.11}$$

which goes to zero as ϵ goes to zero if we take $\kappa=\kappa(\epsilon)=\epsilon$. □

5 Around spherical integrals: proof of Theorem 1.1

We will assume throughout this section, the existence of a compact subset $\mathcal{K}$ of $\mathbb{R}$ such that $\operatorname{supp}\hat{\mu}^N_{D_N}\subset\mathcal{K}$ for all $N\in\mathbb{N}$. Further, we shall suppose that $\hat{\mu}^N_{E_N}(x^2)$ is uniformly bounded, and therefore $\mu_E(x^2)$ is finite by lower semi-continuity of $\mu\to\mu(x^2)$.

Our starting point is the observation that if $(\lambda_i^E)_{1\le i\le N}$ denotes the eigenvalues of E_N, then for any $\delta>0$,

$$\begin{aligned}
&\int_{d(\hat{\mu}^N_{E_N},\mu_E)<\delta} e^{-\frac{N^2}{2}\hat{\mu}^N_{E_N}(x^2)}\prod_{i<j}|\lambda_i^E-\lambda_j^E|^\beta I_N^{(\beta)}(D_N,E_N)\prod_i d\lambda_i^E\\
=\;& e^{\frac{N^2}{2}\hat{\mu}^N_{D_N}(x^2)}\int_{d(\hat{\mu}^N_{E_N},\mu_E)<\delta}\int e^{-\frac{N}{2}\mathrm{tr}(UE_NU^*-D_N)^2}\prod_{i<j}|\lambda_i^E-\lambda_j^E|^\beta\prod_i d\lambda_i^E dm_N^\beta(U)\\
=\;& Z_N^\beta e^{\frac{N^2}{2}\hat{\mu}^N_{D_N}(x^2)}\mathbb{P}^\beta_{D_N}\left(d(\hat{\mu}^N_{X_N(1)},\mu_E)<\delta\right)\,.
\end{aligned}$$

Recall that by Corollary 1.6, we have that when μ_D is compactly supported,

$$\begin{aligned}
&\lim_{\delta\downarrow 0}\liminf_{N\uparrow\infty}\frac{1}{N^2}\log\mathbb{P}^\beta_{D_N}\left(d(\hat{\mu}^N_{X_N(1)},\mu_E)<\delta\right)\\
=\;&\lim_{\delta\downarrow 0}\limsup_{N\uparrow\infty}\frac{1}{N^2}\log\mathbb{P}^\beta_{D_N}\left(d(\hat{\mu}^N_{X_N(1)},\mu_E)<\delta\right)=-J_\beta(\mu_D,\mu_E)\,,
\end{aligned} \tag{5.1}$$

with a good rate function $J_\beta(\mu_D,\cdot)$. Further, if we let

$$I_\beta(\mu)=\frac{1}{2}\int x^2d\mu(x)-\frac{\beta}{2}\int\log|x-y|d\mu(x)d\mu(y),$$

it was proved in [1] that

$$\lim_{N\uparrow\infty}\frac{1}{N^2}\log Z_N = \inf_{\mu\in M_1(I\!R)} I_\beta(\mu).$$

Moreover, since we assumed that D_N has uniformly bounded eigenvalues, there is a constant D such that

$$N^{-2}\log I_N^{(\beta)}(D_N,E_N) \le D\hat\mu^N_{E_N}(|x|) \le D\sqrt{\hat\mu^N_{E_N}(x^2)}\,.$$

Thus,

$$\int_{\hat\mu^N_{E_N}(x^2)\ge L} e^{-\frac{N^2}{2}\hat\mu^N_{E_N}(x^2)}\prod_{i<j}|\lambda_i^E-\lambda_j^E|^\beta I_N^{(\beta)}(D_N,E_N)\prod_i d\lambda_i^E \le \int_{\hat\mu^N_{E_N}(x^2)\ge L} e^{-\frac{N^2}{4}\hat\mu^N_{E_N}(x^2)}\prod_{i<j}|\lambda_i^E-\lambda_j^E|^\beta\prod_i d\lambda_i^E\,,$$

for $L>4D^2$. It follows by the exponential tightness proof in [1] that

$$\lim_{L\to\infty}\limsup_{N\to\infty}\frac{1}{N^2}\log\int_{\hat\mu^N_{E_N}(x^2)\ge L} e^{-\frac{N^2}{4}\hat\mu^N_{E_N}(x^2)}\prod_{i<j}|\lambda_i^E-\lambda_j^E|^\beta\prod_i d\lambda_i^E \;=-\infty\,,$$

and therefore, combining the last two displays,

$$\begin{aligned}
&\lim_{L\uparrow\infty}\lim_{\delta\downarrow 0}\liminf_{N\uparrow\infty}\frac{1}{N^2}\log\int_{d(\hat\mu^N_{E_N},\mu_E)<\delta,\hat\mu^N_{E_N}(x^2)\le L} e^{-\frac{N^2}{2}\hat\mu^N_{E_N}(x^2)}\prod_{i<j}|\lambda_i^E-\lambda_j^E|^\beta I_N^{(\beta)}(D_N,E_N)\prod_i d\lambda_i^E\\
=\;&\lim_{\delta\downarrow 0}\liminf_{N\uparrow\infty}\frac{1}{N^2}\log\int_{d(\hat\mu^N_{E_N},\mu_E)<\delta} e^{-\frac{N^2}{2}\hat\mu^N_{E_N}(x^2)}\prod_{i<j}|\lambda_i^E-\lambda_j^E|^\beta I_N^{(\beta)}(D_N,E_N)\prod_i d\lambda_i^E
\end{aligned}$$

and similarly for the limsup term. We next claim that

$$\begin{aligned}
&-I_\beta(\mu_E)+\lim_{L\uparrow\infty}\lim_{\delta\downarrow 0}\liminf_{N\uparrow\infty}\inf_{d(\hat\mu^N_{E_N},\mu_E)<\delta,\hat\mu^N_{E_N}(x^2)\le L}\frac{1}{N^2}\log I_N^{(\beta)}(D_N,E_N)\\
=\;&\lim_{\delta\downarrow 0}\liminf_{N\uparrow\infty}\frac{1}{N^2}\log\int_{d(\hat\mu^N_{E_N},\mu_E)<\delta} e^{-\frac{N^2}{2}\hat\mu^N_{E_N}(x^2)}\prod_{i<j}|\lambda_i^E-\lambda_j^E|^\beta I_N^{(\beta)}(D_N,E_N)\prod_i d\lambda_i^E\\
=\;&\lim_{\delta\downarrow 0}\limsup_{N\uparrow\infty}\frac{1}{N^2}\log\int_{d(\hat\mu^N_{E_N},\mu_E)<\delta} e^{-\frac{N^2}{2}\hat\mu^N_{E_N}(x^2)}\prod_{i<j}|\lambda_i^E-\lambda_j^E|^\beta I_N^{(\beta)}(D_N,E_N)\prod_i d\lambda_i^E\\
=\;&-I_\beta(\mu_E)+\lim_{L\uparrow\infty}\lim_{\delta\downarrow 0}\limsup_{N\uparrow\infty}\sup_{d(\hat\mu^N_{E_N},\mu_E)<\delta,\hat\mu^N_{E_N}(x^2)\le L}\frac{1}{N^2}\log I_N^{(\beta)}(D_N,E_N).
\end{aligned}\tag{5.2}$$

Indeed, the arguments in [1] yield (5.2) with inequality ($\le$) signs replacing the first and third equality signs. On the other hand, we have the following lemma, whose proof we postpone to the end of this section:

Lemma 5.1 *Assume that there exists $d_{max}\in I\!R^+$ such that for any integer number N, $\hat\mu^N_{D_N}(1_{\{|x|\ge d_{max}\}})=0$. Then, there exists a function $g:[0,1]\times I\!R^+\mapsto I\!R^+$, depending on μ_E only, such that $g(\delta,L)\to_{\delta\to 0}0$ for any $L\in I\!R^+$, and, for $\hat E_N,\bar E_N$ such that*

$$d(\hat\mu^N_{\hat E_N},\mu_E)+d(\hat\mu^N_{\bar E_N},\mu_E)\le\delta/2\,,\tag{5.3}$$

and

$$\int x^2 d\hat\mu^N_{\bar E_N}(x)+\int x^2 d\hat\mu^N_{\hat E_N}(x)\le L,\tag{5.4}$$

it holds that

$$\limsup_{N\to\infty}\left|\frac{1}{N^2}\log\frac{I_N^{(\beta)}(D_N,\hat{E}_N)}{I_N^{(\beta)}(D_N,\bar{E}_N)}\right|\le g(\delta,L)\,.$$

By Lemma 5.1, we conclude that for all $L\in\mathbb{R}^+$

$$\sup_{d(\hat{\mu}^N_{E_N},\mu_E)<\delta,\hat{\mu}^N_{E_N}(x^2)\le L}\frac{1}{N^2}\log I_N^{(\beta)}(D_N,E_N)\le\inf_{d(\hat{\mu}^N_{E_N},\mu_E)<\delta,\hat{\mu}^N_{E_N}(x^2)\le L}\frac{1}{N^2}\log I_N^{(\beta)}(D_N,E_N)+g_N(\delta,L),$$

where $g_N(\delta,L)\to_{\delta\to 0}0$ uniformly in N. From here, equality in (5.2) is clear and, if we let

$$I^{(\beta)}(\mu_D,\mu_E)=\limsup_{N\uparrow\infty}\frac{1}{N^2}\log I_N^{(\beta)}(D_N,E_N)=\liminf_{N\uparrow\infty}\frac{1}{N^2}\log I_N^{(\beta)}(D_N,E_N)\,,$$

(5.1) implies that

$$I^{(\beta)}(\mu_D,\mu_E)=-J_\beta(\mu_D,\mu_E)+I_\beta(\mu_E)-\inf_{\mu\in M_1(\mathbb{R})}I_\beta(\mu)+\frac{1}{2}\int x^2d\mu_D(x)\,,\tag{5.5}$$

completing the proof of Theorem 1.1. □

Proof of Lemma 5.1: Take $\hat{\mu}^N_{\bar{E}_N}$ and $\hat{\mu}^N_{\hat{E}_N}$ satisfying (5.3) and (5.4) with $L\in\mathbb{R}^+$ and $\delta>0$. Fix $\delta'>0$ and then $M=M(\delta')\le L/\delta'\wedge\frac{(\delta')^2}{4\delta}$ such that

$$\int|x|\mathbf{1}_{|x|>M}d(\hat{\mu}^N_{\bar{E}_N}+\hat{\mu}^N_{\hat{E}_N})\le\delta'\,.\tag{5.6}$$

Observe that (δ',M) exists for δ small enough, only case of interest to us. Next, fix a partition of the interval $[-M,M]$ to intervals $\{A_j\}_{j\in\mathcal{J}}$ (with $|\mathcal{J}|\le 2M/\delta'$) such that $|A_j|\in[\delta',2\delta']$ and the endpoints of A_j are continuity points of μ_E. Denote

$$\hat{I}_j=\{i:\hat{E}_N(ii)\in A_j\},\quad \bar{I}_j=\{i:\bar{E}_N(ii)\in A_j\}\,.$$

By (5.3),

$$|\mu_E(A_j)-|\hat{I}_j|/N|+|\mu_E(A_j)-|\bar{I}_j|/N|\le\delta\,.$$

We construct a permutation $\sigma_N:\{1,\dots,N\}\to\{1,\dots,N\}$ as follows: first, if $|\bar{I}_j|\le|\hat{I}_j|$ then $\widetilde{I}_j:=\bar{I}_j$, whether if $|\bar{I}_j|>|\hat{I}_j|$ then $|\widetilde{I}_j|=|\hat{I}_j|$ while $\widetilde{I}_j\subset\bar{I}_j$. Then, choose and fix a permutation σ_N such that $\sigma_N(\widetilde{I}_j)\subset\hat{I}_j$, and denote $\mathcal{J}_0=\cup_j\sigma_N(\widetilde{I}_j)$. Note that $|\mathcal{J}_0|=\sum|\sigma_N(\widetilde{I}_j)|=\sum|\widetilde{I}_j|\wedge|\bar{I}_j|=\sum_j|\bar{I}_j|-\sum_j(|\bar{I}_j|-|\widetilde{I}_j|)\wedge 0$ so that, since $|\widetilde{I}_j|-|\bar{I}_j|\le\delta N$ and $\sharp\{j:|\bar{I}_j|\ne 0\}\le(2M/\delta')$, $|\mathcal{J}_0|\ge\sum_j|\bar{I}_j|-2\delta MN/\delta'\ge\sum_j|\bar{I}_j|-\delta' N/M$, and thus $|\mathcal{J}_0^c|\le 2N\delta'/M$. Next, note the invariance of $I_N^{(\beta)}(D_N,E_N)$ to permutations of the matrix elements of D_N. That is,

$$\begin{aligned}I_N^{(\beta)}(D_N,\bar{E}_N)&=\int\exp\{N\mathrm{tr}(UD_NU^*\bar{E}_N)\}dm_N^\beta(U)=\int\exp\{N\sum_{i,k}u_{ik}^2D_N(kk)\bar{E}_N(ii)\}dm_N^\beta(U)\\&=\int\exp\{N\sum_{i,k}u_{ik}^2D_N(kk)\bar{E}_N(\sigma_N(i)\sigma_N(i))\}dm_N^\beta(U)\,.\end{aligned}$$

But, with $d_{\max} = \max_k |D_N(kk)|$ bounded uniformly in N,

$$\begin{aligned}
& N^{-1}\sum_{i,k} u_{ik}^2 D_N(kk)\bar{E}_N(\sigma_N(i)\sigma_N(i)) \\
= \;& N^{-1}\sum_{i\in\mathcal{J}_0}\sum_k u_{ik}^2 D_N(kk)\bar{E}_N(\sigma_N(i)\sigma_N(i)) + N^{-1}\sum_{i\notin\mathcal{J}_0}\sum_k u_{ik}^2 D_N(kk)\bar{E}_N(\sigma_N(i)\sigma_N(i)) \\
\le \;& N^{-1}\sum_{i\in\mathcal{J}_0}\sum_k u_{ik}^2 D_N(kk)\hat{E}_N(ii) + N^{-1}\sum_{i\in\mathcal{J}_0}\sum_k u_{ik}^2 D_N(kk)(\bar{E}_N(\sigma_N(i)\sigma_N(i)) - \hat{E}_N(ii)) \\
& +N^{-1}\sum_{i\notin\mathcal{J}_0}\sum_k u_{ik}^2 D_N(kk)\bar{E}_N(\sigma_N(i)\sigma_N(i))(\mathbf{1}_{|\bar{E}_N(\sigma_N(i)\sigma_N(i))|\le M} + \mathbf{1}_{|\bar{E}_N(\sigma_N(i)\sigma_N(i))|>M}) \\
:= \;& N^{-1}\sum_{i\in\mathcal{J}_0}\sum_k u_{ik}^2 D_N(kk)\hat{E}_N(ii) + \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3\,.
\end{aligned}$$

Since $|\bar{E}_N(\sigma_N(i)\sigma_N(i)) - \hat{E}_N(ii)| \le 2\delta'$ and $\sum_i u_{ik}^2 = 1$, $\mathcal{E}_1 \le 2d_{\max}\delta'$. Moreover,

$$\mathcal{E}_2 \le d_{\max} M|\{i : i \notin \mathcal{J}_0\}|/N \le 2d_{\max}\delta'$$

and, by (5.6), we have $\mathcal{E}_3 \le d_{\max}\delta'$. Hence, we have proved

$$N^{-1}\sum_{i,k} u_{ik}^2 D_N(kk)\bar{E}_N(\sigma_N(i)\sigma_N(i)) \le N^{-1}\sum_{i\in\mathcal{J}_0}\sum_k u_{ik}^2 D_N(kk)\hat{E}_N(ii) + 5d_{\max}\delta'.$$

Repeating the argument in order to replace $\sum_{i\in\mathcal{J}_0}\sum_k u_{ik}^2 D_N(kk)\hat{E}_N(ii)$ by $\sum_{i,k} u_{ik}^2 D_N(kk)\hat{E}_N(ii)$, we conclude that

$$N^{-1}\sum_{i,k} u_{ik}^2 D_N(kk)\bar{E}_N(\sigma_N(i)\sigma_N(i)) \le N^{-1}\sum_{i,k} u_{ik}^2 D_N(kk)\hat{E}_N(ii) + 10d_{\max}\delta'\,.$$

Since δ' is arbitrary, and the reverse inequality is obtained by interchanging the role of $\bar{E}_N$ and $\hat{E}_N$, we are done. $\square$

6 Concluding remarks

We conclude with some comments on the relation of our variational problem, in the case $\beta = 2$, to the one obtained by [16]. Indeed, suppose that μ_D, μ possess densities with respect to Lebesgue measure that we denote by ρ_0, ρ_1, and further *assume* that the minimizer to the variational problem in (1.6) is achieved by a path $\nu_\cdot \in \mathcal{C}([0,1]; M_1(\mathbb{R}))$ with density $\rho_\cdot$, which is smooth enough to satisfy the equation

$$S^{0,t}(\nu, f) = \int_0^t \int \partial_x h(x,u)\partial_x f(x,u) d\nu_u(x) du\,, \forall f \in \mathcal{C}_b^{2,1}(\mathbb{R}\times[0,1])\,, \tag{6.1}$$

for some nice field $h(x,t)$ (see [6, Section 2.4] for a discussion of assumptions on the field h that ensure the uniqueness of solutions to (6.1)). Denoting $k(x,t) = \partial_x h(x,t)$, it is easy to check that (6.1) is equivalent to the statement

$$\partial_t \rho_t(x) = -\partial_x(\rho(x,t)(H\rho)(x,t)) - \partial_x(k(x,t)\rho(x,t))\,, \tag{6.2}$$

where $H\rho$ denotes the Stieljes transform of ρ:

$$(H\rho)(x,t) = \int \frac{1}{x-y}\rho(y,t)dy\,.$$

In this case, Corollary 1.6 tells us that

$$I^{(2)}(\mu_D,\mu) = \frac{1}{2}\inf\{\int_0^1 \int k(x,t)^2\rho(x,t)dxdt\ :\ \rho(\cdot,\cdot)\,\text{satisfies (6.2) for some nice field}\ h\}\,, \tag{6.3}$$

where the infimum is taken over fields h. The Euler-Lagrange equation for the variational problem (6.3) reads

$$\partial_t \int_x^\infty k(x,t) + k(x,t)(H\rho)(x,t) + (H(k\rho))(x,t) + \frac{k(x,t)^2}{2} = 0\,.$$

Defining

$$\Pi(x,t) = \int \log|x-y|\rho(y,t)dy + \int_x^\infty k(y,t)dy\,,$$

one finds that

$$\partial_t\Pi(x,t) = -k(x,t)(H\rho)(x,t) - \frac{k(x,t)}{2} - (H(\rho(H\rho)))(x,t)\,.$$

Using the relation (see [22, Th. IV])

$$\frac{\pi^2}{2}\rho(x,t)^2 = \frac{1}{2}(H\rho)^2(x,t) - H(\rho(H\rho))(x,t)\,,$$

and the relation

$$\partial_x\Pi(x,t) = (H\rho)(x,t) + k(x,t)\,, \tag{6.4}$$

we conclude that

$$\partial_t\Pi(x,t) = -\frac{1}{2}\left(\partial_x\Pi(x,t)\right)^2 + \frac{\pi^2}{2}\rho(x,t)^2\,.$$

Together with (6.2), which in view of (6.4) can be rewritten as

$$\partial_t\rho(x,t) = -\partial_x\left(\rho(x,t)\partial_x\Pi(x,t)\right)\,,$$

we thus recover the Hamilton-Jacobi equations of [16, Pg. 810].

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