A Metric Entropy Bound is Not Sufficient for Learnability

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Submitted: Oct. 16, 1992
Revised: Sept. 17, 1993

Abstract

We prove by means of a counterexample that it is not sufficient, for PAC learning under a class of distributions, to have a uniform bound on the metric entropy of the class of concepts to be learned. This settles a conjecture of Benedek and Itai.

Key Words: learning, estimation, PAC, metric entropy, class of distributions

1 Introduction

Let $(\mathcal{X}, \mathcal{B})$ be a measurable space. Let $\mathcal{P}$ be a class of probability measures on $(\mathcal{X}, \mathcal{B})$. Let $\mathcal{C}$ (the "concept class" in the language of learning theory, as introduced in [6]) be a subset of $\mathcal{B}$. Suppose one is given a sequence of i.i.d., $\mathcal{X}$ valued random variables $X_1, \ldots, X_n$ distributed according to P^n, where $P \in \mathcal{P}$. In addition, for some unknown $c \in \mathcal{C}$, one is given data $(X_1, I_c(X_1)), \ldots, (X_n, I_c(X_n))$ which we henceforth denote by $\mathcal{D}_n(c)$. The problem of learning consists roughly of the question "given $\mathcal{C}, \mathcal{P}$, how large should n be for approximating c with high accuracy and low probability of error based on the data $\mathcal{D}_n(c)$?" In mathematical terms,

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assume that $(\mathcal{X}, \mathcal{B})$ is a Borel space, and define on $\mathcal{B}$ the pseudo metric $d_P(c_1, c_2) = P(c_1 \triangle c_2)$. Let $\mathcal{T}$ be the algebra of all four subsets of $\{0,1\}$. A learning rule is a map $T^n : (\mathcal{X} \times \{0,1\})^n \to \mathcal{C}$ such that, for any $c \in \mathcal{C}$, any $P \in \mathcal{P}$, and any $\epsilon > 0$,

$$\{(X_1, \ldots, X_n, i_1, ..., i_n) : \; d_P(c, T^n((X_1, i_1), \ldots, (X_n, i_n))) > \epsilon\} \; \in \; \mathcal{B}^n \; \otimes \mathcal{T}^n. \tag{1}$$

It follows that for any $c, d \in \mathcal{C}$,

$$\{(X_1, \ldots, X_n) : \; d_P(d, T^n(\mathcal{D}_n(c))) > \epsilon\} \in \mathcal{B}^n. \tag{2}$$

We say that the concept class $\mathcal{C}$ is PAC learnable under the class of probability measures $\mathcal{P}$ (in short: $\mathcal{C}$ is PAC learnable under $\mathcal{P}$) if, for every $\epsilon > 0$, $\delta > 0$, there exist an integer $n = n(\mathcal{P}, \mathcal{C}, \epsilon, \delta)$ and a learning rule T^n such that, for any $P \in \mathcal{P}$ and $c \in \mathcal{C}$,

$$P^n(\{(X_1, \ldots, X_n) : \; d_P(c, T^n(\mathcal{D}_n(c))) > \epsilon\}) < \delta \, . \tag{3}$$

The notion of learnability in the form (3) has recently received much attention (e.g., see [1, 4, 6]), and in the learning literature is referred to as Probably Approximately Correct (PAC) learning, for reasons obvious from its definition. Intuitively, in PAC learning one attempts to achieve a good prediction on future samples, after seeing some finite number of samples, uniformly in $P \in \mathcal{P}$ and $c \in \mathcal{C}$.

Sufficient and necessary conditions for PAC learnability are by now well known for some cases. Let $B(c, \epsilon) = \{\tilde{c} \in \mathcal{B} : \; d_P(c, \tilde{c}) < \epsilon\}$, and define the *$\epsilon$-covering number* of $\mathcal{C}$ with respect to P by

$$N(\epsilon, \mathcal{C}, P) = \inf\{N : \; \exists c_1, \ldots, c_N \in \mathcal{B} \text{ such that } \mathcal{C} \subset \cup_{i=1}^{N} B(c_i, \epsilon)\} \, .$$

The balls $B(c_i, \epsilon)$ above are said to form an *ϵ-cover* of $\mathcal{C}$, and $\log N(\epsilon, \mathcal{C}, P)$ is often referred to as the *metric entropy* of $\mathcal{C}$ with respect to P. A necessary and sufficient condition for PAC learnability of $\mathcal{C}$ in the special case where $\mathcal{P}$ is a singleton, namely $\mathcal{P} \equiv \{P\}$, is that $N(\epsilon, \mathcal{C}, P) < \infty$ for all $\epsilon > 0$ (see [2] and, in greater generality, [7], pp. 149–151). Moreover, if

$\mathcal{P} = M_1(\mathcal{X})$, the space of Borel probability measures on $\mathcal{X}$, then (under suitable measurability conditions) a well known necessary and sufficient condition for PAC learnability of $\mathcal{C}$ under $\mathcal{P}$ is that the VC dimension of $\mathcal{C}$ be finite, which turns out to be equivalent to the condition that, for all $\epsilon > 0$, $\sup_{P \in M_1(\mathcal{X})} N(\epsilon, \mathcal{C}, P) < \infty$ (see [1, 3, 4, 7, 8, 9] for proofs and additional background on the VC dimension and metric entropy). The similarity between these two extreme cases led Benedek and Itai to conjecture in [2] that the condition

$$\forall \epsilon > 0, \ \sup_{P \in \mathcal{P}} N(\epsilon, \mathcal{C}, P) < \infty \tag{4}$$

is necessary and sufficient for the PAC learnability of $\mathcal{C}$ under $\mathcal{P}$. While necessity is fairly obvious, the sufficiency part is less so because of the difficulty in simultaneously approximately determining $c \in \mathcal{C}$ and $P \in \mathcal{P}$. (We mention that if (4) is replaced by the stronger condition that there exists a fixed finite ϵ-cover of $\mathcal{C}$ under all $P \in \mathcal{P}$, then the sufficiency is just a standard extension of the single measure case. Some cases where (4) is sufficient are described in [5].) It is the purpose of this note to show, by a counterexample, that (4) is not sufficient in general for learnability. The question of finding a necessary and sufficient condition for PAC learnability of $\mathcal{C}$ under $\mathcal{P}$ remains open.

2 A Counterexample

Let $\Omega = \mathcal{X} = \{0,1\}^\infty$, let X^i denote the coordinate map of $X \in \mathcal{X}$, and let $\mathcal{B}$ be the Borel σ-field over $\mathcal{X}$. Let $(p_1, p_2, \ldots) \in [0,1]^\infty$ be defined by $p_i = 1/\log_2(i+1) \leq 1$, and note that for every finite n, $\sum_{i=1}^\infty p_i^n = \infty$. Identifying $p_i = P(X^i = 1)$, the vector $p_1, p_2, \ldots$ induces a product measure P_I on the product space $\mathcal{X}$. For any measure P on $\mathcal{X}$, P^n denotes the product measure on $\mathcal{X}^n$ obtained from P.

Let σ denote a permutation (possibly infinite) of the integers, i.e. $\sigma : N \to N$ is one to one and onto, and define P_σ as the measure on $\mathcal{X}$ induced by $(p_{\sigma^{-1}(1)}, p_{\sigma^{-1}(2)}, \ldots)$. The ensemble of all permutations is denoted Σ. Thus, $P_\sigma(X^{\sigma(i)} = 1) = p_i$ and, if σ is the identity map, then

P_σ equals the P_I defined above.

Now let $\mathcal{P} \equiv \{P_\sigma,\, \sigma \in \Sigma\}$, let $c_i \equiv \{X \in \mathcal{X} :\, X^i = 1\}$, and let $\mathcal{C} \equiv \{c_i,\, i \in N\}$. It is easy to check that for any $P \in \mathcal{P}$, $N(\epsilon, \mathcal{C}, P) < \infty$. Since any c_i with $p_{\sigma^{-1}(i)} < \epsilon$ satisfies $d_{P_\sigma}(c_i, \emptyset) < \epsilon$, we have that for any $P \in \mathcal{P}$,

$$N(\epsilon, \mathcal{C}, P) < 2^{1/\epsilon}.$$

It follows that $\sup_{P \in \mathcal{P}} N(\epsilon, \mathcal{C}, P) < \infty$. We now claim

Theorem 1 *$\mathcal{C}$ is not PAC learnable under $\mathcal{P}$.*

Proof: We use a random coding argument. Suppose that the theorem's assertion is false. Then, for each $\epsilon > 0, \delta > 0$, it is possible to find an $n = n(\epsilon, \delta)$ and a learning rule T^n which satisfy (3) for all $c \in \mathcal{C}$ and $P \in \mathcal{P}$. In particular, for any finite k, it satisfies (3) for $c \in \mathcal{C}^k$ and $P \in \mathcal{P}^k$, where $\mathcal{C}^k = \{c_i,\, i = 1, \ldots, k\}$, $\Sigma^k = \{\sigma :\; \sigma(i) = i \;\forall i > k\}$, and $\mathcal{P}^k = \{P_\sigma,\, \sigma \in \Sigma^k\}$, i.e. $\mathcal{P}^k$ are all possible permutations of the vector $(p_1, p_2, \ldots)$ which involve only the first k coordinates. Let the error event be defined as

$$\mathrm{er}_\sigma^c = \{(X_1, \ldots, X_n) :\; d_{P_\sigma}(c, T^n(\mathcal{D}_n(c))) > \epsilon\}.$$

(It follows from (2) that er_σ^c is a measurable event.) Then, for each $c \in \mathcal{C}^k$ and $P_\sigma \in \mathcal{P}^k$,

$$P_\sigma^n(\mathrm{er}_\sigma^c) < \delta.$$

In particular, if Q is any probability measure on the finite set $\{(\sigma, c) : \sigma \in \Sigma^k, c \in \mathcal{C}^k\}$, then

$$E_Q(P_\sigma^n(\mathrm{er}_\sigma^c)) < \delta. \tag{5}$$

Now choose Q such that $Q|_\Sigma$ is uniform over Σ^k while $c = c_{\sigma(1)}$ (i.e., $Q(\sigma, c) = 1/k!$ if $\sigma \in \Sigma^k$ and $c = c_{\sigma(1)}$, and $Q(\sigma, c) = 0$ otherwise). This Q forces the true concept to involve the coordinate of maximal probability (where in fact the probability is 1) in P_σ. Note that by our choice of Q, if $\epsilon < 1 - 1/\log_2(3) = \min_{j>1} d_{P_I}(c_1, c_j)$, then, when (σ, c) are distributed according to Q,

$$d_{P_\sigma}(c, \tilde{c}) < \epsilon \Rightarrow c = \tilde{c} = c_{\sigma(1)} \quad Q \text{ a.s.} \quad .$$

Thus, in this set-up, Q a.s.,

$$\mathrm{er}_\sigma^c = \{(X_1, \ldots, X_n) : \; c \neq T^n(\mathcal{D}_n(c))\} \, .$$

Using the notation σX to denote the element of $\mathcal{X}$ with coordinates $(\sigma X)^i = X^{\sigma^{-1}(i)}$ and $\sigma \mathcal{D}_n$ to denote the corresponding permutation on $\mathcal{D}_n(c)$ when $c = c_{\sigma(1)}$, i.e.,

$$\begin{aligned} \sigma \mathcal{D}_n &= ((\sigma X_1, I_{c_{\sigma(1)}}(\sigma X_1)), \ldots, (\sigma X_n, I_{c_{\sigma(1)}}(\sigma X_n))) \\ &= ((\sigma X_1, I_{c_1}(X_1)), \ldots, (\sigma X_n, I_{c_1}(X_n))), \end{aligned} \tag{6}$$

we have

$$\begin{aligned} E_Q(P_\sigma^n(\mathrm{er}_\sigma^c)) &= E_Q(P_\sigma^n(c \neq T^n(\mathcal{D}_n(c)))) \\ &= E_Q(P_\sigma^n(c_{\sigma(1)} \neq T^n(\mathcal{D}_n(c_{\sigma(1)})))) \\ &= E_Q(P_I^n(c_{\sigma(1)} \neq T^n(\sigma \mathcal{D}_n))) \\ &= E_{P_I^n} E_Q(1_{c_{\sigma(1)} \neq T^n(\sigma \mathcal{D}_n)}) \, . \end{aligned} \tag{7}$$

For given vectors $\vec{x} = (x_1, \ldots, x_n) \in \mathcal{X}^n$ and $\vec{X} = (X_1, \ldots, X_n) \in \mathcal{X}^n$, denote by $S(\vec{X}, \vec{x})$ the set of permutations $\sigma \in \Sigma^k$ such that $\sigma \vec{X} = \vec{x}$. (Note that for many pairs $(\vec{X}, \vec{x})$, $S(\vec{X}, \vec{x})$ is empty.) It follows from the definition that, for $\sigma \in S(\vec{X}, \vec{x})$,

$$\sigma \mathcal{D}_n = ((x_1, I_{c(1)}(X_1), \ldots, (x_n, I_{c(1)}(X_n))).$$

By the construction of Q, the distribution of σ conditioned on $S(\vec{X}, \vec{x})$ is uniform there. Let now

$$J^{\vec{X}} = \{i \leq k : \; X_j^i = 1 \;\; \forall j = 1, \ldots, n\}$$

and

$$J^{\vec{x}} = \{i \le k : x_j^i = 1 \ \ \forall j = 1, \ldots, n\}.$$

$S(\vec{X}, \vec{x})$ is non-empty only if $|J^{\vec{x}}| = |J^{\vec{X}}|$. When $\vec{X}$ has distribution P_I^n, we have $1 \in J^{\vec{X}}$ almost surely, so $|J^{\vec{X}}| \ge 1$. Let $\sigma_c \in \Sigma^k$ be a fixed permutation such that $\sigma_c(i) \in J^{\vec{x}}$ if $i \in J^{\vec{X}}$. Decompose each permutation $\sigma \in S(\vec{X}, \vec{x})$ into $\sigma = \sigma_c \circ \sigma_b \circ \sigma_a$, with $\sigma_a : J^{\vec{X}} \to J^{\vec{X}}$, and σ_a equals the identity on $\{1, \ldots, k\} \setminus J^{\vec{X}}$ while $\sigma_b : \{1, \ldots, k\} \setminus J^{\vec{X}} \to \{1, \ldots, k\} \setminus J^{\vec{X}}$ and σ_b equals the identity on $J^{\vec{X}}$. This is always possible because all permutations in $S(\vec{X}, \vec{x})$ must satisfy $\sigma\vec{X} = \vec{x}$. Note that whenever $S(\vec{X}, \vec{x})$ is non-empty then $|\sigma_A| = |J^{\vec{X}}|!$, where

$$\sigma_A \triangleq \{\sigma_a : \sigma \in S(\vec{X}, \vec{x})\}, \quad \sigma_B \triangleq \{\sigma_b : \sigma \in S(\vec{X}, \vec{x})\}.$$

Using now (7),

$$\begin{aligned} E_Q(P_\sigma^n(\mathrm{er}_\sigma^c)) &= E_{P_I^n}\left(\sum_{\vec{x}} E_Q(1_{T^n(\sigma\mathcal{D}_n) \ne c_{\sigma(1)}} | \sigma \in S(\vec{X}, \vec{x})) Q(S(\vec{X}, \vec{x}))\right) \\ &= E_{P_I^n}\left(\sum_{\vec{x}} Q(S(\vec{X}, \vec{x})) \frac{\sum_{\sigma_b \in \sigma_B} \sum_{\sigma_a \in \sigma_A} 1_{T^n(\sigma\mathcal{D}_n) \ne c_{\sigma(1)}}}{\sum_{\sigma_b \in \sigma_B} \sum_{\sigma_a \in \sigma_A} 1}\right), \end{aligned} \tag{8}$$

where in the last equality we have used the uniformity of the conditional distribution over $S(\vec{X}, \vec{x})$, and the sum over $\vec{x}$ is taken over all *different* vectors in $\mathcal{X}^n$. By (6), $\sigma\mathcal{D}_n$ is constant for $\sigma \in S(\vec{X}, \vec{x})$, so

$$T^n(\sigma\mathcal{D}_n) = c_T$$

for some $c_T = c_T(\vec{X}, \vec{x}) \in \mathcal{C}$ not depending on $\sigma \in S(\vec{X}, \vec{x})$. Here $c_T(\cdot, \cdot)$ is measurable by (2). Thus, since the number of permutations $\sigma \in \sigma_A$ for which $T^n(\sigma\mathcal{D}_n) = c_T$ is at most equal to the number of permutations in σ_A which have a prescribed index in $J^{\vec{X}}$ unchanged,

$$\sum_{\sigma_a \in \sigma_A} 1_{T^n(\sigma\mathcal{D}_n) \ne c_{\sigma(1)}} \ge (|J^{\vec{X}}| - 1)(|J^{\vec{X}}| - 1)!$$

whereas

$$\sum_{\sigma_a \in \sigma_A} 1 = |J^{\vec{X}}|!.$$

It follows that, for any $\eta > 1$,

$$E_Q(P_\sigma^n(\mathrm{er}_\sigma^c)) \geq E_{P_I^n} \frac{(|J^{\vec{X}}|-1)(|J^{\vec{X}}|-1)!}{|J^{\vec{X}}|!} = (1 - E_{P_I^n} \frac{1}{|J^{\vec{X}}|}) \geq (1 - \frac{1}{\eta} - P_I^n(|J^{\vec{X}}| \leq \eta)) \,.$$

It remains therefore only to show that $|J^{\vec{X}}|$ may, with high probability, be made arbitrarily large by choosing a k large enough. But this is obvious because, by the Borel-Cantelli lemma, using $\vec{X}^i \triangleq (X_1^i, \ldots, X_n^i)$,

$$P_I^n(\vec{X}^i = (1, \ldots, 1) \ \text{ infinitely often}) = 1$$

since $\sum_{i=1}^\infty P_I^n(\vec{X}^i = (1, \ldots, 1)) \geq \sum_{i=1}^\infty p_i^n = \infty$. Thus, for any η, one may find a k large enough such that $P_I^n(|J^{\vec{X}}| \leq \eta)$ is arbitrarily small. $\square$

Remark: Note that we have actually shown that, for any fixed n and any $\epsilon < 1 - 1/\log_2(3)$, one may construct a $\mathcal{P}$ and a $\mathcal{C}$ such that the probability of error is arbitrarily close to 1. By defining $p_i, \ i \geq 2$ to be smaller, we could also take any $\epsilon < 1$.

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