LECTURE 6: LOOP SOUPS

We continues in the setup where V is finite. The material in this lecture is again borrowed, essentially verbatim, from Sznitman's lecture notes.

1. Poisson process of loops

We consider the base space L^* with its natural σ-algebra and measure μ^*. We introduce the Poisson measure with speed measure $\alpha\mu^*$, with $\alpha > 0$ a parameter, and denote it $\mathbb{P}_\alpha$. The base space is $\Omega = \{\text{pure point measures on } L^*\}$, which can be identified with countable configurations of loops.

A slight complication is that the measure μ^* is infinite, but we have already seen that it is σ-finite and finite on the event $N > 1$, and that it is finite on loops with $\xi(\gamma) > s$, so under the Poisson measure there will be only finitely many such loops, almost surely. So a sample point under the measure μ^* is of the form $\nu = \sum_{i=1}^\infty \delta_{\gamma_i^*}$ where γ_i^* are loops (which, e.g., can be ordered in terms of decreasing lifetime). An impportant remark is that while μ^* is a σ-finite measure, $\mathbb{P}_\alpha$ is a *probability* measure.

A word on measurability issues: we equip Ω with the σ algebra generated by the evaluation maps $\omega(A)$, with A measurable subsets of L^*. Note that $\omega(A) \in \mathbb{Z}_+ \cup \{\infty\}$.

A characterization of the Poisson measure is via its Laplace functional: let $\Phi : L^* \to \mathbb{R}_+$ be measurable. Then,

$$\mathbb{E}_\alpha(e^{-\int \Phi d\nu}) = \mathbb{E}_\alpha(e^{-\sum_i \Phi(\gamma_i^*)}) = e^{-\alpha \int_{L^*}(1-e^{-\Phi})} d\mu^*. \tag{1}$$

The formula (1) cannot be extended to a Fourier transform (i.e., replacing Φ by $i\Phi$) because μ^* is not finite; However, if Φ vanishes on loops with short lifetime, then the formula extends naturally.

We can now introduce the occupation field $\{\mathcal{L}_x\}_{x\in V}$ as

$$\mathcal{L}_x(\omega) = \int L_x(\gamma) d\omega(\gamma) = \sum_i L_x(\gamma_i^*),$$

recalling that L_x is an invariant function and thus well defined on γ^*. It is a-priori not clear that $\mathcal{L}_x$ is finite, since μ^* is σ-finite. However, we already saw that $E_{\mu^*}(\xi(\gamma)) < \infty$, and hence

$$\mathbb{E}_\alpha(\sum_x \mathcal{L}_x) = \alpha E_{\mu^*}(\xi(\gamma)) < \infty,$$

which makes $\mathcal{L}_x$ well defined. In fact, we can compute its Laplace transform.

Lemma 1.1. *Let $F : V \to \mathbb{R}_+$. Then,*

$$\mathbb{E}_\alpha(e^{-\sum_{x\in V} F(x)\mathcal{L}_x}) = \det(I+GF)^{-\alpha} = \left(\frac{\det G_F}{\det G}\right)^\alpha. \tag{2}$$

Proof. By (1), the left hand side in (2) equals

$$e^{-\alpha \int (1-e^{-\sum_{x\in V} F(x) L_x(\gamma)}) d\mu_r(\gamma)}.$$

The result then follows from Lemma 3.1 in Lecture 5, as the integral can be represented as the logarithm of determinants. ☐

One important remark is in order: we see that $\alpha = 1/2$ is special, because that case leads to square-roots of determinants, which is consistent with expressions involving Gaussian processes. We state this important consequence as a theorem.

Theorem 1.2. *The field $(\mathcal{L}_x)_{x\in V}$ under $\mathbb{P}_{1/2}$ has the same law as $(\frac{1}{2}\phi_x^2)$ under the free field with covariance G.*

Proof. From Lemma 1.1 we compute the Laplace transform of the left side (i.e., $\mathbb{E}_\alpha e^{-\sum F(x)\mathcal{L}_x}$) to be $\det(I+GF)^{-1/2}$. On the other hand, the Laplace transform of the right side is, by a Gaussian computation, the same expression. ☐

Remark 1.3. In the same way that we did when dealing with cover times, we can use Lupu's coupling (involving metric graphs) to couple the field of occupation measures too the GFF in such a way that the clusters of loops (i.e., those clusters of points belonging to the same loops in the loop soup) are precisely the clusters of same sign in the GFF on the metric graph.

We will explore representation formulas based on Theorem 1.2 in the next section. For the moment, mention only that by taking $F(z) = \mathbf{1}_x$, one gets that

$$\mathbb{E}_\alpha(e^{-\eta\mathcal{L}_x}) = (1+\eta g(x,x))^{-\alpha},$$

i.e. $\mathcal{L}_x$ has a Gamma$(\alpha, g(x,x))$ distribution. More explicitely, density $C_\alpha y^{\alpha-1}e^{-y/g(x,x)}dy$ with $C_\alpha = 1/(\Gamma(\alpha)g(x,x)^\alpha)$. In particular, we obtain that $\mathbb{E}_\alpha(\mathcal{L}_x) < \infty$, as we already know.

Exercise 1. *Let $\hat{\mathcal{L}}_x(\omega) = \sum_i \mathbf{1}_{\{N_i>1\}}\mathcal{L}_x(\gamma_i)$, where as before $\omega = \sum_i \delta_{\gamma_i}$. Prove that*

$$\mathbb{E}_\alpha(e^{-\sum F(x)\hat{\mathcal{L}}_x}) = \left(\frac{\det(I-P)}{\det(I-P^F)}\right)^\alpha,$$

where P^F is obtained from P by changing λ_x to $\lambda_x + F(x)$. This looks at the occupation time of the non-trivial loops in the loop soup. For details, see Sznitman, Page 90.

2. Szymanzik's formula

We recall that for a centered Gaussian field ϕ_x with covariance $R(\cdot,\cdot)$, one has the Feynman moment formula, as follows. Let $\mathcal{P}_k$ denote the collection of pairings of $\{1,\ldots,2k\}$, that is of *disjoint* unions of the form $\{1,\ldots,2k\} = \cup_{i=1}^k\{(x_i,y_i)\}$. Then,

$$\mathbb{E}\prod_{i=1}^{2k}\phi_{x_i} = \sum_{\Pi\in\mathcal{P}_k}\prod_{i=1}^{k}R(x_i,x_{\pi(i)}).$$

The proof goes by differentiating the characteristic function.

Szymanzik's formula gives a similar formula, for a perturbation of the Gaussian measure. Let ν be a probability measure on $\mathbb{R}_+$ and consider its Laplace transform

$$h(u) = \int_0^\infty e^{-uy}d\nu(y).$$

We are interested in the measure on $\{\varphi_x\}_{x\in V}$ with density

$$\bar{\mathbb{P}}^h = \frac{1}{Z^h} e^{-\frac{1}{2}\mathcal{E}(\varphi,\varphi)} \prod_{x\in V} h(\frac{\varphi_x^2}{2}).$$

To describe the analogue of Feynman's formula with respect to $\bar{\mathbb{P}}^h$, we need to consider the Poisson loop measure with parameter $1/2$, as well as additional k paths $w_1,\ldots,w_k$, as follows. Write $\mathcal{P}_k^{\mathbf{z}}$ for the pair partitions of $(z_1,\ldots,z_{2k})$.

Let $\eta(x)$ be i.i.d. random variables with law ν, denote their joint law by $\tilde{\mathbb{P}}$.

Theorem 2.1. *With notation as above, we have*

$$\begin{aligned}&\mathbb{E}^h \prod_{i=1}^{2k} \phi_{z_i}\\ &= \sum_{\Pi=\cup_{i=1}^k\{x_i,y_i\}\in\mathcal{P}_k^{\mathbf{z}}} \frac{E_{x_1,y_1}\otimes\cdots\otimes E_{x_k,y_k}\otimes\tilde{\mathbb{E}}\otimes\mathbb{E}_{1/2}(e^{-\sum_{x\in V}\eta_x(\mathcal{L}_x+L_x(w_1)+\ldots+L_x(w_k))})}{\tilde{E}\otimes\mathbb{E}_{1/2}(e^{-\sum_{x\in V}\eta_x\mathcal{L}_x})}.\end{aligned}$$

Proof. Let $S=S(\phi_{z_1},\ldots,\phi_{z_{2k}})$ be an arbitrary test function. By the definition of the η_x we have that

$$\mathbb{E}^h(S) = \frac{\tilde{E}\int S e^{-\frac{1}{2}(\mathcal{E}(\varphi,\varphi)+\sum\eta_x\varphi_x^2)}\prod d\varphi_x}{\tilde{E}\int e^{-\frac{1}{2}(\mathcal{E}(\varphi,\varphi)+\sum\eta_x\varphi_x^2)}\prod d\varphi_x} =: \frac{(S)_h}{(1)_h}. \tag{3}$$

Note next that, with E^η denoting the Gaussian measure with quadratic form $\mathcal{E}(\varphi,\varphi)+\sum_x\eta_x\varphi_x^2$,

$$(S)_h = C\tilde{E}\otimes\mathbb{E}^\eta(S\det(G_\eta)^{1/2}) \tag{4}$$

where $C=(2\pi)^{|V|/2}$ does not depend on S and therefore will disappear in the ratio computation.

Now, by the isomorphism result in Lemma 1.1, we get that

$$\tilde{E}\otimes\mathbb{E}^\eta(S\det {G_\eta}^{1/2}) = (\det G)^{1/2}\tilde{E}\otimes\mathbb{E}_{1/2}\otimes\mathbb{E}^\eta(Se^{-\sum\eta_x\mathcal{L}_x}).$$

Again, the $(\det G)$ term is an immaterial constant that will disappear in the ratio. Note that now the Gaussian field $\{\phi_x\}$ only appears as argument of S. Using now the Feynman formula, we can write $\mathbb{E}^\eta S$ in terms of G_η.

We next recall that by formula (8) in Lecture 4,

$$E_{x,y}(e^{-\sum_x\eta_x L_x}) = (I-G\eta)^{-1}G(x,y) = G_\eta(x,y).$$

(We had proved it for small η, but the extension to general *positive* η follows the argument we used in lecture 5, see the proof of Lemma 3.1 there. This gives the desired formula. $\square$

3. Avoidance probabilities

Fix a subset $A\subset V$. We can define $\mathcal{L}_x^A$ as the field of occupation measure of those (unrooted) loops *that remain in* A. We write $K=V\setminus A$ for the complement of A in V. Let $G_A=G_{\mathbf{1}_A}$ and, for any function $F:V\to\mathbb{R}$, $G_{F,A}=G_{F\cdot\mathbf{1}_A}$. Further, for any matrix C with index set V, write C^K for the restriction of C to K. Note that C_K is a $|K|\times|K|$ matrix. It will be particularly useful when K is a singleton.

Theorem 3.1. *Let $F : V \to \mathbb{R}_+$. Then,*

$$\mathbb{E}_\alpha(e^{-\sum_{x\in V} F(x)(\mathcal{L}_x - \mathcal{L}_x^A)}) = \left(\frac{\det G_F \cdot \det G_A}{\det G \cdot \det G_{F,A}}\right)^\alpha = \left(\frac{\det (G_F)^K}{\det G^K}\right)^\alpha. \tag{5}$$

Further, if $K_1 \cap K_2 = \emptyset$ then

$$\mathbb{P}_\alpha(\text{no loop intersects both } K_1 \text{ and } K_2) = \left(\frac{\det G \cdot \det G_{(K_1\cup K_2)^c}}{\det G_{K_1^c} \cdot \det G_{K_2^c}}\right)^{-\alpha} = \left(\frac{\det (G_{K_2^c})^{K_1}}{\det G^{K_1}}\right)^\alpha. \tag{6}$$

Proof. We begin with (5). Since $\mathcal{L}_x^A$ involves loops fully contained in A and $\mathcal{L}_x - \mathcal{L}_x^A$ involves only loops that are not fully contained in A, these sets are disjoint. By the Poisson property, these fields are therefore independent. By Lemma 1.1 we have that

$$\mathbb{E}_\alpha(e^{-\sum F(x)\mathcal{L}_x}) = \left(\frac{\det G_F}{\det G}\right)^\alpha.$$

On the other hand, the restriction property tells us that

$$\mathbb{E}_\alpha(e^{-\sum F(x)\mathcal{L}_x^A}) = \left(\frac{\det G_{F,A}}{\det G_A}\right)^\alpha.$$

Together with the stated independence, this gives the first equality in (5). To see the second equality there, recall Schur's complement formula: if

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

then $M_{11}^{-1} = (A - BD^{-1}C)^{-1}$ and $\det M = \det D / \det M_{11}^{-1}$. In particular, setting $M = -L$, $M^{-1} = G$ and $D = -L^A$, we get from this formula (using that $K = A^c$) that $(\det G)^{-1} = (\det G_A)^{-1}(\det (G^K)^{-1}$, i.e.

$$\det(G^K) = \frac{\det G}{\det G_A}.$$

Similarly,

$$\det(G_F^K) = \frac{\det G_F}{\det G_{F,A}}.$$

This gives the second equality in (5).

Turning to the proof of (6), note first that trivial loops cannot intersect both K_1 and K_2. In particular, the event in the RHS of (6) implies that for any loop γ in the Poisson ensemble,

$$\mathbf{1}_{N>1} \sum_{x\in K_1} L_x \sum_{y\in K_2} L_y = 0.$$

By the Poisson property, the probability of the last event is simply

$$e^{-\alpha\mu^*(\{\gamma^* : N(\gamma^*)>1, \sum_{x\in K_1} L_x(\gamma^*)>0, \sum_{x\in K_2} L_x(\gamma^*)>0\})} =: e^{-\alpha Q}.$$

We evaluate Q. By inclusion-exclusion,

$$Q = Q_{K_1} + Q_{K_2} - Q_{K_1\cup K_2}$$

where, for any set $M \subset V$, $Q_M = \mu^*(\{\gamma^* : N(\gamma^*) > 1, \sum_{x\in M} L_x(\gamma^*) > 0\})$. We now claim that

$$Q_M = \log(\det G^M) + \sum_{x\in M} \log \lambda_x. \tag{7}$$

Thus,

$$Q = \log(\det G^{K_1}) + \log(\det G^{K_2}) - \log(\det G^{K_1 \cup K_2}).$$

Recall that $\det(G^M) = \frac{\det G}{\det G_{M^c}}$. Hence,

$$Q = \log(\det(G) \cdot \det(G_{(K_1 \cup K_2)^c})) - \log(\det(G_{K_1^c}) \cdot \det(G_{K_2^c})).$$

Another application of Schur's complement gives

$$\det(G^{K_1}_{K_2^c}) = \frac{\det(G_{K_2^c})}{\det(G_{K_1^c \cap K_2^c})} = \frac{\det(G_{K_2^c})}{\det(G)/\det(G^{K_1 \cup K_2})}.$$

This completes the proof.

It only remains to prove (7). Using the Poisson nature of $\mathbb{P}_\alpha$, we have that

$$Q_M = -\frac{1}{\alpha} \log \mathbb{P}_\alpha(\{\omega = \sum \delta_{\gamma_i} : \sum_{x \in M} \sum_i \mathbf{1}_{N(\gamma_i)>1} L_x(\gamma_i) = 0\}).$$

In the notation of Exercise 1, we need to evaluate

$$\mathbb{P}_\alpha(\sum_{x \in M} \hat{\mathcal{L}}_x = 0).$$

Using this exercise with $F(x) = \rho 1_{x \in M}$ and taking $\rho \to \infty$, we obtain that

$$\mathbb{P}_\alpha(\sum_{x \in M} \hat{\mathcal{L}}_x = 0)^{1/\alpha} = \lim_{\rho \to \infty} (\mathbb{E}_\alpha(e^{-\rho \sum_{x \in M} \hat{\mathcal{L}}_x}))^{1/\alpha} = \lim_{\rho \to \infty} \frac{\det(I-P)}{\det(I-P^\rho)}$$

where $P^\rho = P^{\rho \mathbf{1}_M}$. By the definition,

$$P^\rho(x,y) \to_{\rho \to \infty} \mathbf{1}_{x \in M^c} P(x,y) =: \bar{P}_M.$$

Note that

$$I - \bar{P}_M = \begin{bmatrix} I_M & 0 \\ * & (I-P)^{M^c} \end{bmatrix}.$$

Hence, $\det(I - P_M) = \det((I-P)^{M^c})$. Recalling that $-L = \lambda(I-P)$, we have

$$\det(G_{M^c}) = \det((-L)^{M^c})^{-1} = \frac{\det(I-P)^{M^c}}{\prod_{x \in M^c} \lambda_x}.$$

Similarly, $\det G = \det(I-P)^{-1}/\prod_{x \in V} \lambda_x$. Hence,

$$\det(I - P_M) = \frac{\det(G_{M^c})}{\det(G)} \cdot \frac{1}{\prod_{x \in M} \lambda_x},$$

which completes the proof.

□

4. Interlacements

We restrict attention in this section to $\mathbb{Z}^d$, $d \geq 3$. It is not a finite graph, but random walk is transient, so we can hope to simply approximate $\mathbb{Z}^d$ by a sequence of finite boxes V_n of side n, centered at 0, with the random walk killed when exiting the box. We take all conductances in V_n to be equal to $1/2d$, so that there is no difference between the variable rate and fixed rate random walk.

Let $\hat{X}.$ denote the (continuous time) random walk on $\mathbb{Z}^d$. Define the Green function

$$g(x,y) = g(x-y) = E^x \int_0^\infty \mathbf{1}_{\hat{X}_s = y} ds.$$

It is well known (and easy to check by Fourier analysis, aka local limit theorems) that

$$g(x) \sim c_d |x|^{2-d},$$

with explicit constant c_d.

We next introduce the notion of equilibrium measure. For $K \subset Z^d$, let $T_K = \inf\{t : \hat{X}_t \in K\}$. Let $\tau_K = \inf\{t : \hat{X}_t \in K, \exists s \in (0,t] \text{s.t.} \hat{X}_s \neq \hat{X}_0\}$. The *equilibrium measure of* K, is the measure

$$e_K(y) = P^y(\tau_K = \infty)\mathbf{1}_K(y), \tag{8}$$

i.e. $e_K(y)$ is the escape probability from y (recall that the walk is transient!). Necessarily, $e_K(y) > 0$ only if y lives on the boundary of K. The equilibrium measure is easily checked to satisfy

$$P^x(T_K < \infty) = \sum_{y \in K} g(x,y) e_K(y), x \in \mathbb{Z}^d. \tag{9}$$

Exercise 2. *Check the last display, using that the number of visits to $y \in \partial K$ before escaping is geometric with parameter $e_K(y)$.*

We define the *capacity* of K as the total mass of the equilibrium measure e_K, i.e. $\text{Cap}(K) = \sum_{z \in K} e_K(z)$.

We next introduce loops. We take a point x^* and insist on considering only those loops that pass through x^*. We will take $x^* \to \infty$ after we take $n \to \infty$. Of course, in order to visit a compact set we need to adjust α. Pick $u > 0$ and set

$$\alpha = \alpha(u, x^*) = u \frac{g(0)}{c_d^2} |x^*|^{2(d-2)}.$$

Note that a-priori, when starting from x^* the probability to hit 0 (say) is roughly $c/|x^*|^{d-2}$, but the probability to hit 0 and return to x^* is only $c/|x^*|^{2(d-2)}$, hence the scaling. We write $\mathbb{P}^{u,x^*,n} = \mathbb{P}_{\alpha(u,x^*),V_n}$.

Fix n and x^* (with $x^* \in V_n$). Let $\mathcal{I}_{n,x^*}$ denote the vertices $z \in V_n$ that have been visited by an (unrooted) loop in V_n that passed through x^*. Under the measure $\mathbb{P}^{u,x^*,n}$, this is a random set. We will care about its distribution. We begin with avoidance probabilities.

Theorem 4.1. *For any fixed compact $K \subset \mathbb{Z}^d$,*

$$\lim_{x^* \to \infty} \lim_{n \to \infty} \mathbb{P}^{u,x^*,n}(\mathcal{I}_{n,x^*} \cap K = \emptyset) = e^{-u\text{Cap}(K)}. \tag{10}$$

Before providing the proof, we recall a consequence of Theorem 3.1.

Lemma 4.2. *Fix $K \subset V$ and $x^* \in V \setminus K$. Then,*

$$\mathbb{P}_\alpha(\text{All loops through } x \text{ do not intersect } K) = \left(1 - \frac{E^{x^*}(\mathbf{1}_{\{T_K<\infty\}}\, g(X_{T_K}, x^*)}{g(x^*,x^*)})\right)^\alpha.$$

Proof. We take in (6) of Theorem 3.1 $K_2 = K$ and $K_1 = \{x^*\}$. Then,

$$\begin{aligned}
&\mathbb{P}_\alpha(\text{All loops through } x \text{ do not intersect } K)^{1/\alpha}\\
&= \mathbb{P}_\alpha(\text{No loop intersects both } K_1 \text{ and } K_2)^{1/\alpha}\\
&= \frac{g_{K^c}(x^*,x^*)}{g(x^*,x^*)} = \frac{g(x^*,x^*) - E^{x^*}(\mathbf{1}_{\{T_K<\infty\}}\, g(X_{T_K}, x^*))}{g(x^*,x^*)},
\end{aligned}$$

where the last equality is due to a first time decomposition. $\square$

Proof of Theorem 4.1. From Lemma 4.2 we have that, with $g^{(n)}$ denoting the Green function of the random walk killed at exiting V_n,

$$\mathbb{P}^{u,x^*,n}(\mathcal{I}_{n,x^*} \cap K = \emptyset) = \frac{g^{(n)}(x^*,x^*) - E^{x^*}(\mathbf{1}_{\{T_K < T_{V_n^c}\}}\, g^{(n)}(X_{T_K},x^*))}{g^{(n)}(x^*,x^*)}. \tag{11}$$

By transience, $g^{(n)}(x,y) \to_{n\to\infty} g(x,y)$, uniformly in compact subsets of $\mathbb{Z}^d$. Thus, taking the limit as $n \to \infty$ in (11) one gets $g^{(n)}(x^*,x^*) \to g(0)$ and thus

$$\lim_{n\to\infty} \mathbb{P}^{u,x^*,n}(\mathcal{I}_{n,x^*} \cap K = \emptyset) = \left(1 - \frac{E^{x^*}(\mathbf{1}_{\{T_K<\infty\}}\, g(X_{T_K},x^*))}{g(0)}\right)^{\alpha}.$$

As $x^* \to \infty$ we have that $g(x_{T_K},x^*) \sim g(x^*) \sim c_d|x^*|^{2-d}$. Hence,
(12)

$$\lim_{x^*\to\infty}\lim_{n\to\infty} \mathbb{P}^{u,x^*,n}(\mathcal{I}_{n,x^*} \cap K = \emptyset) = \lim_{x^*\to\infty}\left(1 - \frac{c_d|x^*|^{2-d}P^{x^*}(T_K<\infty)}{g(0)}\right)^{u\frac{g(0)}{c_d^2}|x^*|^{2(d-2)}}.$$

We recall (9):

$$P^x(T_K<\infty) = \sum_{y\in K} g(x,y)e_K(y), x \in \mathbb{Z}^d.$$

when $x = x^* \to \infty$, we have that $g(x^*,y) \sim g(x^*) \sim c_d|x^*|^{2-d}$ and hence $P^{x^*}(T_K < \infty) \sim c_d \mathrm{Cap}(K)|x^*|^{2-d}$. Substituting in (12) we obtain

$$\begin{aligned}\lim_{x^*\to\infty}\lim_{n\to\infty} \mathbb{P}^{u,x^*,n}(\mathcal{I}_{n,x^*} \cap K = \emptyset) &= \lim_{x^*\to\infty}\left(1 - \frac{c_d^2|x^*|^{2(2-d)}\mathrm{Cap}(K)}{g(0)}\right)^{u\frac{g(0)}{c_d^2}|x^*|^{2(d-2)}} \\ &= e^{-u\mathrm{Cap}(K)}.\end{aligned}$$

□

While this is not obvious, this computation completely characterizes the limit process of interlacements. The reason is that by inclusion-exclusion, one can evaluate from it the limit of

$$\mathbb{P}^{u,x^*,n}(\mathcal{I}_{n,x^*} \cap K = \emptyset, K' \in I_{n,x^*})$$

for disjoint K, K' and thus characterize the occupied set distribution.

4.1. **Another construction.** We sketch another construction of interlacements, which does not use the loop soop but rather works directly with path measures.

Let V_n be as above, but now identify the boundary of V_n with the point x^*. Let $\tau_u^n = \inf\{t \geq 0 : L_t^{n,x^*} > u\}$.

Proposition 4.3. *For $d \geq 3$, one has that*

$$\left\{L^{n,x}_{\tau_u^n}\right\}_{x\in\mathbb{Z}^d} \to^d_{n\to\infty} \left\{L_x^{\mathrm{int},u}\right\}_{x\in\mathbb{Z}^d},$$

where $L_x^{\mathrm{int},u}$ is the local time of the interlacement process defined above, and the convergence is with respect to cylinder test functions.

A proof can be obtained by computing the Laplace transform $E(\exp(-\sum F(x)L^{n,x}_{\tau_u^n}))$, noting that it has a limit as $n \to \infty$, and then equating that limit with

$$E(\exp(-\sum F(x)L_x^{\mathrm{int},u})) = \exp(-u\sum_{x,y}(I+GF)^{-1}(x,y)F(x)).$$

One convenient way to compute the limit is by using the GRK2 for $L^{n,x}_{\tau^n_u}$.

Exercise 3 (*)**.** *Prove Proposition 4.3.*

Once the limit is obtained, and rrecalling that for $d \geq 3$ the GFF in V_n has a limit $\{\phi_x\}$ due to transience of the random walk, we obtain the following.

Corollary 4.4.
$$\left\{L_x^{\text{int},u} + \frac{1}{2}\phi_x^2\right\}_{x\in\mathbb{Z}^d} \stackrel{d}{=} \left\{\frac{1}{2}(\phi_x + \sqrt{2u})^2\right\}_{x\in\mathbb{Z}^d}.$$

Let $\mathcal{V}_u$ denote the complement of the interlacement set, i.e. the vacant set. Let now u_* be such that $\mathcal{V}_u$ percolates for $u < u^*$ (monotonicity here is clear due to the Poisson construction of interlacements). Let h_* be the supremum of the h such that the set $\{\phi_x > h\}$ percolates. By applying Lupu's isomorphism (i.e. moving to the metric graph of $\mathbb{Z}^d$), we see that the connected clusters of $-\phi + \sqrt{2u^*} < 0$ are dominated by the vacant set. In particular, $h_* \leq \sqrt{2u_*}$.

Remark 4.5. An interesting question is whether $h_* > 0$. This was shown in large enough dimension for d large enough by Rodriguez and recently in all dimensions $d \geq 3$ by Drewitz, Prevost and Rodriguez (arXiv:1708.03285).

4.2. **Yet another construction.** This sub-section is devoted to Sznitman's original construction of the interlacement process, which was motivated by certain disconnection problems, and in particular the claim that the time to disconnect the cylinder $(\mathbb{Z}/N\mathbb{Z})^d \times \mathbb{Z}$ by random walk is of order N^{2d}. This is done in discrete time, directly on $\mathbb{Z}^d$, $d \geq 3$, as follows.

Let W_+ (W) denote the space of infinite (doubly infinite) paths, let $W^* = W/\sim$ and $\pi^* : W \to W^*$ where the equivalence relation is equality modulu time shifts. We consider the space $W^* \times \mathbb{R}_+$, where the second coordinate corresponds to intensity. Sznitman constructs a Poisson measure $\mathbb{P}$ on $W^* \times \mathbb{R}_+$, with intensity $\hat{\nu} = \nu \times \text{Leb}$ (a σ-finite measure), as follows. For a compact $K \subset \mathbb{Z}^d$, let W_K^0 denote the subset of trajectories entering K for the first time at time 0, i.e. $w(0) \in K$, $w(n) \notin K$ for $n < 0$. Let $W_K^* = \pi^* W_K^0$.

Theorem 4.6 (Sznitman)**.** *There exists a unique σ-finite measure ν as above so that, for any compact K,*
$$1_{W_K^*}\nu = Q_K(\pi^*)^{-1}$$
and Q_K satisfies:

1. *$Q_K(w_0 = x) = e_K(x)$.*
2. *Fix $x \in \partial K$. Conditionally on $w_0 = x$, under Q_K the paths $(w_n)_{n\geq 0}$ and $\{w_{-n}\}_{n\geq 0}$ are independent, of law P^x and $P^x(\cdot|\tau_K = \infty)$.*

In particular, under conditioning by $w_0 = x$, Q_K is a probability measure.

Note that from the existence of the measure ν we obtain that $\hat{\nu}(W_K^* \times [0, u]) = u\text{Cap}(K)$ and hence, with $\mathcal{V}_u$ denote the vertices in $\mathbb{Z}^d$ that have not been visited by any path of intensity smaller than u, one has $\mathbb{P}(K \subset \mathcal{V}_u) = e^{-u\text{Cap}(K)})$. Thus, we obtain an alternative construction of the interlacement process.

(Sketch). The uniqueness is clear since there exists an increasing sequence $K_n \nearrow \mathbb{Z}^d$. So the only issue is the construction of ν. For that, it is enough to show that the measures defining ν are compatible. That is, define $\nu_K = Q_K(\pi^*)^{-1}$. It is

enough to show that if $K \subset K'$ then $\mathbf{1}_{W^*_K} \nu_{K'} = \nu_K$. This is a computation: let $\Sigma_{K,K'}$ denote the collection of paths of finite length connecting a point in $\partial K'$ to a point in ∂K, entering K only at the endpoint; let $L(\sigma)$ denote the length of a path $\sigma \in \Sigma_{K,K'}$. Let $A_i \subset \mathbb{Z}^d$. Then, we need to compute

$$\mathcal{Q} := \sum_{\sigma \in \Sigma_{K,K'}} Q_{K'}(w_{i+L(\sigma)} \in A_i, i \in \mathbb{Z}, w_j = \sigma(j), j \in [0, L(\sigma)])$$

and compare this to $Q_K(w_i \in A_i, i \in \mathbb{Z})$. (Picture!).

Relabeling, we have

$$\mathcal{Q} = \sum_{\sigma \in \Sigma_{K,K'}} Q_{K'}(w_i \in A_{i-L(\sigma)}, i \in \mathbb{Z}, w_j = \sigma(j), j \in [0, L(\sigma)]).$$

By definition of $Q_{K'}$, this equals

$$\begin{aligned}
\mathcal{Q} &= \sum_{\sigma \in \Sigma_{K,K'}} \sum_{x \in \partial K'} e_{K'}(x) P^x(w_i \in A_{-i-L(\sigma)}, i \geq 0 | \tau_{K'} = \infty) \\
&\qquad \cdot P^x(w_j = \sigma_i \in A_{j-L(\sigma)}, j \in [0, L(\sigma)]) P^{\sigma(L(\sigma))}(w_i \in A_{i-L(\sigma)}, i > L(\sigma)) \\
&= \sum_{\sigma \in \Sigma_{K,K'}} \sum_{x \in \partial K'} P^x(w_i \in A_{-i-L(\sigma)}, i \geq 0, \tau_{K'} = \infty) \\
&\qquad \cdot P^x(w_j = \sigma(j) \in A_{j-L(\sigma)}, j \in [0, L(\sigma)]) P^{\sigma(L(\sigma))}(w_i \in A_{i-L(\sigma)}, i > L(\sigma)) \qquad (13) \\
&=: \sum_{\sigma \in \Sigma_{K,K'}} \sum_{x \in \partial K'} P_1^x P_2^x P_3^{\sigma(L(\sigma))}.
\end{aligned}$$

Let $\Sigma^{x,y}_{K,K'}$ denote the paths in $\Sigma_{K,K'}$ that start at $x \in \partial K'$ and end in $y \in \partial K$. By time reversal (we are dealing with random walk!), we have for $\sigma(\cdot) \in \Sigma^{x,y}_{K,K'}$,

$$P_2^x = P^y(w_j = \sigma(L(\sigma) - j) \in A_{-j}, j \in [0, L(\sigma)], \tau_K > L(\sigma)).$$

Therefore, for such $\sigma(\cdot)$,

$$P_1^x P_2^x = P^y(w_j \in A_{-j}, j \geq 0; w_j = \sigma(L(\sigma) - j), j \in [0, L(\sigma)]; w_{L(\sigma)} = x; \tau_K = \infty).$$

Summing the last expression over x and those $\sigma \in \Sigma^{x,y}_{K,K'}$ with $\sigma(L(\sigma)) = y$, we obtain that the sum equals

$$P^y(w_j \in A_{-j}, j \geq 0; \tau_K = \infty).$$

Substituting in (14), we obtain that

$$\mathcal{Q} = \sum_{y \in \partial K} P^y(w_j \in A_{-j}, j \geq 0; \tau_K = \infty) P^y(w_j \in A_j, j \geq 0),$$

which equals

$$\sum_{y \in \partial K} P^y(w_j \in A_{-j}, j \geq 0 | \tau_K = \infty) e_K(y^y_P(w_j \in A_j, j \geq 0) = Q_K(w_j \in A_j, j \in \mathbb{Z}),$$

as claimed. $\square$

It is easy to see from the construction that the measure ν is translation invariant and invariant under time reversal. It is a bit more delicate, but still true, that its restriction to $W^* \times [0, u]$ is ergodic with respect to spatial shifts. From this it follows that, with $\mathbb{P}^u(\cdot) := \mathbb{P}(\cdot \times [0, u])$, one has

$$\mathbb{P}^u(\mathcal{V}^u \text{ contains an infinite cluster}) \in \{0, 1\}.$$

With u_* denoting the supremum of us for which the probability that 0 is in an infinite cluster is positive, Sznitman showed that $u_* < \infty$ (and that $u_* > 0$ for $d \geq 7$.) The proof that $u_* < \infty$ involves a renormalization scheme, using a small increase of $u_n \to u_{n+1}$ (and independence coming with it) to overcome the long-range dependence of the model.

We remark that Teixeira showed that the infinite cluster, when it exists, is unique.

Let us sketch the main part of Sznitman's proof of non-percolation for u large. He starts with a sequence of scales growing fast, $L_n = L_0^{(1+a)^n}$, with $\ell_n = L_{n+1}/L_n = L_n^a$. Tile $\mathbb{Z}^d$ with L_n-boxes, consider for each such box V_n its neighbors at scale L_n, let $\tilde{V}_n$ denote this bigger box (of side $3L_n$) and then consider the event

There exists a vacant crossing from V_n to $\partial\tilde{V}_n$, with intensity $\leq u_n$,

and mark by $p_n(u_n)$ the probability of that event. One now chooses the intensities $u_n = u_0 \prod_{i=0}^{n}(1 + c/\ell_i^{d-2})^r$ for some c and r tbd. The main estimate is

$$p_n(u_n) \leq C\ell_n^{2(d-1)}/L_n$$

which goes to 0 if the constants are chosen right, while $u_n \to \bar{u} < \infty$.

To see the main estimate, one goes a level down, and notes that such a vacant path must connect a point in V_n through a box at scale $n-1$ on its boundary to a box of scale $n-1$ at distance $3L_n/2$ to $\partial\tilde{V}_n$, while staying inside the "annulus" between these two. To start with, by an induction hypothesis there are not too many such boxes in the u_{n-1} intensity level that are crossed by a vacant path. The union over the number of choices of the intermediate boxes gives the factor $(\ell_n^{d-1})^2$, and now we can fix two such boxes (picture!). Now use the increased intensity to see that the event of having a crossing path between them is of low probability, about $1/L_n$. The actual computation is long and difficult and is beyond these notes.