LECTURE 5: LOOPS

We continues in the setup where V is finite. The material in this lecture is again borrowed, essentially verbatim, from Sznitman's lecture notes.

1. Rooted loops

We introduce the space of rooted loops, which are trajectories with finite lifetime that, instead of going to the cemetery at the lifetime, are then at the starting point. More precisely, let $L_{r,t}$ denote the space of right continuous trajectories $\{\gamma(s)\}_{s\leq t}$ with finitely many jumps, satisfying $\gamma(t) = \gamma(0)$. As long as we keep the information t, we can also think of $\gamma \in L_{r,t}$ as an infinite path by extending it periodically, and then we can define in a natural way the shift θ_s, $s \in \mathbb{R}$.

Let $L_r = \cup_t L_{r,t}$, noting that the spaces $L_{r,t}$ are pairwise disjoint. We refer to a $\gamma \in L_r$ as a *rooted loop of length* $\xi(\gamma)$. For $\gamma \in L_r$, there is a unique t so that $\gamma \in L_{r,t}$, and we set $\xi(\gamma) = t$ as the lifetime of the process. The product σ-algebra on L_r is obtained after linearly stretching all paths to be of length 1; measurability issues are in general never a serious issue in this context and we will simply ignore them.

Let $n(\gamma)$ denote the total number of jumps. Since $\gamma(\xi(\gamma)) = \gamma(0)$, we have $n(\gamma) \in \{0, 2, \ldots\}$. We thus set $N(\gamma) = n(\gamma) \vee 1$, and let T_i denote the successive jump times of the path (no such T_i exist if $N(\gamma) = 1$). We call a loop *trivial* if $N(\gamma) = 1$, and *pointed* if $N(\gamma) > 1$ and $T_N = \xi(\gamma)$, i.e. if the last jump occurs at $\xi(\gamma)$, which by periodicity could be thought of as occuring at time 0. (Due to right continuity, there can't be a jump at time 0.)

Recall the measures $P^t_{x,y}$ that we introduced in the last lecture. We only need now the measures $P^t_{x,x}$, and define

$$\mu_r(B) = \sum_{x\in V}\int_0^\infty P^t_{x,x}(B)\lambda_x\frac{dt}{t} = \sum_{x\in V}\int_0^\infty P^x(\{\tilde{X}_t = x\}\cap B)\frac{dt}{t}.$$

(We will see later the reason for the division by t; in a nutshell, this reflects the uniform choice of the root of the loop.)

The following facts are simple consequences of the definitions.

(1) The lifetime under μ_r has a finite distribution. Indeed, since the total mass of $P^t_{x,x}$ is $r_t(x,x)$, we have

$$\mu_r(\xi(\tilde{X}) \geq s) = \sum_{x\in V}\int_s^\infty r_t(x,x)\lambda_x\frac{dt}{t} \leq \frac{1}{s}\sum_{x\in V}\lambda_x g(x,x) \leq \frac{1}{s}C,$$

for some constant C.

Note that the same argument shows that μ_r is not a finite measure, only a σ-finite one.

(2) For $k > 1$,

$$\mu_r(\tilde{X}_{t_1} = x_1, \ldots, \tilde{X}_{t_k = x_k}, \xi \in (t, t+dt) = r_{t_2-t_1}(x_1,x_2)\lambda_{x_2}\ldots r_{t+t_1-t_k}(x_k,x_1)\lambda_{x_1}\frac{dt}{t}.$$

We bring a proof: for $t > t_k$,

$$P^t_{x,x}(\tilde{X}_{t_1} = x_1, \ldots, \tilde{X}_{t_k = x_k})\lambda_x = r_{t_1}(x, x_1)\lambda_{x_1} r_{t_2 - t_1}(x_1, x_2)\lambda_{x_2} \ldots r_{t_k - t_{k-1}}(x_{k-1}, x_k)\lambda_{x_k} r_{t - t_k}(x_k, x)\lambda_x.$$

Summing over $x \in V$, since $\sum_{x \in V} r_{t-t_k}(x_k, x)\lambda_x r_{t_1}(x, x_1) = r_{t+t_1-t_k}(x_k, x_1)$,
we obtain

$$\sum_{x \in V} P^t_{x,x}(\tilde{X}_{t_1} = x_1, \ldots, \tilde{X}_{t_k = x_k})\lambda_x = r_{t_2 - t_1}(x_1, x_2)\lambda_{x_2} \ldots r_{t_k - t_{k-1}}(x_{k-1}, x_k)\lambda_{x_k} r_{t - t_k}(x_k, x_1)\lambda_{x_1}. \tag{1}$$

From the definition of μ_r, this gives the claim.

(3) For $k = 1$,

$$\mu_r(\tilde{X}_{t_1} = x_1, \xi(\tilde{X}) \in (t, t + dt) = r_t(x_1, x_1)\lambda_{x_1} \frac{dt}{t}.$$

(4) Let $Z_0 = \gamma(0)$ and $Z_i = \gamma(T_i)$, $i = 1, \ldots, N(\gamma)$. (Recall that $Z_N = \gamma(0)$ under μ_r.) Then, for $n > 1$,

$$\mu_r(N = n, Z_i = x_i, i = 0, \ldots, n-1, T_i \in t_i + dt_i, i = 1, \ldots, n, \xi(\gamma) \in t + dt))$$
$$= P(x_0, x_1) \cdots P(x_{n-1}, x_0)\mathbf{1}_{\{0 < t_1 < \ldots < t_n < t\}} \frac{e^{-t}}{t} \prod_{i=1}^n dt_i \cdot dt.$$

and, for $n = 1$,

$$\mu_r(N = 1, Z_0 = x_0, \xi(\gamma) \in t + dt) = \frac{e^{-t}}{t} dt. \tag{2}$$

We skip the proof.

An important property of μ_r is its stationarity, which is summarized in the next lemma.

Lemma 1.1. *With notation as above, with $n > 1$, recalling that $x_n = x_0$,*

$$\mu_r(N = n, Z_i = x_i, i = 0, \ldots, n-1) = \frac{1}{n} \prod_{i=1}^n P(x_{i-1}, x_i). \tag{3}$$

(Recall that $P(x_{i-1}, x_i) = \omega_{x_{i-1}, x_i}/\lambda_{x_i}$, and may sum to less than 1.)

Further, $\mu_r(N = n) = \frac{1}{n}\mathrm{Tr}(P^n)$ (therefore, $\mu_r(N > 1) = -\log(\det(I - P))$), and (recall that we extended path to infinite paths by periodicity), for any $k \in \mathbb{Z}$,

$$\mu_r(N > 1, \{Z_{k+m}\}_{m \in \mathbb{Z}} \in \cdot) = \mu_r(N > 1, \{Z_m\}_{m \in \mathbb{Z}} \in \cdot). \tag{4}$$

Finally, $\mu_r \theta_v^{-1} = \mu_r$, for any $v \in \mathbb{R}$.

Proof. The equality (3) follows from point (4) above by integration. Summing over $\{\mathbf{x}\}$ gives the trace formula. From invariance of the right side of (3) with respect to cyclic shifts of the sequence $(x_0, x_1, \ldots, x_{n-1})$, we obtain (4). So it only remains to prove the continuous-time stationarity. This follows in turn from the following stationarity: Consider $P^t_{x,x}$ as a measure on infinite path (extended by periodicity from $[0, t]$ since $\tilde{X}_0 = \tilde{X}_t$ under this measure). We claim that $\sum_{x \in V} \lambda_x P^t_{x,x} \cdot \theta_v^{-1} = \sum_{x \in V} \lambda_x P^t_{x,x}$. It is enough to consider $v \le t$ since the general case reduces to it by periodicity. But for $v < t$, the claim follows at once from the right side of (1).

Now, recalling that $\mu_r(\cdot) = \int_0^\infty \sum_{x \in V} P^t_{x,x}(\cdot)\lambda_x \frac{dt}{t}$, the claimed stationarity follows. ∎

Similar computations lead to invariance under time reversal, as follows. For $\gamma \in L_r$, let $\check{\gamma} \in L_r$ be such that $\xi(\check{\gamma}) = \xi(\gamma)$ and $\check{\gamma}(s) = \gamma(-s)$ (where we recall that we extended γ by periodicity.)

Lemma 1.2.

$$\mu_r[N > 1, \{Z_{-m}\}_{m=1}^N \in \cdot) = \mu_r[N > 1, \{Z_m\}_{m=1}^N \in \cdot).$$

Further, $\mu_r \cdot (\check{\cdot})^{-1} = \mu_r$.

2. Pointed loops

Recall that a rooted loop is pointed if it jumps at its lifetime. We extend the definition of pointed loops by declaring all trivial loops to be pointed. Let $L_p \subset L_r$ denote the space of pointed loops. Set, for $\gamma \in L_r$ with $N = n$,

$$\sigma_0(\gamma) = T_1(\gamma) + \xi(\gamma) - T_n(\gamma), \sigma_i(\gamma) = T_{i+1} - T_i, i = 1, \ldots, n-1,$$

the time epoch between jumps; note that $\sigma_0 = T_1$ for $\gamma \in L_p$.

Introduce the measure μ_p on L_p in a way similar to μ_r, by

$$\mu_p(N = 1, Z_0 = x_0, \zeta \in t + dt) = e^{-t}\frac{dt}{t},$$

$$\begin{aligned} &\mu_p(N = n, Z_i = x_i, \sigma_i \in s_i + ds_i, i = 0, \ldots, n-1) \\ &= \frac{1}{n} P(x_0, x_1) \cdots P(x_{n-1}, x_0) e^{-\sum_{i=0}^{n-1} s_i} \prod_{i=1}^{n-1} ds_i, \end{aligned}$$

. There are natural maps from L_r to L_p given by $\theta_{T_m(\gamma)}\gamma$, for $m = 1, \ldots, n$. It maps naturally $\mathbf{1}_{\{N=n\}}\mu_r$ to a measure on L_p. As in the case of palm measures, there is a relation between stationary measures in the continuum and (size bias versions of) stationary measures on the discrete, as follows.

Proposition 2.1. *For* $1 \leq m \leq n$,

$$\begin{aligned} &\mathbf{1}_{\{N=n\}}\mu_r \circ \theta_{T_m}^{-1}(Z_i = x_i, \sigma_i \in s_i + ds_i, i = 0, \ldots, n-1) \\ &= \frac{\mathbf{s_{n-m}}}{\sum \mathbf{s_i}} P(x_0, x_1) \cdots P(x_{n-1}, x_0) e^{-\sum_{i=0}^{n-1} s_i} \prod_{i=1}^{n-1} ds_i, \end{aligned} \tag{5}$$

and therefore

$$\mathbf{1}_{\{N=n\}}\mu_r \circ \theta_{T_m}^{-1} = n \frac{\sigma_{n-m}}{\sum \sigma_i} \mathbf{1}_{\{N=n\}}\mu_p.$$

Further, if $F : L_r \to \mathbb{R}$ *is a shift invariant bounded measurable function, i.e.* $F \cdot \theta_v = F$, *then*

$$\int \mathbf{1}_{\{N=n\}} F d\mu_r = \int \mathbf{1}_{\{N=n\}} F d\mu_p. \tag{6}$$

Proof. The proof of (5) involves a change of variables. Recall that

$$\mu_r(Z_i = x_i, T_i \in t_i + dt_i, \xi(\gamma) \in t + dt) = \prod_{i=1}^{n} P(x_{i-1}, x_i) \mathbf{1}_{0 < t_1 < \ldots < t_n < t} \frac{e^{-t}}{t} dt \prod dt_i.$$

Now, make the change of variables

$$(t_1, \ldots, t_n, t) \mapsto (t_1, s_1 = t_2 - t_1, \ldots, s_{n-1} = t_n - t_1 - \sum_{i=1}^{n-2} s_i, s_0 = t - \sum_{i=1}^{n-1} s_i).$$

The new variables satisfy $0 < t_1 < s_0$, $s_i \geq 0, i = 1, \ldots, n-1$. The Jacobian of the transformation is 1, and the s_i are precisely the definition of the σ_i. So, in the original variables, the new density is

$$\prod_{i=1}^{n} P(x_{i-1}, x_i) \mathbf{1}_{\{0<t_1<s_0\}} \frac{e^{-\sum s_i}}{\sum s_i} dt_1 \prod_{i=1}^{n-1} ds_i$$

Integrating over t_1 one picks up a factor s_0. Since the expresions in the density except for this factor are invariant under cyclic transformations of the indices, the shift T_m simply replaces s_0 by s_{n-m}. This proves (5).

To see (6), use the fact that F is invariant to write that $F(\gamma) = \frac{1}{n} \sum_{i=1}^{n} F(\theta_{T_i} \gamma)$, and use the above expression. Using (5), one picks the factor

$$\frac{1}{n} \frac{\sum \sigma_i}{\sum \sigma_i} = \frac{1}{n},$$

as needed. □

The measures μ_r and μ_p possess also good restriction properties. Let $A \subset V$. Let $L_{r,A}$ and $L_{p,A}$ denote the sets of rooted (pointed) loops that stay in A up to their lifetimes. It is not hard to check that

$$\mathbf{1}_{\{L_{r,A}\}} \mu_r = \mu_{r,A}, \quad \mathbf{1}_{\{L_{p,A}\}} \mu_r = \mu_{p,A} \tag{7}$$

where $\mu_{r,A}$ and $\mu_{p,A}$ are defined as mu_r, μ_p on the restricted graph A with weights $\lambda_x^A = \lambda_x + \sum_{y \notin A} \omega_{x,y}$.

Exercise 1. *Prove* (7).

3. Local times for loops

The definition of local times for loops is analogous to that for paths, we simply set

$$L_x = L_x(\gamma) = \frac{1}{\lambda_x} \int_0^{\xi(\gamma)} \mathbf{1}_{\{\gamma_s = x\}} ds.$$

By definition, the local time is invariant under time shifts, and under time reversals, and therefore has the same "law" under μ_r or μ_p.

The next lemma gives a Laplace trasform for the law of the local time. We saw similar computations in the context of Dynkin's isomorphism.

Lemma 3.1. *For any* $\eta > 0$*,*

$$\int (1 - e^{-\eta L_x}) \mathbf{1}_{\{N=1\}} d\mu_r = \log \left(1 + \frac{\eta}{\lambda_x} \right), \tag{8}$$

and more generally, for $F \geq 0$*,*

$$\int (1 - e^{-\sum_x F(x) L_x}) d\mu_r = \log \det(I + GF) = -\log \left(\frac{\det G^F}{\det G} \right). \tag{9}$$

Proof. If $N = 1$ then the loop is a trivial loop and $L_x = \xi/\lambda_x$ on the event that $\gamma_0 = x$. By (2) it then follows that

$$\int (1 - e^{-\eta L_x}) \mathbf{1}_{\{N=1\}} d\mu_r = \int_0^\infty (1 - e^{-\eta t/\lambda_x}) \frac{e^{-t}}{t} dt,$$

from which the conclusion (8) follows after using the trick

$$\frac{1}{t}dt = \int_0^\infty e^{-ts}ds$$

and Fubini.

To see (9), we use the definitions for $P^t_{x,x}$, and recall that

$$E^t_{x,x}(1 - e^{-\sum_{y\in V} F(y)L_y}) = P_x(\tilde{X}_t = x)/\lambda_x - E_x(\mathbf{1}_{\{\tilde{X}_t = x\}} e^{-\int_0^t (F/\lambda)(\tilde{X}_s)ds})$$
$$= \frac{1}{\lambda_x}\left(e^{t(P-I)}\mathbf{1}_x - e^{t(P-I-F/\lambda)}\mathbf{1}_x\right)$$

where the Feynman-Kac formula was used in the last step. Summing over x and integrating with respect to the weight dt/t we obtain that

$$\int (1 - e^{-\sum_{y\in V} F(y)L_y(\gamma)})d\mu_r = \int_0^\infty \mathrm{Tr}(e^{t(P-I)} - e^{t(P-I-F/\lambda)})\frac{dt}{t}. \tag{10}$$

Recall that the eigenvalues of P (a self adjoint matrix in $L^2(d\lambda)$), denoted λ_i^P, satisfy

$$\max_i |\lambda_i^P| < 1. \tag{11}$$

By Weyl's inequalities[1], with $\lambda_i^{P-F/\lambda}$ denoting the eigenvalues of $P - F/\lambda$, we also have that $\max_i |\lambda_i^{P-F/\lambda}| < 1$ if $\|F\|$ is small enough. Therefore, we can expand the exponential and obtain that

$$\begin{aligned}\mathrm{Tr}(e^{t(P-I)} - e^{t(P-I-F/\lambda)}) &= e^{-t}\sum_{k\geq 1}\frac{t^k}{k!}\mathrm{Tr}(P^k - (P-F/\lambda)^k)\\ &= e^{-t}\sum_i\sum_{k\geq 1}\left(\frac{(\lambda_i^P t)^k}{k!} - \frac{(\lambda_i^{P-F/\lambda}t)^k}{k!}\right).\end{aligned}$$

Now, note that for $|a| < 1$,

$$\int_0^\infty \sum_{k\geq 1}\frac{t^k a^k}{k!}\frac{e^{-t}dt}{t} = \sum_{k\geq 1}\frac{a^k}{k!}\int_0^\infty t^{k-1}e^{-t}dt = \sum_{k\geq 1}\frac{a^k(k-1)!}{k!} = -\log(1-a).$$

Therefore, from (10) we obtain that

$$\begin{aligned}\int (1 - e^{-\sum_{y\in V} F(y)L_y(\gamma)})d\mu_r &= -\sum_i[\log(1-\lambda_i^P) - \log(1-\lambda_i^{P-F/\lambda})]\\ &= \log\det(I - P + F/\lambda) - \log\det(I-P) \qquad (12)\\ &= \log\det(I + (I-P)^{-1}F/\lambda) = \log\det(I + GF), \qquad (13)\end{aligned}$$

where in the last equality we used that $(I-P)^{-1}\lambda^{-1} = G$. Recall from Lecture 4 that $(I+GF)^{-1} = (-L+F)^{-1}(-L)$ if $\|F\|$ is small, and that $L = \lambda(P-I)$. Thus,

$$(I+GF)^{-1} = (\lambda(I-P+F/\lambda))^{-1}(\lambda(I-P))$$

and therefore

$$\det(I+GF)^{-1} = \det(\lambda^{-1})\det(\lambda)\det(I-P+F/\lambda)^{-1}\det(I-P) = \det(G_F)/\det(G).$$

[1]See Remark 3.2 below for an alternative, shorter argument

It follows from the last display and (13) that

$$\int (1 - e^{-\sum_{y\in V} F(y)L_y(\gamma)}) d\mu_r = \log(\det(G)/\det(G_F)). \tag{14}$$

To extend the proof from $\|F\|$ small to general bounded positive F, we argue by the general principle that if the formula makes sense, it should extend. To begin, note that

$$0 < \beta \mapsto \int (1 - e^{-\beta \sum_{y\in V} F(y)L_y(\gamma)}) d\mu_r \tag{15}$$

is monotone increasing. It is also bounded for small $\beta > 0$ and finite for all $\beta > 0$ since $1 - e^{-(a+b)} \le (1 - e^{-a}) + (1 - e^{-b})$ for $a, b > 0$ (differentiate in b to see that).

We next claim that the function in (15) is actually analytic on the right half plane of $\mathbb{C}$. To see that it is enough to show that the integrand is dominated by an integrable function. Note that the integrand is dominated by 2 (if $\Re\beta \ge 0$) and, since $\mu_r(\{N > 1\}) < \infty$, see Lemma 1.1, we only have to worry about the event $N = 1$. But, on this event,

$$|1 - e^{-\beta \sum_y F(y)L_y(\gamma)}| = |1 - e^{-\beta F(\gamma_0)\xi(\gamma)}| \le |\beta| \|F\|_\infty \xi(\gamma),$$

where we uses that $\Re\beta$. Since $\xi(\gamma)\mathbf{1}_{\{N=1\}}$ is integrable under μ_r, we obtain the desired uniform integrability on compact subsets of $\mathbb{C}_R := \{z \in \mathbb{C} : \Re z > 0\}$. It follows that (15) is analytic on $\mathbb{C}_R$. On the other hand, taking first $F > 0$, write $I = F^{-1/2}F^{1/2}$ and use the formula $\det(AB) = \det(BA)$ to get

$$\det(I + \beta GF) = \det(I + \beta\sqrt{F}G\sqrt{F}),$$

noting that the formula remains true even if certain entries of F vanish by continuity. Now, $\sqrt{F}G\sqrt{F}$ is a symmetric matrix, with nonnegative eigenvalues, and the analyticity of the log-determinant on $\mathbb{C}_R$ follows. Thus, the equality (14) extends from $\beta \in (0, \beta_0)$ to $\beta > 0$, and in particular to $\beta = 1$. □

Note that substituting in (14) $F = 1_x$ gives

$$\int (1 - e^{-\eta L_x}) d\mu_r = \log(1 + \eta g(x, x)), \tag{16}$$

since $\det(I + GF) = \det(I + F^{1/2}GF^{1/2}) = 1 + \eta g(x, x)$.

Remark 3.2. A shorter argument, that avoids the analytic continuation, was suggested by Liying Li and is as follows. From (11) one has that $1 - \lambda_i^P > 0$. Similarly, $1 - \lambda_i^{P-F/\lambda} > 0$. Since

$$\mathrm{Tr}(e^{t(P-I)} - e^{t(P-I-F/\lambda)}) = \sum_i (e^{-(1-\lambda_i^P)t} - e^{-(1-\lambda_i^{P-F/\lambda})t}.$$

Since

$$\int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt = \log(b/a),$$

and the number of eigenvalues is finite, the integrability and hence the conclusion follow.

Exercise 2. *Use the restriction property to show that for $A \subset V$ and F as above,*

$$\int_{\gamma_\cdot \in A} (1 - e^{-\sum_x F(x)L_x}) d\mu_r = \log\det(I + G^A F),$$

where G^A is the green function of the random walk killed at exiting A.

Exercise 3. *(*) Prove the following variant of Lemma 3.1, using (8) of lecture 4: for $F : V \to \mathbb{R}_+$,*

$$E_{x,y}(e^{-\sum F(z)L^z}) = (F - L)^{-1}(x, y).$$

4. Unrooted loops

We have so far considered loops with a distinguish point, the root. Sometimes it is better to forget that distinguished vertex. We can introduce an equivalence relation $\sim$ on L_r by declaring two loops equivalent if their lifetime is equal and one it a shift of the other. Let L^* be the space of equivalent classes of loops, endow it with the σ-algebra induced from L_r vi the map $\pi^* : L_r \to L^*$, and define $\mu^* = \mu_r(\pi^*)^{-1}$. (Since equivalence classes are translation invariant, we can use μ_r or μ_p in the last definition).

We do one calculation with respect to the measures μ^*.

Definition 4.1. *A unit weight is a (measurable) function $T : L_r \to \mathbb{R}_+$ satisfying*

$$\int_0^{\xi(\gamma)} T(\theta_v \gamma) dv = 1.$$

An example (besides the trivial one $T_0(\gamma) = 1/\xi(\gamma)$) is $T_1(\gamma) = T_0(\gamma)\mathbf{1}_{\gamma.\cap\{x\}=\emptyset} + \mathbf{1}_{\gamma(0)=x}/(\lambda_x L^x)$.

Lemma 4.2. *If T is a unit weight and $F : L^* \to \mathbb{R}_+$ measurable then*

$$\int F d\mu^* = \sum_{x\in V} F \circ \pi^*(\gamma)T(\gamma)dP_{x,x}(\gamma)\lambda_x. \tag{17}$$

Note first that the choice of unit weight does not affect the left side in (17), and thus the right side does not depend on the unit weight chose. This can be useful in computations.

Proof. By definition, the right side in (17) equals

$$\int_0^\infty \sum_{x\in V} \lambda_x E^t_{x,x}(F \circ \pi^* \cdot T)dt.$$

Using stationarity of $\sum_x \lambda_x P^t_{x,x}$ (as in the proof of Lemma 1.1, we have

$$\sum_{x\in V} \lambda_x E^t_{x,x}(F \circ \pi^* \cdot T) = \frac{1}{t}\int_0^t dv(\sum_{x\in V} \lambda_x E^t_{x,x}(F \circ \pi^* \circ \theta_v) \cdot (T \circ \theta_v).$$

Using that $(F \circ \pi^* \circ \theta_v) = F \circ \pi^*$, we can perform the integration over v and use the fact that T is a unit weight (note that under $E^t_{x,x}$, $\xi(\gamma) = t$) to obtain that the expression in the last display equals

$$\int_0^\infty \sum_{x\in V} \lambda_x E^t_{x,x}(F \circ \pi^*)\frac{dt}{t} = \int_{L_r} F \circ \pi^* d\mu_r = \int_{L^*} F d\mu^*.$$

□

Using the weight T_1, a consequence of the last lemma is that

$$\mathbf{1}_{x\in\gamma^*} d\mu^* = \frac{1}{L_x} dP_{x,x} \circ (\pi^*)^{-1},$$

that is, the meausre μ^*, on the event that a point x was visited, can be obtained from $P_{x,x}$ by a random rescale (corresponding to local time) together with “forgetting the root”. By invariance of L_x to shifts, this is the same as the equality

$$\mathbf{1}_{x\in\gamma^*} L_x d\mu^* = dP_{x,x} \circ (\pi^*)^{-1},$$

Integrating, we obtain that $\int_{x\in\gamma^*} L_x d\mu^* = g(x,x)$.