LECTURE 4: DYNKIN ISOMORPHISM THEOREMS

We continues in the setup where V is finite, with a distinguished vertex x_0. For $A \subset V$, we recall the hitting time

$$\tau_A = \inf\{t \geq 0 : X_t \in A\}$$

and the modified hitting time

$$T_A = \inf\{t > 0 : X_t \in A, X_s \neq X_0 \text{ for some } s \in (0,t)\}.$$

Recall also that if we let $\{Y_n\}$ denote the discrete Markov chain corresponding to $\{X_t\}$, then for $x, y \in U = E \setminus x_0$ and the Markov chain killed at hitting x_0,

$$g(x,y) = \int_0^\infty P^x(X_t = y)dt = \frac{1}{\lambda_y} \sum_{n=0}^\infty P^x(Y_n = y)\,.$$

Remark 0.1. We could also work with an excess killing at each (or some vertex), i.e. take $\lambda_x > \sum W_{x,y}$ for at least one x. This point of view will be sometimes useful, in particular when deriving Dynkin's isomorphism. All the formula we derived are valid for that case.

1. Conditioning the GFF

One of the nice properties of the GFF is its (spatial) Markov properties, i.e. its behavior under conditioning. We develop these in this section.

1.1. **Gaussian preliminaries.** We recall the following facts concerning Gaussian vectors.

Lemma 1.1. *If $\mathbf{Z} = (X, \mathbf{Y})$ is a centered Gaussian vector then $\hat{X}_Y := E[X|\mathbf{Y}]$ is a Gaussian random variable, and $\hat{X}_Y = T\mathbf{Y}$ for a deterministic matrix T. If $\det(R_{YY}) \neq 0$ then $T = R_{XY}R_{YY}^{-1}$.*

Proof. Assume first that $\det(R_{YY}) \neq 0$. Set $W = X - T\mathbf{Y}$. Then, since $T\mathbf{Y}$ is a linear combination of entries of $\mathbf{Y}$ and since $\mathbf{Z}$ is Gaussian, we have that $(W, \mathbf{Y})$ is a (centered) Gaussian vector. Now,

$$E(W\mathbf{Y}) = R_{XY} - TR_{YY} = 0\,.$$

Hence, W and $\mathbf{Y}$ are independent. Thus, $E[W|\mathbf{Y}] = EW = 0$, and the conclusion follows from the linearity of the conditional expectation.

In case $\det(R_{YY}) = 0$ and $\mathbf{Y} \neq 0$, let Q denote the projection to range(R_{YY}), a subspace of dimension $d \geq 1$. Then $\mathbf{Y} = Q\mathbf{Y} + Q^\perp \mathbf{Y} = Q\mathbf{Y}$ since $\mathrm{Var}(Q^\perp \mathbf{Y}) = 0$. Changing bases, one thus finds a matrix B with $n-d$ zero rows so that $\mathbf{Y} = \hat{Q}B\mathbf{Y}$ for some matrix $\hat{Q}$, and the covariance matrix of the d dimensional vector of non-zero entries of $B\mathbf{Y}$ is non-degenerate. Now repeat the first part of the proof using the non-zero entries of $B\mathbf{Y}$ instead of $\mathbf{Y}$. ∎

1.2. **Conditioning via the Green function representation.** The Green function representation allows one to give probabilistic representation for certain conditionings. Recall that $\{\phi_x\}_{x\in V}$ denotes the GFF associate with g. Let $A \subset U$ and set $\phi^A = E[\phi|\phi_x, x \in A]$. Of course, $\phi^A_x = \sum_{z\in A} a(x,z)\phi_z$ for some matrix $\{a(x,z)\}$. We clearly have that for $x \in A$, $a(x,y) = \mathbf{1}_{x=y}$. On the other hand, because g_A (the restriction of g to A) is non-degenerate, we have from Lemma 1.1, that for $x \notin A$, $a(x,y) = \sum_{w\in A} g(x,w)g_A^{-1}(w,y)$. It follows that for any $y \in A$, $a(x,y)$ (as a function of $x \notin A$) is harmonic, i.e. $\sum P(x,w)a(w,y) = a(x,y)$ for $x \notin A$. Hence, a satisfies the equations

$$\left\{ \begin{array}{ll} (I-P)a(x,y) = 0, & x \notin A\,, \\ a(x,y) = \mathbf{1}_{\{\mathbf{x=y}\}}\,, & x \in A\,. \end{array} \right. \tag{1}$$

By the maximum principle, the solution to (1) is unique. On the other hand, one easily verifies that the function $\hat{a}(x,y) = P^x(\tau_A < \tau, S_{\tau_A} = y)$ satisfies (1). Thus, $a = \hat{a}$.

The difference $\hat{\phi}^A = \phi - \phi^A$ is independent of $\{\phi_x\}_{x\in A}$ (see the proof of Lemma 1.1). What is maybe surprising is that $\hat{\phi}^A$ can also be viewed as a GFF.

Lemma 1.2. *$\hat{\phi}^A$ is the GFF associated with $(P, x_0 \cup A)$.*

(Here, we mean that we identify all vertices in $x_0 \cup A$.)

Proof. Let g_A denote the Green function restricted to A (i.e., with $\tau_A \wedge \tau$ replacing τ). By the strong Markov property we have

$$g(x,y) = \sum_{y'\in A} a(x,y')g(y',y) + g_A(x,y)\,, \tag{2}$$

where the last term in the right side of (2) vanishes for $y \in A$. On the other hand,

$$E(\hat{\phi}^A_x \hat{\phi}^A_{x'}) = g(x,x') - E(\phi_x \phi^A_{x'}) - E(\phi_{x'}\phi^A_x) + E\phi^A_x\phi^A_{x'}\,.$$

Note that

$$E\phi_x\phi^A_{x'} = \sum_{y\in A} a(x',y)g(x,y) = g(x',x) - g_A(x',x)$$

while

$$\begin{aligned} E\phi^A_x\phi^A_{x'} &= \sum_{y,y'\in A} a(x,y)a(x',y)g(y,y') \\ &= \sum_{y'\in A} a(x,y')g(x',y') = g(x,x') - g_A(x,x')\,. \end{aligned}$$

Substituting, we get $E(\hat{\phi}^A_x \hat{\phi}^A_{x'}) = g_A(x,x')$, as claimed. $\square$

From the hitting time representation one sees that if $A \subset V$ then the law of $\{\phi_x\}_{x\in A}$ conditioned on $\sigma(\phi_x : x \in A^c)$ depends only on $\sigma(\phi_x : x \in \partial A)$. This is the spatial Markov property alluded to above.

2. Dynkin paths measures

Much of the content of this and the next sections is taken from the published lecture notes of A.-S. Sznitman.

Let $\tilde{X}_t$ denote the unit-rate continuous time random walk associated with W, that is, take the discrete time random walk $\{Y_n\}$ and make jumps with rate 1. Note that

$$R_t f(x) := E^x f(\tilde{X}_t) = \sum_{n\geq 0} e^{-t}\frac{t^n}{n!}P^n f(x) = e^{t(P-I)}f(x)$$

is a self-adjoint bounded operator on $L^2(d\lambda)$.

Exercise 1. *Check that indeed R_t is a self adjoint operator.*

Define the density $r_t(x,y) = \frac{1}{\lambda_y}(R_t\mathbf{1}_y)(x)$.

We add to the process killing, either at hitting of a particular vertex or at a certain rate at each vertex. That is, we let $\lambda_x \geq \sum_y W_{x,y}$. It will be convenient to speak of right continuous paths γ with finitely many jumps and a lifetime $\xi(\gamma)$. Often, we will take $\xi(\gamma) = \tau(x_0)$.

We consider now the measure $P^t_{x,y}$ on paths of length t, given by

$$P^t_{x,y} = \mathbf{1}_{\tilde{X}_t = y}\frac{P^x}{\lambda_y}.$$

(We always think of the lifetime of the path as t, and whenever convenient extend the path to an infinite path by making it equal to a cemetery point for $s > t$. With this extension, the path is not necessarily right continuous at t.) Note that $r_t(x,y)$ is the total mass of $P^t_{x,y}$. Then, we introduce the measure $P_{x,y}$ on paths of arbitrary (finite) length by setting

$$P_{x,y} = \int_0^\infty P^t_{x,y}dt.$$

The total mass of $P_{x,y}$ is $\int_0^\infty r_t(x,y) = g(x,y)$. The following are simple properties of $P_{x,y}$.

Lemma 2.1. *For any $0 < t_1 < t_2 \ldots < t_n$, $x_1, \ldots, x_n \in V$ and setting $x_0 = x$, we have*

$$\begin{aligned} (3) \qquad P_{x,y}(\tilde{X}_{t_i} = x_i, i = 1,\ldots,n) &= P^x(\tilde{X}_{t_i} = x_i, i = 1,\ldots,n)g(x_n,y) \\ &= \left(\prod_{i=1}^n \lambda_{x_i} r_{t_i - t_{i-1}}(x_{i-1},x_i)\right)\cdot g(x_n,y) \end{aligned}$$

Further, if $K \subset V$ and ξ is the lifetime of the process then for any set B measurable wrt $\sigma(\tilde{X}_s, s \leq \xi)$, we have

$$(4) \qquad E^x(g(\tilde{X}_{\tau_K},y)\mathbf{1}_{B\cap\tau_K<\infty}) = P_{x,y}(B\cap\{\tau_K \leq \xi\}).$$

Finally, if T_i are the successive jumps of $\tilde{X}$ and N denotes the number of jumps of $\tilde{X}$ strictly before the lifetime ξ then, for $n \geq 1$,

$$\begin{aligned} (5) \qquad & P_{x,y}(N = n, \tilde{X}_{T_i} = x_n, T_i \in t_i + dt_i, i = 1,\ldots,n, \xi \in t + dt) \\ &= \frac{W_{x_n,y}\cdot\prod_{i=1}^n W_{x_{i-1},x_i}}{\lambda_y\cdot\prod_{i=1}^n \lambda_{x_{i-1}}}\delta_{x_n,y}\mathbf{1}_{0<t_1<\ldots<t_n<t}e^{-t}\cdot\prod_{i=1}^n dt_i. \end{aligned}$$

The last point of the lemma gives the interpretation of $P_{x,y}$ as the un-normalized version of the law of the path conditioning on y being the last state visited.

Proof. We begin with the proof of (3). By definition,

$$\begin{aligned} P_{x,y}(\tilde{X}_{t_i} = x_i, i = 1, \ldots, n) &= \int_0^\infty P^t_{x,y}(\tilde{X}_{t_i} = x_i, i = 1, \ldots, n) \\ &= \int_{t_n}^\infty P^x(\tilde{X}_{t_i} = x_i, i = 1, \ldots, n, \tilde{X}_t = y))\frac{dt}{\lambda_y}, \end{aligned}$$

where in the last equality we used the fact that necessarily $t > t_n$. Applying now the Markov at t_n, we obtain that the last term equals

$$\int_{t_n}^\infty E^x(\mathbf{1}_{\tilde{X}_{t_i}=x_i, i=1,\ldots,n} r_{t-t_n}(x_n, y)dt = P^x(\tilde{X}_{t_i} = x_i, i = 1, \ldots, n)g(x_n, y)),$$

which is the first equality in (3). The second equality is just the Markov property of P^x.

Turning to the proof of (4), we have

$$P_{x,y}(B \cap \tau_K \le \xi) = \int_0^\infty P^t_{x,y}(B \cap \{\tau_K \le \xi\}) = \int_0^\infty P^x(B \cap \{\tau_K \le t\} \cap \{\tilde{X}_t = y\})\frac{dt}{\lambda_y},$$

where the second equality is due to the definition. Applying the Markov property at τ_K, the last expression equals

$$\int_0^\infty E^x(\mathbf{1}_{B\cap\{\tau_K \le t\}} \cdot r_{t-\tau_K}(\tilde{X}_{\tau_K}, y)dt = E^x(\mathbf{1}_{B\cap\{\tau_K<\infty\}} \cdot g(\tilde{X}_{\tau_K}, y)dt.$$

We leave the proof of (5) as an exercise. $\square$

The following are analogues of Kac's moment formulas, for the measures $P_{x,y}$.

Lemma 2.2. *(1) Let $F : V \to \mathbb{R}$ and fix $n \ge 0$. Then,*

$$E_{x,y}\left((\int_0^\infty F(\tilde{X}_s)ds)^n\right) = n! \sum_z (QF)^n(x, z)g(z, y), \tag{6}$$

where $QF(a, b) = g(a, b)F(b)\lambda_b$.

(2) Let $L^x = L^x(\infty)$ (recalling that the process $\tilde{X}$ is killed a.s. at a finite time). Let $x_i \in V$, $i = 1, \ldots, n$. Then, with S_n denoting the set of permutations on n letters,

$$E_{x,y}\left(\prod_{i=1}^n L^{x_i}\right) = \sum_{\sigma\in S_n} g(x, x_{\sigma(1)})g(x_{\sigma(n)}, y) \prod_{i=1}^{n-1} g(x_{\sigma(i)}, x_{\sigma(i+1)}). \tag{7}$$

Similarly, if $\|Q|F|\|_\infty < 1$ then

$$E_{x,y}(\exp(\sum_{z\in V} F(z)L^z)) = \sum_{z\in V}(I - GF)^{-1}(x, z)g(z, y). \tag{8}$$

Proof. Consider $F_i : V \to \mathbb{R}$, $i = 1, \ldots, n$. Write

$$E_{x,y}(\prod_{i=1}^n \int_0^\infty F_i(\tilde{X}_s)ds) = \sum_{\sigma\in S_n} \int_{0<s_{\sigma(1)}<\ldots<s_{\sigma(n)}} E_{x,y}F_1(\tilde{X}_{s_1}) \cdots F_n(\tilde{X}_{s_n})ds_1 \cdots ds_n.$$

Applying (3), the last expression equals

$$\sum_{\sigma\in S_n}\int_{0<s_{\sigma(1)}<\ldots<s_{\sigma(n)}}\sum_{x_i} r_{s_{\sigma(1)}}(x,x_1)F_{\sigma(1)}(x_1)\lambda_{x_1}$$
$$\cdot\prod_{i=1}^{n-1} r_{s_{\sigma(i+1)}-s_{\sigma(i)}}(x_i,x_{i+1})F_{\sigma(i+1)}(x_{i+1})\lambda_{x_{i+1}}g(x_n,y)ds_{\sigma(1)}\cdots ds_{\sigma(n)}.$$

Integrating over $s_{\sigma(n)}$ gives for the last factor $\sum_{x_n} g(x_{n-1},x_n)F_{\sigma(n)}(x_n)g(x_n,y)$, and iterating this gives

$$E_{x,y}(\prod_{i=1}^n\int_0^\infty F_i(\tilde{X}_s)ds)=\sum_{\sigma\in S_n} QF_{\sigma(1)}\cdots QF_{\sigma(n)}g(\cdot,y)(x). \tag{9}$$

Choosing $F_i=F$ gives then (6).

To see (7), we take $F_i=\lambda_{x_i}^{-1}\mathbf{1}_{x_i}$. In this case, $QF_i(x,z)=g(x,x_i)\mathbf{1}_{z=x_i}$, and substituting in (9) gives that

$$E_{x,y}(\prod_{i=1}^n L^{x_i})=\sum_{\sigma\in S_n} g(x,x_{\sigma(1)})g(x_{\sigma(1)},x_{\sigma(2)})\cdots g(x_{\sigma(n-1)},x_{\sigma(n)})g(x_{\sigma(n)},y),$$

as claimed.

Finally, to obtain (8), we have by expanding the exponential function that

$$E_{x,y}(\exp(\sum_{z\in V}F(z)L^z))=E_{x,y}\left(\sum_n\frac{1}{n!}\left(\int_0^\infty\frac{F}{\lambda}(\tilde{X}_s)ds\right)^n\right).$$

Because $\|G|F|\|_\infty<1$ one can apply dominated convergence and interchange the order of summation and $E_{x,y}$. Applying (9) then gives that the last expression equals

$$\sum_n (GF)^n g(\cdot,y)(x),$$

which equals the claimed expression since $\sum_n (GF)^n=(I-GF)^{-1}$. □

3. Dynkin isomorphism

Recall that we abbreviated $L^x=L^x_\infty$. The goal of this section is to prove the following.

Theorem 3.1 (Dynkin)**.** *The field $(L^x+\frac{1}{2}\phi_x^2)_{x\in V}$ under $P_{x,y}\otimes\mathbb{P}$ has the same law as $(\frac{1}{2}\phi_x^2)_{x\in V}$ under $\phi_x\phi_y\mathbb{P}$.*

Note that the measures in Theorem 3.1 are not probability measures (and $\phi_x\phi_y\mathbb{P}$ is not even a positive measure!). What we mean by law is that for any test function F,

$$E_{x,y}\otimes\mathbb{E}(F((L^x+\frac{1}{2}\phi_x^2)_{x\in V}))=\mathbb{E}(\phi_x\phi_y F((\frac{1}{2}\phi_x^2)_{x\in V})). \tag{10}$$

Proof. The proof is not hard but is somewhat mysterious, as the heart of it is a computation, which appears in the proof of the following equality of Laplace transforms.

Proposition 3.2. *There exists $\delta > 0$ such that for any $F : V \to \mathbb{R}$ with $\|F\|_\infty < \delta$,*
(11)
$$E_{x,y} \otimes \mathbb{E}\left(\exp\left(\sum_{z\in V} F(z)(L^z + \frac{1}{2}\phi_z^2)\right)\right) = \mathbb{E}\left(\phi_x\phi_y \exp\left(\sum_{z\in V} F(z)(\frac{1}{2}\phi_z^2)\right)\right).$$

Assuming Proposition 3.2, we show how Theorem 3.1 follows. Let F be given. Applying Cauchy-Shwartz, we find that for some $\epsilon > 0$, $\exp(\pm\epsilon \sum_{z\in V} F(z)\phi_z^2/2)$ is integrable under $|\phi_x\phi_y|\mathbb{P}$, and therefore, because of (11), so is $\exp(\pm\epsilon \sum_{z\in V} F(z)(L^z + \frac{1}{2}\phi_z^2))$ under $P_{x,y} \otimes \mathbb{P}$. Let $D = (-\epsilon, \epsilon) + i\mathbb{R} \subset \mathbb{C}$. It follows from this integrability that the functions $f_1, f_2 : D \to \mathbb{C}$ given by

$$f_1(\theta) = E_{x,y} \otimes \mathbb{E}(\exp(\theta \sum_{z\in V} F(z)(L^z + \frac{1}{2}\phi_z^2)), f_2(\theta) = \mathbb{E}(\phi_x\phi_y \exp(\theta \sum_{z\in V} F(z)\phi_z^2/2))$$

are analytic on D. Since they are equal for $\theta \in \mathbb{R} \cap D$, they are equal on D, and in particular $f_1(i\eta) = f_2(i\eta)$ for $\eta \in \mathbb{R}$. The equality of the characteristic functions implies then the equality of the laws. □

Proof of Proposition 3.2. Recall from (8) that for F small enough in norm,

$$E_{x,y}(\exp(\sum_{z\in V} F(z)L^z)) = (I - GF)^{-1}g(\cdot, y). \tag{12}$$

Define the quadratic form

$$\mathcal{E}^F(\varphi, \varphi) = \mathcal{E}(\varphi, \varphi) - \sum_{z\in V} F(z)\varphi_z^2 = \langle -L\varphi, \varphi\rangle - \langle F\varphi, \varphi\rangle = \langle(-L - F)\varphi, \varphi.$$

Since the form $\mathcal{E}$ is strictly positive (recall the spectral gap!), there is a δ so that if $\|F\|_\infty < \delta$ then $\mathcal{E}^F$ is stritcly positive definite. Let $\mathbb{P}^F$ denote the Gaussian law with the quadratic form given by $\mathcal{E}^F$. The covariance under $\mathbb{P}^F$ is given by the matrix

$$(-L - F)^{-1} = (I - GF)^{-1}(-L)^{-1} = (I - GF)^{-1}G. \tag{13}$$

Therefore, using (12),

$$\mathbb{P}^F(\phi_x\phi_y) = (I - GF)^{-1}g(\cdot, y)(x) = E_{x,y}(\exp(\sum_{z\in V} F(z)L^z)). \tag{14}$$

But

$$\mathbb{P}^F = \mathbb{P} \cdot \exp(-\frac{1}{2}\sum_{z\in V} \phi_z^2 F(z)) \cdot \mathbb{E}(\exp(\frac{1}{2}\sum_{z\in V} V(z)\phi_z^2))^{-1}, \tag{15}$$

and therefore, using (14),

$$\begin{aligned}
\mathbb{E}(\phi_x\phi_y \exp(\frac{1}{2}\sum_{z\in V} F(z)\phi_z^2)) &= \mathbb{P}^F(\phi_x\phi_y)\mathbb{E}(\exp(\frac{1}{2}\sum_{z\in V} F(z)\phi_z^2)) \\
&= E_{x,y}(\exp(\sum_{z\in V} F(z)L^z))\mathbb{E}(\exp(\frac{1}{2}\sum_{z\in V} V(z)\phi_z^2)) \\
&= E_{x,y} \otimes \mathbb{E}(\exp(\sum_{z\in V} F(z)(L^z + \frac{1}{2}\phi_z^2))), \qquad (16)
\end{aligned}$$

as claimed. □

4. Eisenbaum's version of the isomorphism theorem

We would like to replace the measures $P_{x,y}$ by the measures P_x. Kac's moment formula takes then the following form, compare with Lemma 2.2.

Lemma 4.1. *(1) Let $F : V \to \mathbb{R}$ and fix $n \geq 0$. Then,*

$$E^x\left(\left(\int_0^\infty F(\tilde{X}_s)ds\right)^n\right) = n! \sum_{z\in V}(QF)^n(x,z). \tag{17}$$

(2) Let $x_i \in V$, $i = 1, \dots, n$. Then,

$$E^x\left(\prod_{i=1}^n L^{x_i}\right) = \sum_{\sigma\in S_n} g(x, x_{\sigma(1)}) \prod_{i=1}^{n-1} g(x_{\sigma(i)}, x_{\sigma(i+1)}). \tag{18}$$

Similarly, if $\|Q|F|\|_\infty < 1$ then

$$E^x(\exp(\sum_{z\in V} F(z)L^z)) = \sum_{z\in V}(I - GF)^{-1}(x,z). \tag{19}$$

We now have the following version of the isomorphism theorem.

Theorem 4.2 (Eisenbaum)**.** *Fix $x \in V, s > 0$. Then, the field $(L^z + \frac{1}{2}(\phi_z + s)^2)_{z\in V}$ under $P^x \otimes \mathbb{P}$ has the same law as $(\frac{1}{2}(\phi_z + s)^2)_{z\in V}$ under $(1 + \phi_x/s)\mathbb{P}$.*

(One would like to be able to force $s \to 0$, and recover GRK2. However, this is not obvious directly because $(1 + \phi_x/s)\mathbb{P}$ does not force $\phi_x = 0$ due to the non-positivity of the measure. But one could look at $\mathbb{P}$ *conditioned* on $\phi_x \in (s^2, \epsilon s)$, say, and consider the limit as $s \to 0$ followed by $\epsilon \to 0$.)

Proof. It is enough to show that for small $F : V \to \mathbb{R}$,

$$\begin{aligned} & E^x \otimes \mathbb{P}\left(\exp\left(\sum_{z\in V} F(z)\left(L^z + \frac{(\phi_z + s)^2}{2}\right)\right)\right) \\ &= \mathbb{E}\left(\left(1 + \frac{\phi_x}{s}\right)\exp\left(\sum_{z\in V} F(z)\frac{(\phi_z + s)^2}{2}\right)\right). \end{aligned} \tag{20}$$

By (19) we have

$$E^x\left(\exp\left(\sum_{z\in V} F(z)L^z\right)\right) = \sum_{z\in V}(I - GF)^{-1}(x,z). \tag{21}$$

Recall $\mathbb{P}^F$, see (15). We turn to computing the right side in (20). Because $\mathbb{P}^F$ absorbs the quadratic term $\sum F(z)\phi_z^2$ in the exponent, we have

$$\begin{aligned} & \frac{\mathbb{E}\left((1 + \phi_x/s)\exp\left(\frac{1}{2}\sum_{z\in V} F(z)(\phi_z + s)^2\right)\right)}{\mathbb{E}\left(\exp\left(\frac{1}{2}\sum_{z\in V} F(z)(\phi_z + s)^2\right)\right)} \\ &= 1 + \frac{\mathbb{E}\left(\phi_x \exp\left(\frac{1}{2}\sum_{z\in V} F(z)(\phi_z + s)^2\right)\right)}{s\mathbb{E}\left(\exp\left(\frac{1}{2}\sum_{z\in V} F(z)(\phi_z + s)^2\right)\right)} \\ &= 1 + \frac{\mathbb{E}^F\left(\phi_x \exp\left(s\sum_{z\in V} F(z)\phi_z\right)\right)}{s\mathbb{E}^F\left(\exp\left(s\sum_{z\in V} F(z)\phi_z\right)\right)}. \end{aligned} \tag{22}$$

The following identity involving Laplace transforms is somewhat opaque.

Lemma 4.3. *Let (X, Y) be a two dimensional centered Gaussian vector. Then, for anyy $s \neq 0$,*

$$E(XY) = \frac{E(X \exp(sY))}{sE(\exp(sY))}. \tag{23}$$

We postpone the proof of Lemma 4.3. Applying it with $X = \phi_x$ and $Y = \sum_{z \in V} F(z)\phi_z$ and the Gaussian measure $\mathbb{P}^F$, and recalling that $\mathbb{E}^F \phi_x \phi_y = (-L - F)^{-1}(x, y) = (I - GF)^{-1}G$, see (13), we see that the right side of (22) equals

$$1 + \mathbb{E}^F \left(\phi_x \sum_{z \in V} F(z)\phi_z \right) = 1 + \sum_{z \in V} F(z)\mathbb{E}^F (\phi_x \phi_z) = 1 + \sum_{z \in V} (I - GF)^{-1}GF(x, z).$$

Since $(I - GF)^{-1} = (I - GF)^{-1}(I - GF + GF) = I + (I - GF)^{-1}GF$, substituting in the last expression we get that the right side of (22) equals $\sum_{z \in V}(I - GF)^{-1}(x, z)$, i.e. the right side of (21). Rearranging gives the claim. $\square$

Proof of Lemma 4.3. This is a Laplace computation. We have that

$$E(e^{tX+sY}) = e^{E(tX+sY)^2/2} = e^{t^2 EX^2/2 + s^2 EY^2/2 + stE(XY)}.$$

Differentiate in t and set $t = 0$ to obtain

$$E(Xe^{sY}) = sE(XY)e^{s^2 EY^2/2} = E(XY)Ee^{sY},$$

as claimed. $\square$