LECTURE 2: LOCAL TIME FOR BROWNIAN MOTION

We will define local time for one-dimensional Brownian motion, and deduce some of its properties. We will then use the generalized Ray-Knight theorem proved in Lecture 1 in order to obtain the classical result (proved originally and separately by Ray and Knight).

1. Construction of local time

Throughout, we let B_t denote a standard one dimensional Brownian motion, which may start at $B_0 = a$ with a arbitrary. The *occupation measure* associated with it is the collection of random measures $\mu_t(A) = \int_0^t \mathbf{1}_{\{B_s \in A\}} ds$.

Lemma 1.1. *For each t, $\mu_t <<$ Leb.*

Proof. By regularity of Leb, it is enough to show that a.s., for μ_t-a.e. x,

$$\liminf_{r\to 0} \frac{\mu_t(B(x,r))}{|B(x,r)|} < \infty.$$

A sufficient condition for the latter is that

$$E \int d\mu_t(x) \liminf_{r\to 0} \frac{\mu_t(B(x,r))}{|B(x,r)|} < \infty. \tag{1}$$

By Fatou, we have that the left side in (1) is bounded above by

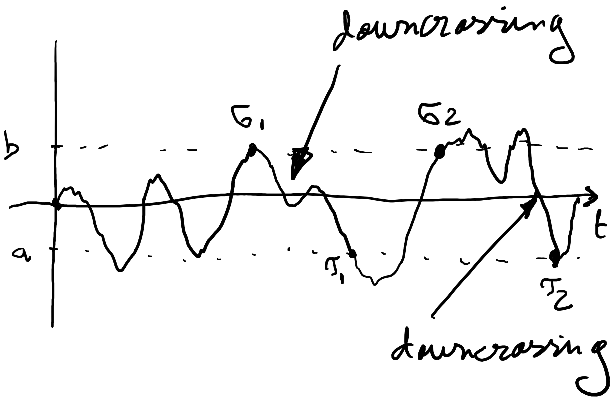
$$\begin{aligned}
\liminf_{r\to 0} \frac{1}{2r} E \int \mu_t(B(x,r)) d\mu_t(x) &= \liminf_{r\to 0} \frac{1}{2r} E \int d\mu_t(x) \int_0^t \mathbf{1}_{\{x-r\le B_s \le x+r\}} ds \\
&= \liminf_{r\to 0} \frac{1}{2r} E \int_0^t du \int_0^t \mathbf{1}_{\{B(u)-r\le B_s \le B(u)+r\}} ds \\
&= \liminf_{r\to 0} \frac{1}{2r} E \int_0^t du \int_0^t \mathbf{1}_{\{|B(u)-B(s)|\le r\}} ds \\
&= \liminf_{r\to 0} \frac{1}{2r} \int_0^t du \int_0^t P(|B(u)-B(s)|\le r) ds \\
&\le \int_0^t \int_0^t duds \frac{C}{\sqrt{|s-u|}} < \infty.
\end{aligned}$$

□

Note that this proof does not work for $d \ge 2$. For $d = 2$, a renormalization procedure will be needed in order to define local time. No such procedure is possible for $d \ge 3$.

By Lemma 1.1, the occupation measure μ_t has a density on $\mathbb{R}$. To get access to it, we begin by introducing a related quantity, the number of downcrossings of an interval around 0.

Definition by picture:



Let $D(a, b, t) = \max\{i : \tau_i \leq t\}$ denote the number of downcrossings of (a, b) by time t.

Theorem 1.2. *There exists a stochastic process $L(t)$ so that, for any sequence $a_n \nearrow 0$ and $b_n \searrow 0$ with $a_n < 0 < b_n$, we have*

$$2(b_n - a_n)D(a_n, b_n, t) \to_{n\to\infty} L(t), \quad a.s. \tag{2}$$

Further, $L(t)$ is a.s. Hölder $(\frac{1}{2}^-)$.

We call $L(t) = L_B(t)$ the local time at 0 of the Brownian motion B.

The next theorem gives an alternative representation of $L(t)$.

Theorem 1.3. *For any sequence $a_n \nearrow 0$ and $b_n \searrow 0$ with $a_n < 0 < b_n$,*

$$\frac{1}{b_n - a_n}\int_0^t \mathbf{1}_{\{a_n \leq B_s \leq b_n\}} ds = L(t), \quad a.s. \tag{3}$$

Note that Theorem 1.3 represents the local time at 0 as the Lebesgue derivative of μ_t at 0.

We postpone the proof of Theorems 1.2 and 1.3 until after we derive the Ray-Knight theorem.

Since Theorem 1.2 is stated with arbitrary starting point a, we may define the local time at a by $L^a(t) = L_{B-a}(t)$. We call the pair (a, ω) *good* if the conclusions of Theorems 1.2 and 1.3 hold. Note that since the construction works for any a, we have that

$$P \times \text{Leb}(\{(\omega, a) : L^a(t) \text{ not good}\} = 0).$$

In particular, it follows by Fubini that the conclusions of these theorems holds a.s. for Lebesgue almost every a. This observation, together with the Lebesgue differentiation theorem, immediately give the following.

Corollary 1.4. *For any bounded measurable function g,*

$$\int g(a)d\mu_t(a) = \int_0^t g(B(s))ds = \int_{\mathbb{R}} g(a)L^a(t)da, \quad a.s.$$

2. Brownian local times and random walk - a dictionary

Fix R, N and set $G_{R,N} := \mathbb{Z} \cap [-RN, RN]$. Set weights $W_{i,i+1} = 1$, $i = -RN, \ldots, RN-1$. Consider both the discrete and continuous time random walks on $G_{R,N}$. Note that DRW can be coupled with the Brownian motion B_s (started at 0 and reflected at R) in a natural way, using lattice spacing $1/N$. Note also that the DRW and CRW are naturally coupled, but we never couple the triple (B_s, DRW, CRW).

We introduce the discrete local time

$$L^z_{k,R} = \#\text{visits of DRW to site } z \text{ by (discrete) time } k.$$

Remark 2.1. Let $T_k(z)$ denote the total time spent by CRW at site z by the time of the k-th jump from z. If $|z| < RN$ then $T_k/k \to 1/2$ (since the jump rate of CRW is 2) and in fact, for such z,

$$P(|\frac{T_k(z)}{k} - \frac{1}{2}| > \delta) \leq e^{-I(\delta)k}. \tag{4}$$

If $k \sim N$ then one can apply a union bound and see that there exist $\epsilon_N, \delta_N \to 0$ so that $|T_k(z)/k - 1/2| < \epsilon_N$ at once for all $|z| < RN$ and $k > \delta_N N$, with probability approaching 1 as $N \to \infty$ with R fixed. (One can also allow $R \to \infty$ with N, but we will not need that.)

We choose R large and let τ_R denote the hitting time of $\{-R, R\}$ by the Brownian motion. Let $D_N(x,t)$ denote the number of downcrossings from $([xN]+1)/N$ to $[xN]$ by time t. Let $T(N,t)$ denote the total number of steps of the coupled DRW by (Brownian) time t. The coupling of the BM to DRW gives that for x which is not a multiple of $1/N$,

$$\begin{aligned} D_N(x,t) &= \#\text{downcrossings of DRW from } [xN]+1 \text{ to } [xN] \text{ by time } T(N,t) \\ &\sim \frac{1}{2}\#\text{visits to } [xN] \text{ by time } T(N,t) = \frac{1}{2}L^{[xN]}_{T(N,t),R} \end{aligned} \tag{5}$$

where here and later, $\sim$ is in the sense of Remark 2.1. Since $T(N,t)/N^2 t \sim 1$, the last term in the RHS of (5) is $\sim \frac{1}{2}L^{[xN]}_{tN^2,R} =: \frac{N}{2}\tilde{L}_{R,N}(x,t)$.

Note that for a.e. irrational x (which are never multiples of $1/N$), we have that $\tilde{L}_{R,N}(x, t\wedge\tau_R) \to L^x(t\wedge\tau_R)$, by Theorem 1.2 (at least on a subsequence of N which guarantees monotonicity).

On the other hand, for CRW, the Green function (with $x_0 = 0$) is, with $x, y < R$, $G_R(xN, yN) = (x\wedge y)N$, since for $y < x$

$$E^x \sum_{i=0}^{\tau_0} \mathbf{1}_{X_i=y} = E^y \sum_{i=0}^{\tau_0} \mathbf{1}_{X_i=y} = \frac{1}{P^y(\tau_0 < \tau_y)} = 2y$$

and $\lambda_x = 2$ for $|x| < RN$. In particular, we have that $\phi_{[xN]}/\sqrt{N} \xrightarrow{d} W_x$ where $W_x, |x| < R$ is a two sided Brownian motion.

3. The classical Ray-Knight theorem

The object in the left side of the generalized discrete Ray-Knight theorem can be written, in our notation, as

$$\frac{1}{2}L^{[xN]}_{2k,R}+\frac{1}{2}\phi^2_{[xN]}\ |_{2k:\frac{1}{2}L^0_{2k,R}=u}$$

(The factor 2 arises in translating number of visits to downcrossings.) Take $u=sN$. Take R large enough so that, with high probability, the hitting time of R is larger than the time it takes to accumulate occupation measure of sN at 0. Divide by N and take limits to get that the above converges (in the sense of finite dimensional distributions) to

$$\frac{1}{2}(L^x(\theta_s)+W_x^2).$$

On the other hand, the right side converges to $\frac{1}{2}(W_x+\sqrt{s})^2$. This works for any finite collection of irrational x, and extends by continuity to all x. We thus obtained:

Theorem 3.1 (Generalized Ray-Knight for Brownian motion)**.** *Let $\theta_u=\inf\{t: L^0(t)=u\}$. Then,*

$$(L^x(\theta_u)+W_x^2)\stackrel{d}{=}(W_x+\sqrt{u})^2.$$

4. Bessel processes and the classical (second) Ray-Knight theorem

A square bessel process of parameter δ started at z (denoted $BESQ^\delta(z)$) is the solution to the stochastic differential equation

$$dZ_t=2\sqrt{Z_t}dW_t+\delta dt,\quad Z_0=z. \tag{6}$$

(That a unique strong, positive solution exists is not completely trivial, but in the cases of interest to us ($\delta=0,1$) we show it below.)

The reason for the name is as follows. If B_t is a d-dimensional Brownian motion then Ito's lemma implies that

$$d|B_t|^2=2\sum_{i=1}^d B_t^i dB_t^i+d\cdot dt$$

But $\int_0^t\sum_{i=1}^d B_s^i dB_s^i$ equals in distribution $\int_0^t|B_s|^2dW_s$. Thus, the square of the modulus of d-dimensional Brownian motion is a $BESQ^\delta$ process with $\delta=d$. For $\delta=1$, this also gives a strong solution to (6). Weak uniqueness of a positive solution can then be deduced by taking square-root, and strong uniqueness follows by from weak uniqueness + strong existence.

To see a strong solution for $\delta=0$, note that a strong solution exists up to τ_0, the first hitting time of 0, and then extend it by setting $Z_t=0$ for all $t>\tau_0$.

Lemma 4.1. *Let X_t be a $BESQ^\delta(y)$ and let Y_t be an independent $BESQ^{\delta'}(y')$. Then $Z_t=X_t+Y_t$ is a $BESQ^{\delta+\delta'}(y+y')$.*

Proof. Write $dX_t=2\sqrt{X_t}dW_t+\delta dt$, $dY_t=2\sqrt{Y_t}dW_t'+\delta' t$, where W and W' are independent Brownian motions. From Ito's lemma we get

$$dZ_t=2(\sqrt{X_t}dW_t+\sqrt{Y_t}dW_t')+(\delta+\delta')dt.$$

The martingale part in the last equation has quadratic variation $X_t+Y_t=Z_t$, from which the conclusion follows. $\square$

In fact, the following converse also holds.

Lemma 4.2. *Let $y, y', \delta, \delta' \geq 0$. If X_t is a $BESQ^\delta(y)$ process, $Y_t \geq 0$ is independent of it, and $X_t + Y_t$ is a $BESQ^{\delta'}(y + y')$ process, then Y_t is a $BESQ^{\delta'}(y')$ process.*

Proof. Note first that Y_t is necessarily continuous. Hence, its law is characterized by its finite dimensional distributions. Since it is non-negative, the finite dimensional distributions are characterized by their Laplace transform. Now, with $\lambda_i, t_i \geq 0$, and Z_t a $BESQ^\eta(z)$ process, write

$$\psi^{\eta,z}(\lambda_i, t_i, i = 1, \ldots, k) = E(e^{-\sum_{i=1}^d \lambda_i Z_{t_i}}).$$

Then, from independence,

$$E(e^{-\sum_{i=1}^k \lambda_i Y_{t_i}}) = \frac{\psi^{\delta+\delta', y+y'}(\lambda_i, t_i, i = 1, \ldots, k)}{\psi^{\delta,y}(\lambda_i, t_i, i = 1, \ldots, k)}.$$

In particular, the law of Y_t is determined. Now apply Lemma 4.1 to identify it as $BESQ^{\delta'}(y')$. □

Remark 4.3. By the same reasoning, Lemma 4.1 gives immediately (weak) uniqueness for the $BESQ^0$ process.

We can now finally prove the classical version of the Ray-Knight theorem.

Theorem 4.4 (Classical second Ray-Knight theorem)**.** *$L^x(\theta_u)$ is a $BESQ^0(\sqrt{u})$ process.*

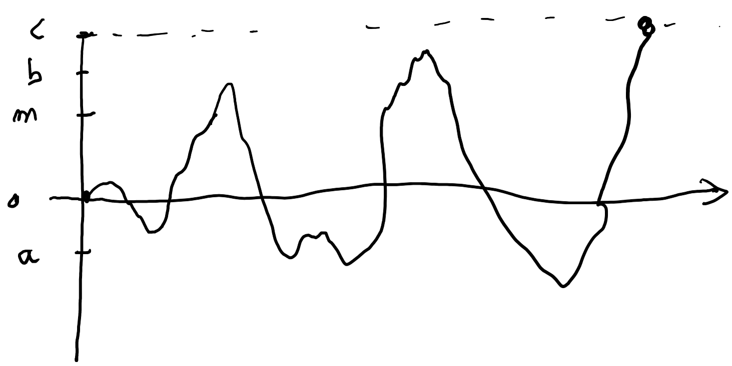
Proof. Note that, by Ito's lemma, $(W_x + \sqrt{u})^2$ is a $BESQ^1(\sqrt{u})$ process. It follows from Theorem 3.1 that $L^x(\theta_u) + W_x^2$ is a $BESQ^1(\sqrt{u})$ process. Noting that W_x^2 is a $BESQ^1(0)$ process, the conclusion follows from Lemma 4.2. □

5. Proof of Theorem 1.2

We follow the treatment in the Mörters-Peres book. We will sketch some of the arguments, see Chapter 6 there if you have troubles filling in the details. (Beware that some of the computations there are not quite right, this will be corrected in the forthcoming paperback edition.) We begin with a decomposition lemma for downcrossings.

Lemma 5.1. *Fix $a < m < b < c$. Let $T_c = \inf\{t : X_t = c\}$. Let $D_1 = D(a, m, T_c)$, $D_u = D(m, b, T_c)$, $D = D(a, b, T_c)$. Then there exist independent sequences of independent random variables X_i and Y_i, independent of D with the following properties: For $i \geq 1$, X_i is Geometric($\frac{b-a}{m-a}$) and Y_i is Geometric($\frac{(b-a)(c-m)}{(b-m)(c-a)}$) (both starting at 1), and*

$$D_1 = X_0 + \sum_{i=1}^{D} X_i, \quad D_u = Y_0 + \sum_{i=1}^{D} Y_i.$$



Proof. The proof is an exercise in the strong Markov property.

(a) Let $X_0 = D(a, m; T_{down}(a, b))$ where $T_{down}(a, b)$ is the time of the first downcrossing of (a, b). Note that it is possible that $X_0 = 0$.

(b) Each downcrossing of (a, b) has exactly one downcrossing of (a, m) (since a can't be reached twice).

(c) Each upcrossing of (a, b) has a (Geometric-1) downcrossings of (a, m), with parameter $(b - a)/(m - a)$ (start at m to see that!).

All these are of course independent of D. Facts (a)–(c) give the statement for D_1.

The claim concerning D_u is similar: Take Y_0 to be the number of downcrossings of (m, b) *after* the last downcrossing of (a, b). Note that there are no downcrossings of (m, b) during an upcross of (a, b), and that every downcross of (a, b) gives a geometric number of downcrossings of (m, b). □

We next prove Theorem 1.2 in the special case that $t = T_c$ and $c > b_1$. For that we need the following lemma.

Lemma 5.2. *Let $a_n \nearrow 0$ and $b_n \searrow 0$ with $a_n < 0 < b_n$. Then $S_n := 2(b_n - a_n)D(a_n, b_n, T_c)/(c - a_n)$ is a submartingale (with respect to its natural filtration $\mathcal{F}_n$) which is bounded in L^2.*

Proof. We use the representation of Lemma 5.1. Assume wlog that either $a_n = a_{n+1}$ or $b_n = b_{n+1}$ (otherwise, augment sequence).

Assume first that $a_n = a_{n+1}$. Recall that from Lemma 5.1 we have that

$$D(a_n, b_{n+1}, T_c) = X_0 + \sum_{i=1}^{D(a_n, b_n, T_c)} X_i$$

where X_i are independent of $\mathcal{F}_n$ and $D(a_n, b_n, T_c)$ is measurable on $\mathcal{F}_n$. Note also that $X_0 \geq 0$ and that $E(X_i|\mathcal{F}_n) = EX_i = (b_n - a_n)/(b_{n+1} - a_n)$ for $i \geq 1$. We get that

$$\frac{b_{n+1} - a_n}{c - a_n} E(D(a_n, b_{n+1}, T_c)|\mathcal{F}_n) \geq \frac{b_n - a_n}{c - a_n} D(a_n, b_n, T_c),$$

as needed. The proof for $b_n = b_{n+1}$ is similar, using the representation of D_u in Lemma 5.1.

To see the claimed L^2 bound, note that $D(a_n, b_n, T_c)$ is a geometric random variable with parameter $(b_n - a_n)/(c - a_n)$, and therefore of second moment $C(c - a_n)^2/(b_n - a_n))^2$. This implies that $ES_n^2 \leq C'$ and completes the proof. □

It follows from Lemma 5.2 that

$$L(T_b) = \lim_{n \to \infty} 2(b_n - a_n) D(a_n, b_n, T_c)$$

exists almost surely, and does not depend on the chosen sequence (a_n, b_n) (since one can always interleave sequences and get a contradiction if the limits do not coincide).

To complete the proof of Theorem 1.2, we need to consider a fixed $t > 0$. Choose then c large so that $P(T_c < t)$ is arbitrarily small, and extend the Brownian motion $B_s, s \leq t$ by an independent Brownian motion $\tilde{B}$ started at B_t. Let $\tilde{L}(T_c)$ be the local time at 0 of $\tilde{B}$ before T_c, which exists by the first part of the proof, and let $L(T_c)$ denote the local time at 0 up to time T_c of the Brownian motion B concatenated with $\tilde{B}$ (both equal a.s. to the limit of downcrossings count). Then, it follows from the definitions that $L(t) = L(T_c) - \tilde{L}(T_c)$, a.s., and Theorem 1.2 follows.

Exercise 1. *Use the downcrossing representation to show that, for all $\gamma < 1/2$ there exists $\epsilon = \epsilon(\gamma) > 0$ so that for h small,*

$$P(L(t+h) - L(t) > h^\gamma) \leq \exp(-h^\epsilon).$$

Using that, extend $L(t)$ to all times (almost surely), and show it is Holder continuous.

Exercise 2. *Let $T_1 = \inf\{t : W_t = 1\}$. Use the downcrossing representation to show that $L^0(T_1)$ has an exponential distribution.*

6. Proof of Theorem 1.3

This is an exercise in law of large numbers, of the type we mentioned in the main text. First note that if B is a standard Brownian motion starting at 0 and $\tau_1 = \min\{t > 0 : B_t = 1\}$, then

$$E \int_0^{\tau_1} \mathbf{1}_{B_s \in [0,1]} ds = 1.$$

This can be shown in several ways: either use that removing the negative excursions of Brownian motion gives a reflected Brownian motion, and so τ_1 has the same law as the exit time of B from the interval $[-1, 1]$, which by a martingale argument equals 1. Alternatively, if one sets $v(x) = E^x \int_0^{\tau_1} \mathbf{1}_{B_s \in [0,1]}$ then

$$\frac{1}{2} v''(x) = 1_{[0,1]}(x), x \in (-\infty, 1], \quad \text{and } v(1) = 0. \tag{7}$$

One can check that

$$v(x) = \begin{cases} (1-x)^2, & x \geq 0 \\ 1, & x < 0 \end{cases}$$

(As pointed out in class, this last argument is not quite OK, since there is no uniqueness to (7), even if one postulates that $v(x) = c$ for $x < 0$ and that $v(x)$ is monotone decreasing. To fix that, one can approximate the indicator by a smooth function, solve and take limits. A better option is to note that the exit time from $[0,1]$ when starting at $1/2$ is $1/4$, and use that to deduce the condition $v(1/2) = 1/4 + v(0)/2$, which gives the missing boundary condition when solving (7) on $[0,1]$.)

Note that τ_1 has the law of the time spent in $[0,1]$ during a downcrossing of the Brownian motion from 1 to 0. Now, embed a random walk in the Brownian motion and use the downcrossing representation, together with a law of large numbers and Brownian scaling, to complete the proof. Details are omitted.