EQUIVALENCE OF THE ALEXANDER-KOLMOGOROFF AND CECH COHOMOLOGY THEORIES

A DISSERTATION

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1. Introduction.

By a <u>cohomology theory</u> for spaces (complexes) we mean a triple (H^p, j^*, δ) defined on a class $\mathcal{C}$ of spaces (complexes) such that

(i) For each pair (X,A) in $\mathcal{C}$, i.e., $X, A \in \mathcal{C}$ and $A \subset X$, and each integer $p \geq 0$ there is an attached abelian group $H^p(X,A)$.

(ii) If (X,A) and (Y,B) are pairs in $\mathcal{C}$ and f is a continuous function (simplicial mapping) of (X,A) into (Y,B), there is an attached homomorphism $f^*: H^p(Y,B) \longrightarrow H^p(X,A)$, called the <u>induced homomorphism</u>.

(iii) For each pair (X,A) in $\mathcal{C}$ and each integer $p \geq 0$ there is a homomorphism $\delta : H^p(A,\emptyset) \longrightarrow H^{p+1}(X,A)$, called the <u>coboundary operator</u>.

Two cohomology theories $({}_1H^p, {}_1f^*, {}_1\delta)$ and $({}_2H^p, {}_2f^*, {}_2\delta)$ defined on the same class $\mathcal{C}$ of topological spaces (complexes) are said to be <u>equivalent</u> if and only if for each pair (X,A) in $\mathcal{C}$ and each integer $p \geq 0$ there is an isomorphism $\times$ of ${}_1H^p(X,A)$ onto ${}_2H^p(X,A)$ such that the commutativity holds in the diagrams:

$$\begin{array}{ccc} {}_2H^p(Y,B) & \xrightarrow{{}_2f^{\#}} & {}_2H^p(X,A) \\ \uparrow \varkappa & & \uparrow \varkappa \\ {}_1H^p(Y,B) & \xrightarrow{{}_1f^{\#}} & {}_1H^p(X,A) \end{array} \qquad \begin{array}{ccc} {}_2H^p(A,\emptyset) & \xrightarrow{{}_2\delta} & {}_2H^{p+1}(X,A) \\ \uparrow \varkappa & & \uparrow \varkappa \\ {}_1H^p(A,\emptyset) & \xrightarrow{{}_1\delta} & {}_1H^{p+1}(X,A) \end{array}$$

Many definitions have been given extending the cohomology theory from complexes to spaces. The theories most commonly used are the Čech, the singular and the Alexander-Kolmogoroff theories [6;3;7]. It is well-known that finiteness conditions (e.g., finite open coverings) lead to non-intuitive results for very simple spaces. In order to avoid this situation (as well as for other reasons) it is customary to introduce compactness in some form, compact supports, compact cohomologies and so on. This, however, introduces difficulties in applications. Very few function spaces, for example, are provided with a sufficient number of compact subsets. For reasons now familiar the singular theory is inadequate. Even for locally compact connected finite-dimensional groups satisfactory results about regularity in the small have not yet been obtained in sufficient amount to permit application of the singular theory. One is then inclined toward the Čech theory (using quite arbitrary coverings) [2] or the Alexander-Kolmogoroff theory. Dowker [2] has shown that the unrestricted Čech cohomology theory for general spaces satisfies the Eilenberg-Steenrod axioms [4]; therefore it has an advantage in certain applications. But unfortunately an elaborate machinery of complexes, orientation (or ordering) and limit-groups is essential to even the definition of the Čech groups. However, the Alexander-Kolmogoroff theory is more immediate and direct. The Eilenberg-Steenrod axioms except the homotopy axiom are known to be satisfied in this theory with no restrictions at all on the spaces and much more is known [11] when the space is fully normal [9,p.53]. It is only recently [1;10;11] that the usefulness of fully normal spaces in algebraic topology has been recognized. We note that this category of spaces contains both metric and compact Hausdorff spaces. Further, A. H. Stone [8] has shown that for Hausdorff spaces full normality is the same as paracompactness. Therefore it is desirable to have a suitable cohomology theory on this category of spaces.

In this paper it is shown that, for fully normal spaces, the unrestricted Čech cohomology theory is equivalent to the Alexander-Kolmogoroff cohomology theory. Hence for fully normal spaces the Alexander-Kolmogoroff theory is a reasonable choice, since it has the advantage of a simple definition and it has all the properties the Čech theory may have.

An immediate corollary of our result is that for compact Hausdorff spaces the Alexander-Kolmogoroff cohomology theory is equivalent to the restricted Čech cohomology theory [7]. Moreover, we obtain the homotopy theorem for the Alexander-Kolmogoroff theory over fully normal spaces by using Dowker's result. Therefore the groups of convex subsets of linear metric spaces and thus the groups of Euclidean spaces all are trivial. Further, we see at once that if a fully normal space has Lebesgue dimension [5,p.4] at most n, then its groups in dimensions above n all vanish. Looking at our result from another direction we also know that the extension and reduction theorems [11] hold for the unrestricted Čech cohomology theory over fully normal spaces. Hence the map excision theorem holds.

Section 2 contains a brief sketch of the ordered cohomology theory of simplicial complexes. The Eilenberg-Steenrod axioms for such a theory are stated. Section 3 deals with direct systems of groups and some elementary properties are given.

Both Sections 2 and 3 are used to develop the machinery to define the unrestricted Čech cohomology theory which is given in Section 4. Our treatment is essentially the same as Dowker's because of Eilenberg's result [3,p.418].

In Section 5 we sketch the Alexander-Kolmogoroff co-homology theory and state some results which we need in Section 7. (5.9) is exactly the lemma 9.1 in Spanier [7]; however we atate in a manner suitable for our purpose.

Section 6 is preparation for the following section. We discuss full normality and certain homomorphisms between the cochains of a simplicial complex and the cochains of a space. In order to avoid the confusion which may arise in the definition of a canonical covering as well as in some places in Section 7, a more precise definition of a cover (i.e., an open covering) of a spaces is used in this paper.

Finally, the main theorem and several corollaries are proved in Section 7.

In the appendix we give a proof of the homotopy theorem for the Alexander-Kolmogoroff theory over fully normal spaces without involving any simplicial complexes. We prove also an analogue of Eilenberg's result for the Alexander-Kolmogoroff theory, that is, the theory based on "ordered" cochains is equivalent to that based on alternative cochains.

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2. Cohomology Theory of Simplicial Complexes.

Let S be a set. A non-null finite subset of S is called a simplex in S. A collection K of simplexes in S is said to form a simplicial complex if and only if any non-null subset of a simplex of K is also a simplex of K. The union of the simplexes of K will be denoted by S(K). A subcomplex of a simplicial complex K is a subcolliction of K which is also a simplicial complex. A pair (K,L) of complexes consists of a simplicial complex K and a subcomplex L of K.

Let K be a simplicial complex and let p be a non-negative integer. Denote by $S(K)^{p+1}$ the (p+1)-fold cartesian product of S(K) with itself. Then every element of $S(K)^{p+1}$ is an ordered (p+1)-tuple $(a_0,\ldots,a_p)$ with its coordinates $a_0,\ldots,a_p$ in S(K) and $S(K)^{p+1}$ consists of all these ordered (p+1)-tuples. There is a function on $S(K)^{p+1}$ into the subsets of S(K) defined as follows: Whenever $\xi \in S(K)^{p+1}$ its image $|\xi|$ is the set of coordinates of ξ. An element ξ of $S(K)^{p+1}$ is an ordered p-simplex of K if $|\xi|$ is a simplex of K. The set of all the ordered p-simplexes of K will be denoted by K(p).

Let G be a fixed additive abelian group used as coefficient group throughout this paper.

An ordered p-cochain of a simplicial complex K is a function from K(p) to G. The set of all the ordered p-cochains of K is a group $C^p(K)$ with functional addition as its group operation.

There is a homomorphism $\bar{\delta} : C^p(K) \longrightarrow C^{p+1}(K)$ defined by

$$(2.1) \quad (\bar{\delta}\varphi)(a_0,\dots,a_{p+1}) = \sum_{i=0}^{p+1}(-1)^i\varphi(a_0,\dots,\hat{a}_i,\dots,a_{p+1}),$$

where $(a_0,\dots,a_{p+1}) \in K(p+1)$ and $(a_0,\dots,\hat{a}_i,\dots,a_{p+1}) = (a_0,\dots,a_{i-1},a_{i+1},\dots,a_{p+1})$, which is clearly in K(p). Strictly speaking, $\bar{\delta}$ is a homomorphism dependent on K and p. But in this paper we follow the usual convention that the same notation $\bar{\delta}$ may denote any one of these homomorphisms. (see, for examples, [6,p.116], [7,p.409], etc.)

Let $\bar{\delta}\bar{\delta}$ be the composite function of $\bar{\delta}: C^p(K) \longrightarrow C^{p+1}(K)$ and $\bar{\delta}: C^{p+1}(K) \longrightarrow C^{p+2}(K)$ defined by $\bar{\delta}\bar{\delta}\varphi = \bar{\delta}(\bar{\delta}\varphi)$. Then

$$(2.2) \qquad \bar{\delta}\bar{\delta} = 0.$$

Let (K,L) be a pair of complexes. Then L(p) is a subset of K(p). An ordered p-cochain φ of K is called an ordered p-cochain of K mod L if $\varphi = 0$ on L(p). The set of all the ordered p-cochains of K mod L form a subgroup $C^p(K,L)$ of $C^p(K)$. Obviously we have

$$(2.3) \qquad C^p(K,L) \subset C^{p+1}(K,L).$$

The group $Z^p(K,L)$ of <u>ordered</u> p-<u>cocycles</u> of K mod L is defined by

$$(2.4) \qquad Z^p(K,L) = \{\varphi \mid \varphi \in C^p(K,L) \text{ and } \bar{\delta}\varphi = 0\}.$$

The group $B^p(K,L)$ of <u>ordered</u> p-<u>coboundaries</u> of K mod L is defined by

$$(2.5) \qquad B^p(K,L) = \begin{cases} \bar{\delta}C^{p-1}(K,L) & \text{if } p > 0, \\ \{0\} & \text{if } p = 0. \end{cases}$$

Using (2.2), (2.3), (2.4) and (2.5) it is trivial that $B^p(K,L) \subset Z^p(K,L)$, that is, $B^p(K,L)$ is a subgroup of $Z^p(K,L)$. We may therefore introduce the factor group

(2.6) $H^p(K,L) = Z^p(K,L)/B^p(K,L)$.

It is known as the p-th ordered cohomology group of K mod L, and its elements as the p-th ordered cohomology classes of K mod L.

If $L = \emptyset$ (The symbol $\emptyset$ will be used to denote the null set), then $C^p(K,\emptyset) = C^p(K)$. The groups $Z^p(K,\emptyset)$, $B^p(K,\emptyset)$ and $H^p(K,\emptyset)$ will be respectively designated by $Z^p(K)$, $B^p(K)$ and $H^p(K)$.

Let (K,L) and (K_1,L_1) be two pairs of complexes. A simplicial mapping $j: K \longrightarrow K_1$ is a function from S(K) to $S(K_1)$ such that whenever $\sigma \in K$, $j(\sigma) \in K_1$. A simplicial mapping $j: (K,L) \longrightarrow (K_1,L_1)$ is a simplicial mapping $j: K \longrightarrow K_1$ such that whenever $\sigma \in L$, $j(\sigma) \in L_1$. Therefore every simplicial mapping $j: (K,L) \longrightarrow (K_1,L_1)$ induces a simplicial mapping $(j|L): L \longrightarrow L_1$ defined by $(j|L)(a) = j(a)$, $a \in S(L)$. If $K \subset K_1$, and $L \subset L_1$, then the injection of S(K) into $S(K_1)$ defines a simplicial mapping $i: (K,L)$

$\longrightarrow (K_1, L_1)$, called the <u>injection</u> of (K, L) into (K_1, L_1). The injection of (K, L) into itself is called the <u>identity mapping</u>.

Let $j: K \longrightarrow K_1$ be a simplicial mapping. Then j induces a function $j_\#: K(p) \longrightarrow K_1(p)$ defined by $j_\#(a_0, \ldots, a_p) = (j(a_0), \ldots, j(a_p))$, $(a_0, \ldots, a_p) \in K(p)$. Therefore j induces a homomorphism $j^\#: C^p(K_1) \longrightarrow C^p(K)$ defined by $(j^\# \varphi)(\xi) = \varphi(j_\# \xi)$, $\varphi \in C^p(K_1)$ and $\xi \in K(p)$, or

(2.7) $(j^\# \varphi)(a_0, \ldots, a_p) = \varphi(j(a_0), \ldots, j(a_p))$, $(a_0, \ldots, a_p)$ is an element of $K(p)$.

The composite functions $\bar{\delta} j^\#$ and $j^\# \bar{\delta}$ from $C^p(K_1)$ to $C^{p+1}(K)$ are homomorphisms and

(2.8) $\bar{\delta} j^\# = j^\# \bar{\delta}$.

If $j: K \longrightarrow K_1$ and $j_1: K_1 \longrightarrow K_2$ are simplicial mappings, then so is $j_1 j: K \longrightarrow K_2$. Using (2.7), we have

(2.9) $(j_1 j)^\# = j^\# j_1^\#$.

If L is a subcomplex of K and if i is the injection of L into K, we can easily verify

(2.10) $i^{\#}: C^p(K) \longrightarrow C^p(L)$ <u>is onto and the kernel of</u> $i^{\#}$ <u>is</u> $C^p(K,L)$.

Now let j be a simplicial mapping of (K,L) into (K_1,L_1). Then $j_{\#}L(p) \subset L_1(p)$ and hence

(2.11) $\quad j^{\#}C^p(K_1,L_1) \subset C^p(K,L)$.

From (2.4), (2.5), (2.8) and (2.11) we infer that

(2.12) $\quad j^{\#}Z^p(K_1,L_1) \subset Z^p(K,L), \quad j^{\#}B^p(K_1,L_1) \subset B^p(K,L)$.

It follows from the induced homomorphism theorem that there is a unique homomorphism

$j^{*}: H^p(K_1,L_1) \longrightarrow H^p(K,L)$

such that

$$\begin{array}{ccc} H^p(K_1,L_1) & \xrightarrow{\quad j^{*}\quad} & H^p(K,L) \\ \uparrow{\scriptstyle \gamma_1} & & \uparrow{\scriptstyle \gamma} \\ Z^p(K_1,L_1) & \xrightarrow{\quad j^{\#}\quad} & Z^p(K,L) \end{array}$$

(2.13) $\quad j^{*}\gamma_1 = \gamma j^{\#} \quad$ on $Z^p(K_1,L_1)$,

where $\gamma: Z^p(K,L) \longrightarrow H^p(K,L)$ and $\gamma_1: Z^p(K_1,L_1) \longrightarrow H^p(K_1,L_1)$ are natural homomorphisms. j^{*} is the <u>induced homomorphism</u> of j.

$$\begin{array}{ccccc} H^p(L) & & \xrightarrow{\qquad\qquad \delta \qquad\qquad} & & H^{p+1}(K,L) \\ \uparrow{\scriptstyle \beta} & & & & \uparrow{\scriptstyle \gamma} \\ Z^p(L) & \xleftarrow{\quad i^{\#}\quad} & i^{\#-1}Z^p(L) & \xrightarrow{\quad \bar{\delta}\quad} & Z^{p+1}(K,L) \end{array}$$

Let (K,L) be a pair of complexes and let i be the injection of L into K. Then, by (2.10), the induced homomorphism $i^{\#}: C^p(K) \longrightarrow C^p(L)$ is onto. Therefore

$$H^p(L) = \beta i^{\#} i^{\#-1} Z^p(L).$$

If $\varphi \in i^{\#-1}Z^p(L)$, then $i^{\#}\varphi \in Z^p(L)$ and $i^{\#}\bar{\delta}\varphi = \bar{\delta} i^{\#}\varphi = 0$. It follows by (2.10) that $\bar{\delta}\varphi \in C^{p+1}(K,L)$. Using (2.2) and (2.4), $\bar{\delta}\varphi \in Z^{p+1}(K,L)$. Hence $\bar{\delta} i^{\#-1}Z^p(L) \in Z^{p+1}(K,L)$. If $e \in H^p(L)$ and $\varphi_1, \varphi_2 \in i^{\#-1}\beta^{-1}e$, then for $p = 0$, $\bar{\delta}(\varphi_1 - \varphi_2) \in \bar{\delta} i^{\#-1}B^0(L) = \bar{\delta} C^0(K,L) = B^1(K,L)$ and for $p > 0$, $\bar{\delta}(\varphi_1 - \varphi_2) \in \bar{\delta} i^{\#-1}B^p(L) = \bar{\delta} i^{\#-1}\bar{\delta}C^{p-1}(L) \subset \bar{\delta}\bar{\delta} i^{\#-1}C^{p-1}(L) = \{0\}$. Hence $\gamma\bar{\delta}\varphi_1 = \gamma\bar{\delta}\varphi_2$, where γ is the natural homomorphism of $Z^{p+1}(K,L)$ onto $H^{p+1}(K,L)$. Consequently there is a unique homomorphism $\delta: H^p(L) \longrightarrow H^{p+1}(K,L)$ such that

(2.14) $\quad \delta\beta i^{\#} = \gamma\bar{\delta} \quad$ on $\; i^{\#-1}Z^p(L)$.

δ is the <u>coboundary operator</u>.

The following definitions will be needed. Let K be a simplicial complex and M a subset of K. We denote

$$\text{St } M = \{\sigma \mid \sigma \in K \text{ and } \sigma \supset \sigma' \text{ for some } \sigma' \in M\},$$

$$\text{Cl } M = \{\sigma \mid \sigma \in K \text{ and } \sigma \subset \sigma' \text{ for some } \sigma' \in M\}.$$

M is said to be <u>open</u> or <u>closed</u> (in K) according to St M = M or Cl M = M. A sequence of groups and homomorphisms

$$G_1 \xrightarrow{h_1} G_2 \xrightarrow{h_2} \cdots \xrightarrow{h_{p-1}} G_p \xrightarrow{h_p} G_{p+1} \xrightarrow{h_{p+1}} \cdots$$

is said to be an <u>exact sequence</u> if the kernel of h_{p+1} equals the image of h_p for all $p > 0$ and if the kernel of h_1 is $\{0\}$.

On the class of simplicial complexes, the ordered cohomology groups, the induced homomorphisms and the coboundary operators defined respectively by (2.6), (2.13) and (2.14) form a cohomology theory in the sense of Section 1. This theory satisfies the following Eilenberg-Steenrod axioms:

(2.15) Algebraic axiom 1. *If* $i: (K,L) \longrightarrow (K,L)$ *is the identity mapping, then* $i^*: H^p(K,L) \longrightarrow H^p(K,L)$ *is the identity isomorphism.*

(2.16) Algebraic axiom 2. *If* $j: (K,L) \longrightarrow (K_1,L_1)$ *and* $j_1: (K_1,L_1) \longrightarrow (K_2,L_2)$ *are simplicial mappings, then* $(j_1 j)^* = j^* j_1^*$.

(2.17) Algebraic axiom 3. *If* $j: (K,L) \longrightarrow (K_1,L_1)$ *is a simplicial mapping, then* $j^* \delta = \delta (j|L)^*$.

(2.18) Homotopy axiom. *If* j *and* j_1 *are simplicial mappings of* (K,L) *into* (K_1,L_1) *such that whenever* $\sigma \in K$, $j(\sigma)\cup j_1(\sigma) \in K_1$ *and whenever* $\sigma \in L$, $j(\sigma)\cup j_1(\sigma) \in L_1$, *then* $j^* = j_1^*$.

(2.19) Exactness axiom. *Given* (K,L) *and the injections* $i\colon (L,\emptyset)\longrightarrow(K,\emptyset)$ *and* $j\colon (K,\emptyset)\longrightarrow(K,L)$ *the groups and homomorphisms*

$$H^0(K,L)\xrightarrow{j^*}H^0(K)\xrightarrow{i^*}\cdots\xrightarrow{j^*}H^p(K)\xrightarrow{i^*}H^p(L)$$

$$\xrightarrow{\delta}H^{p+1}(K,L)\xrightarrow{j^*}\cdots$$

form an exact sequence, called the cohomology sequence of (K,L).

(2.20) Excision axiom. *Given* (K,L) *and an open subset* M *of* K *with* $\mathrm{Cl}\,M \subset L$, *the injection* $j\colon (K-M,L-M)\longrightarrow(K,L)$ *induces isomorphisms* $j^*\colon H^p(K,L)\longrightarrow H^p(K-M,L-M)$ *for each* $p \geq 0$.

(2.21) Dimension axiom. *If* K *is a simplicial complex with* $S(K)$ *consisting of a single element, then* $H^p(K) = \{0\}$ for each $p > 0$.

The verification of these axioms will be omitted, since there is no difficulty in doing it by following the idea as indicated in Spanier [7].

Remark. In this section we deal with only the <u>ordered cohomology theory</u> (H^p, j^*, δ) of simplicial complexes. If we use orientation instead of ordering, an <u>oriented cohomology theory</u> $(\tilde{H}^p, \tilde{j}^*, \tilde{\delta})$ of simplicial complexes can be established in a similar way. According to Eilenberg [3, p. 418], these two cohomology theory are equivalent, since Eilenberg has constructed isomorphisms

$$\zeta : \tilde{H}^p(K,L) \approx H^p(K,L)$$

and according to his construction it is easily seen that the following permutability conditions hold:

$$\zeta \tilde{j}^* = j^* \zeta \qquad\qquad \zeta \tilde{\delta} = \delta \zeta$$

$$\begin{array}{ccc} H^p(K_1,L_1) & \xrightarrow{j^*} & H^p(K,L) \\ \uparrow \zeta & & \uparrow \zeta \\ \tilde{H}^p(K_1,L_1) & \xrightarrow{\tilde{j}^*} & \tilde{H}^p(K,L) \end{array} \qquad\qquad \begin{array}{ccc} H^p(L) & \xrightarrow{\delta} & H^{p+1}(K,L) \\ \uparrow \zeta & & \uparrow \zeta \\ H^p(L) & \xrightarrow{\tilde{\delta}} & H^{p+1}(K,L) \end{array}$$

A parallel result holds for the Alexander-Kolmogoroff cohomology theory which will be given in the Appendix.

3. Direct Systems.

A <u>directed set</u> $\{\Lambda, >\}$ consists of a set Λ and a binary (order) relation $>$ such that (i) $\mu > \lambda$ and $\nu > \mu$ imply $\nu > \mu$ and (ii) for any two elements λ, μ of Λ there is a third element ν such that $\nu > \mu$ and $\nu > \lambda$. If Λ_1 is a subset of Λ, Λ_1 is <u>cofinal</u> in $\{\Lambda, >\}$ whenever for every $\lambda \in \Lambda$ there is some $\lambda_1 \in \Lambda_1$ such that $\lambda_1 > \lambda$.

A <u>direct system</u> is a quadruple $\{H; \xi; \Lambda, >\}$ such that (i) $\{\Lambda, >\}$ is a directed set; (ii) for each $\lambda \in \Lambda$ there is given an abelian group H_λ; and (iii) given any $\lambda, \mu \in \Lambda$ there is given a homomorphism $\xi_{\mu\lambda}: H_\lambda \longrightarrow H_\mu$ such that $\mu > \lambda$ and $\nu > \lambda$ imply $\xi_{\nu\lambda} = \xi_{\nu\mu}\xi_{\mu\lambda}$.

Let E' be the weak product of the system $\{H_\lambda \mid \lambda \in \Lambda\}$, that is, the group of all the functions f on Λ to $\cup \{H_\lambda \mid \lambda \in \Lambda\}$ such that $f(\lambda) \in H_\lambda$ for all λ and f is finitely not zero. For each $\mu \in \Lambda$ let p_μ be the natural function on H_μ into E' defined by, for $e_\mu \in H_\mu$,

$$(p_\mu e_\mu)(\lambda) = \begin{cases} e_\mu & \text{if } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly each p_μ is an isomorphism of H_μ into E' and every non-zero element of E' has a unique representation $\sum_{\lambda \in F} p_\lambda e_\lambda$, where F is a finite subset of Λ and each e_λ is a non-zero element of H_λ. Let E'' be the subgroup of E' generated by elements of the form

$$p_\lambda e_\lambda - p_\mu \xi_{\mu\lambda} e_\lambda \qquad \text{for } \lambda, \mu \in \Lambda,\ \mu > \lambda,\ e_\lambda \in H_\lambda.$$

The factor group $E = E'/E''$ is the <u>limit-group</u> of the direct system $\{H;\xi;\Lambda,>\}$, written

$$E = \underset{\longrightarrow}{\lim} \{H;\xi;\Lambda,>\}.$$

Let τ be the natural homomorphism of E' onto E. Let $\eta_\lambda = \tau p_\lambda$, the natural homomorphism of H_λ into E. Whenever $e_\lambda \in H_\lambda$ and $\mu > \lambda$ we have $\eta_\lambda e_\lambda = \tau p_\lambda e_\lambda = \tau(p_\lambda e_\lambda - p_\mu \xi_{\mu\lambda} e_\lambda) + \tau p_\mu \xi_{\mu\lambda} e_\lambda = \eta_\mu \xi_{\mu\lambda} e_\lambda$. Hence

(3.1) $$\eta_\lambda = \eta_\mu \xi_{\mu\lambda} \ (\mu > \lambda).$$

(3.2) <u>For each</u> $e \in E$ <u>there is some</u> $\lambda \in \Lambda$ <u>and some</u> $e_\lambda \in H_\lambda$ <u>such that</u> $e = \eta_\lambda e_\lambda$.

(3.3) <u>Let</u> $e_\lambda \in H$. $\eta_\lambda e_\lambda = 0$ <u>if and only if</u> $\xi_{\mu\lambda} e_\lambda = 0$ <u>for some</u> $\mu > \lambda$.

The proofs of (3.2) and (3.3) are omitted; see, for example, [6, p. 57].

(3.4) *Let* $\{H;\xi;\Lambda,>\}$ *are* $\{\tilde{H};\tilde{\xi};\tilde{\Lambda},\tilde{>}\}$ *be direct systems and* E *and* $\tilde{E}$ *their respective limit-groups. For some pairs* (λ,σ) *with* $\lambda \in \Lambda$ *and* $\sigma \in \tilde{\Lambda}$ *there is given a homomorphism* $\zeta_{\lambda\sigma}: \tilde{H}_\sigma \longrightarrow H_\lambda$; *let* Z *be a set of these homomorphisms. We write* $\lambda \succ \sigma$ *whenever there exists* $\zeta_{\lambda\sigma} \in Z$.

If the following conditions hold:

(i) *For each* $\sigma \in \tilde{\Lambda}$ *there is some* $\lambda \in \Lambda$ *such that* $\lambda \succ \sigma$.

(ii) $\mu > \lambda \succ \sigma \tilde{>} \rho$ *implies* $\mu \succ \sigma$, $\lambda \succ \rho$, $\zeta_{\mu\sigma} = \xi_{\mu\lambda}\zeta_{\lambda\sigma}$ *and* $\zeta_{\lambda\rho} = \zeta_{\lambda\sigma}\tilde{\xi}_{\sigma\rho}$.

Then Z *induces a unique homomorphism* $\check{\zeta}: \tilde{E} \longrightarrow E$ *such that* $\eta_\lambda\zeta_{\lambda\sigma} = \check{\zeta}\tilde{\eta}_\sigma$ *for all* $\lambda \succ \sigma$, *where* $\eta_\lambda: H_\lambda \longrightarrow E$ *and* $\tilde{\eta}_\sigma: \tilde{H}_\sigma \longrightarrow \tilde{E}$ *are natural homomorphisms.*

Proof. Let $\tilde{E}'$, $\tilde{E}''$, $\tilde{E}$, $\tilde{\tau}$, $\tilde{p}_\sigma$, $\tilde{\eta}_\sigma$ be the analogues of E', E'', E, τ, p_λ, η_λ for $\{\tilde{H};\tilde{\xi};\tilde{\Lambda},\tilde{>}\}$. Fix an $e_\sigma \in \tilde{H}_\sigma$, $\sigma \in \tilde{\Lambda}$. By (i), there is some $\lambda \in \Lambda$ such that $\lambda \succ \sigma$; therefore $\eta_\lambda\zeta_{\lambda\sigma}e_\sigma \in E$. If $\lambda \succ \sigma$ and $\mu \succ \sigma$, there is, by the directedness of $\{\Lambda, >\}$, an element ν of Λ such that $\nu > \lambda$ and $\nu > \mu$. It follows by (ii) that $\eta_\lambda \zeta_{\lambda\sigma} e_\sigma = \eta_\nu \xi_{\nu\lambda} \zeta_{\lambda\sigma} e_\sigma = \eta_\nu \zeta_{\nu\sigma} e_\sigma = \eta_\nu \xi_{\nu\mu} \zeta_{\mu\sigma} e_\sigma = \eta_\mu \zeta_{\mu\sigma} e_\sigma$. Hence a homomorphism $\zeta : E' \longrightarrow E$ is defined by, for each $e_\sigma \in \tilde{H}_\sigma$,

(3.5) $$\zeta(\tilde{p}_\sigma e_\sigma) = \eta_\lambda \zeta_{\lambda\sigma} e_\sigma \qquad (\lambda \succ \sigma).$$

Since $\tilde{E}''$ is generated by elements of the form $\tilde{p}_\sigma e_\sigma - \tilde{p}_\rho \tilde{\xi}_{\rho\sigma} e_\sigma$ for $\sigma, \rho \in \tilde{\Lambda}$, $\rho \tilde{>} \sigma$, $e_\sigma \in \tilde{H}_\sigma$, and for each $\tilde{p}_\sigma e_\sigma - \tilde{p}_\rho \tilde{\xi}_{\rho\sigma} e_\sigma$ we have $\zeta(\tilde{p}_\sigma e_\sigma - \tilde{p}_\rho \tilde{\xi}_{\rho\sigma} e_\sigma) = \eta_\lambda \zeta_{\lambda\sigma} e_\sigma - \eta_\lambda \zeta_{\lambda\rho} \tilde{\xi}_{\rho\sigma} e_\sigma = 0$ $(\lambda \succ \rho \tilde{>} \sigma)$, it follows that $\zeta(E'') = 0$. Hence there is a homomorphism $\check{\zeta} : \tilde{E} \longrightarrow E$ such that $\zeta = \check{\zeta}\tilde{\tau}$. If $\lambda > \sigma$, then $\check{\zeta}\tilde{\eta}_\sigma = \check{\zeta}\tilde{\tau}\tilde{p}_\sigma = \zeta\tilde{p}_\sigma = \eta_\lambda \zeta_{\lambda\sigma}$. By (3.1), every element e of $\tilde{E}$ has a representation $e = \tilde{\eta}_\sigma e_\sigma$ for some $\sigma \in \tilde{\Lambda}$ and some $e_\sigma \in \tilde{H}_\sigma$. It follows that $\check{\zeta} e = \check{\zeta}\tilde{\eta}_\sigma e_\sigma = \eta_\lambda \zeta_{\lambda\sigma} e_\sigma$. Hence $\check{\zeta}$ is unique. Q.E.D.

(3.6) Let $\{H;\xi;\Lambda,>\}$ be a direct system and E its limit-group. Let $\tilde{\Lambda}$ be a subset of Λ with $\{\tilde{\Lambda},>\}$ being directed and let $\{H;\xi;\tilde{\Lambda},>\}$ be the direct subsystem attached to $\{\tilde{\Lambda},>\}$ and E its limit-group. Then there is a unique homomorphism $\chi : \tilde{E} \longrightarrow E$ such that $\eta_\lambda = \chi\tilde{\eta}_\lambda$ for

<u>all</u> $\lambda \in \tilde{\Lambda}$, <u>where</u> $\eta_\lambda : H_\lambda \longrightarrow E$ <u>and</u> $\tilde{\eta}_\lambda : H_\lambda \longrightarrow \tilde{E}$ <u>are natural homomorphisms</u>. <u>Moreover, if</u> $\tilde{\Lambda}$ <u>is cofinal in</u> $\{\Lambda, >\}$, <u>then</u> χ <u>is an isomorphism onto</u>.

Proof. Let $Z = \{\xi_{\lambda\mu} | \lambda \in \Lambda, \mu \in \tilde{\Lambda}$ and $\lambda > \mu\}$. Then the conditions (i) and (ii) of (3.4) hold. It follows that Z induces a unique homomorphism $\chi : \tilde{E} \longrightarrow E$ such that for each $\mu \in \tilde{\Lambda}$, $\chi\eta_\mu = \eta_\lambda \xi_{\lambda\mu} = \eta_\mu$ ($\lambda \in \Lambda$ and $\lambda > \mu$).

Now suppose that $\tilde{\Lambda}$ is cofinal in $\{\Lambda, >\}$. Let $e \in \tilde{E}$ be such that $\chi e = 0$. By (3.2), there is some $\lambda \in \tilde{\Lambda}$ and some $e_\lambda \in H_\lambda$ such that $e = \tilde{\eta}_\lambda e_\lambda$. Then $\eta_\lambda e_\lambda = \chi \tilde{\eta}_\lambda e_\lambda = \chi e = 0$. It follows by (3.3) that there is some $\mu > \lambda$, $\mu \in \Lambda$, such that $\xi_{\mu\lambda} e_\lambda = 0$. Since $\tilde{\Lambda}$ is cofinal in $\{\Lambda, >\}$, there is some $\nu \in \tilde{\Lambda}$ such that $\nu > \mu$. Therefore $\xi_{\nu\lambda} e_\lambda = \xi_{\nu\mu}\xi_{\mu\lambda} e_\lambda = 0$ and $e = \tilde{\eta}_\lambda e_\lambda = \tilde{\eta}_\nu \xi_{\nu\lambda} e_\lambda = 0$. This proves that χ is 1-1. Given any $e \in E$ there is, by (3.2), some $\lambda \in \Lambda$ and some $e_\lambda \in H_\lambda$ such that $e = \eta_\lambda e_\lambda$. Let $\mu \in \tilde{\Lambda}$ be such that $\mu > \lambda$. Then $e = \eta_\lambda e_\lambda = \eta_\mu \xi_{\mu\lambda} e_\lambda = \chi \tilde{\eta}_\mu \xi_{\mu\lambda} e_\lambda = \chi e'$ with $e' = \tilde{\eta}_\mu \xi_{\mu\lambda} e_\lambda \in \tilde{E}$. Hence χ is onto. Q.E.D.

(3.7) <u>Let</u> $\{H^{(i)}; \xi^{(i)}; \Lambda^{(i)}, >^{(i)}\}$ <u>be direct systems</u>

with limit-groups $E^{(i)}$, $i = 1, 2, 3$. Let $\{\Lambda, >\}$ be a directed set and let there be given, for each i, a function $t_i : \Lambda \longrightarrow \Lambda^{(i)}$ satisfying the following conditions: For each $\lambda \in \Lambda$ we denote $t_i(\lambda) = \lambda_i$, $i = 1, 2, 3$.

(i) If $\mu > \lambda$, $\lambda, \mu \in \Lambda$, then $\mu_i >^{(i)} \lambda_i$.

(ii) $t_i(\Lambda)$ is cofinal in $\{\Lambda^{(i)}, >^{(i)}\}$.

(iii) For each $\lambda \in \Lambda$ there are given homomorphisms $q_\lambda : H^{(1)}_{\lambda_1} \longrightarrow H^{(2)}_{\lambda_2}$ and $r_\lambda : H^{(2)}_{\lambda_2} \longrightarrow H^{(3)}_{\lambda_3}$ such that (a) the kernel of r_λ is the image of q_λ and (b) whenever $\mu > \lambda$, $\lambda, \mu \in \Lambda$, we have $\xi^{(2)}_{\mu_2 \lambda_2} q_\lambda = q_\mu \xi^{(1)}_{\mu_1 \lambda_1}$ and $\xi^{(3)}_{\mu_3 \lambda_3} r_\lambda = r_\mu \xi^{(2)}_{\mu_2 \lambda_2}$.

Let Q be the set of homomorphisms of the forms q_λ, $\xi^{(2)}_{\rho \lambda_2} q_\lambda$, $q_\lambda \xi^{(1)}_{\lambda_1 \sigma}$, $\xi^{(2)}_{\rho \lambda_2} q_\lambda \xi^{(1)}_{\lambda_1 \sigma}$ with $\lambda \in \Lambda$, $\sigma \in \Lambda^{(1)}$, $\rho \in \Lambda^{(2)}$, $\lambda_1 >^{(1)} \sigma$, $\rho >^{(2)} \lambda_2$, and let R be the set of homomorphisms of the forms r_λ, $\xi^{(3)}_{\rho \lambda_3} r_\lambda$, $r_\lambda \xi^{(2)}_{\lambda_2 \sigma}$, $\xi^{(3)}_{\rho \lambda_3} r_\lambda \xi^{(2)}_{\lambda_2 \sigma}$ with $\lambda \in \Lambda$, $\sigma \in \Lambda^{(2)}$, $\rho \in \Lambda^{(3)}$, $\lambda_2 >^{(2)} \sigma$, $\rho >^{(3)} \lambda_3$. Let $q : E^{(1)} \longrightarrow E^{(2)}$ and $r : E^{(2)} \longrightarrow E^{(3)}$ be the respective induced homomorphisms of Q and R in the sense of (3.5). Then the kernel of r is equal to the image of q.

Proof. We note that (iii), (b) is meaningful because of (i) and the functions q and r are well-defined because of (ii) and (iii), (b). Denote by I[q] the image of q and by K[r] the kernel of r. For each $\lambda \in \Lambda$ let $\eta_\lambda^{(i)}$ be the natural homomorphism of $H_{\lambda_i}^{(i)}$ into $E^{(i)}$, $i = 1, 2, 3$.

Let $e^{(2)} \in I[q]$, say $e^{(2)} = qe^{(1)}$ with $e^{(1)} \in E^{(1)}$. By (ii) and (3.2) there is some $\lambda \in \Lambda$ and some $e_\lambda^{(1)} \in H_{\lambda_1}^{(1)}$ such that $e^{(1)} = \eta_\lambda^{(1)} e_\lambda^{(1)}$. Applying (3.5), we have $re^{(2)} = rqe^{(1)} = rq\eta_\lambda^{(1)} e_\lambda^{(1)} = r\eta_\lambda^{(2)} q_\lambda e_\lambda^{(1)} = \eta_\lambda^{(3)} r_\lambda q_\lambda e_\lambda^{(1)} = 0$. Hence $I[q] \subset K[r]$.

Let $e^{(2)} \in K[r]$. By (ii) and (3.2) there is some $\lambda \in \Lambda$ and some $e_\lambda^{(2)} \in H_{\lambda_2}^{(2)}$ such that $e^{(2)} = \eta_\lambda^{(2)} e_\lambda^{(2)}$. Since $\eta_\lambda^{(3)} r_\lambda e_\lambda^{(2)} = r\eta_\lambda^{(2)} e_\lambda^{(2)} = re^{(2)} = 0$, it follows by (ii) and (3.3) that there is some $\mu > \lambda$, $\mu \in \Lambda$, such that $\xi_{\mu_3\lambda_3}^{(3)} r_\lambda e_\lambda^{(2)} = 0$, or $r_\mu \xi_{\mu_2\lambda_2}^{(2)} e_\lambda^{(2)} = 0$. By hypothesis, there is some $e_\mu^{(1)} \in H_{\mu_1}^{(1)}$ such that $q_\mu e_\mu^{(1)} = \xi_{\mu_2\lambda_2}^{(2)} e_\lambda^{(2)}$. Let $e^{(1)} = \eta_\mu^{(1)} e_\mu^{(1)}$; then $qe^{(1)} = q\eta_\mu^{(1)} e_\mu^{(1)} = \eta_\mu^{(2)} q_\mu e_\mu^{(1)} = \eta_\mu^{(2)} \xi_{\mu_2\lambda_2}^{(2)} e_\lambda^{(2)} = \eta_\lambda^{(2)} e_\lambda^{(2)} = e^{(2)}$. Hence $K[r] \subset I[q]$. Q.E.D.

4. Unrestricted Čech Cohomology Theory.

Let X be a topological space and Q(X) the collection of all the open subsets of X. Let A be a subset of X. A <u>cover</u> $\{\lambda_0; \lambda_1\}$ of A <u>in</u> X consists of a set λ_1 and a function λ_0 on λ_1 to Q(X) such that $\cup\{\lambda_0(u) \mid u \in \lambda_1\} \supset A$. A cover of X is a cover of X in X. Let $\{\lambda_0; \lambda_1\}$ and $\{\mu_0; \mu_1\}$ be covers of X. By $\{\mu_0; \mu_1\} > \{\lambda_0; \lambda_1\}$ we mean that there exists a function $p: \mu_1 \longrightarrow \lambda_1$ such that for each $v \in \mu_1$, $\mu_0(v) \subset \lambda_0(p(v))$.

By a <u>pair</u> (X,A) we mean a topological space X and a subset A of X. A <u>covering</u> $\lambda = \{\lambda_0; \lambda_1, \lambda_2\}$ of a pair (X,A) consists of a set λ_1, a subset λ_2 of λ_1 and a function $\lambda_0: \lambda_1 \longrightarrow Q(X)$ such that $\{\lambda_0; \lambda_1\}$ is a cover of X and $\{\lambda_0 | \lambda_2; \lambda_2\}$ is a cover of A in X. Denote by $\Lambda(X,A)$ the class of all the coverings of (X,A). If $\lambda, \mu \in \Lambda(X,A)$, $\mu > \lambda$ will mean that there exists a function $p: \mu_1 \longrightarrow \lambda_1$ such that (i) for each $v \in \mu_1$, $\mu_0(v) \subset \lambda_0(p(v))$ and (ii) $p(\mu_2) \subset \lambda_2$. It is clear that, if $\mu > \lambda$ and $\nu > \mu$, $\lambda, \mu, \nu \in \Lambda(X,A)$, then $\nu > \lambda$. Given any two $\lambda, \mu \in \Lambda(X,A)$ let $\nu_1 = \lambda_1 \times \mu_1$, $\nu_2 = \lambda_2 \times \mu_2$ (cartesian products) and define $\nu_0: \nu_1 \longrightarrow Q(X)$ by $\nu_0(u,v) = \lambda_0(u) \cap \mu_0(v)$, $(u,v) \in \lambda_1 \times \mu_1$. Then $\nu = \{\nu_0; \nu_1, \nu_2\} \in \Lambda(X,A)$ and $\nu > \lambda$, $\nu > \mu$. Hence $\{\Lambda(X,A), >\}$ is a directed set.

Let $\lambda = \{\lambda_0; \lambda_1, \lambda_2\}$ be a covering of (X,A). A finite non-null subset σ of λ_1 is a simplex of K_λ if and only if $\cap\{\lambda_0(u) \mid u \in \sigma\} \neq \emptyset$. Clearly K_λ is a simplicial complex. Let L_λ be the subcollection of K_λ such that σ is in L_λ if and only if $\sigma \subset \lambda_2$ and $A \cap (\cap\{\lambda_0(u) \mid u \in \sigma\}) \neq \emptyset$. Then L_λ is a subcomplex of K_λ. The pair (K_λ, L_λ) of complexes is called the <u>nerve</u> of λ. Note that $S(K_\lambda) = \{u \mid u \in \lambda_1 \text{ and } \lambda_0(u) \neq \emptyset\}$ and $S(L_\lambda) = \{u \mid u \in \lambda_2 \text{ and } A \cap \lambda_0(u) \neq \emptyset\}$.

Let λ and μ be coverings of (X,A) with $\mu > \lambda$. Then there is a function $p: \mu_1 \longrightarrow \lambda_1$ such that (i) for each $v \in \mu_1$, $\mu_0(v) \subset \lambda_0(p(v))$ and (ii) $p(\mu_2) \subset \lambda_2$. If $v \in S(K_\mu)$, then $\mu_0(v) \neq \emptyset$. Therefore $\lambda_0(p(v)) \neq \emptyset$, or $p(v) \in S(K_\lambda)$. Hence there is a function $\pi_{\mu\lambda}: S(K_\mu) \longrightarrow$

$S(K_\lambda)$ defined by $\pi_{\mu\lambda}(v) = p(v)$, $v \in S(K_\mu)$. If $\sigma \in K_\mu$, then $\cap \{\mu_0(v) \mid v \in \sigma\} \neq \emptyset$. Therefore $\cap \{\lambda_0(\pi_{\mu\lambda}(v)) \mid v \in \sigma\} \neq \emptyset$, or $\pi_{\mu\lambda}(\sigma) \in K_\lambda$. Similarly, if $\sigma \in L_\mu$, then $\pi_{\mu\lambda}(\sigma) \in L_\lambda$. Hence $\pi_{\mu\lambda}$ is a simplicial mapping, called a <u>projection</u>, of (K_μ, L_μ) into (K_λ, L_λ). By (2.13), induces a homomorphism $\pi^*_{\mu\lambda}: H^p(K_\lambda, L_\lambda) \longrightarrow H^p(K_\mu, L_\mu)$. In general, there will be many projections of (K_μ, L_μ) into (K_λ, L_λ). If $\pi'_{\mu\lambda}$ is a second choice, then for any $\sigma \in K_\mu$ we have $(\cap \{\lambda_0(\pi_{\mu\lambda}(v)) \mid v \in \sigma\}) \cap (\cap \{\lambda_0(\pi'_{\mu\lambda}(v)) \mid v \in \sigma\}) \supset \cap \{\mu_0(v) \mid v \in \sigma\} \neq \emptyset$ and hence $\pi_{\mu\lambda}(\sigma) \cup \pi'_{\mu\lambda}(\sigma) \in K_\lambda$. Similarly for each $\sigma \in L_\mu$, $\pi_{\mu\lambda}(\sigma) \cup \pi'_{\mu\lambda}(\sigma) \in L_\lambda$. It follows by (2.18) that $\pi^*_{\mu\lambda} = \pi'^*_{\mu\lambda}$. Consequently, for each $\mu > \lambda$, $\lambda, \mu \in \Lambda(X,A)$, there is a unique homomorphism

$$\pi^*_{\mu\lambda}: \; H^p(K_\lambda, L_\lambda) \longrightarrow H^p(K_\mu, L_\mu)$$

induced by the projections of (K_μ, L_μ) into (K_λ, L_λ). If $\nu > \mu > \lambda$, $\lambda, \mu, \nu \in \Lambda(X,A)$ and $\pi_{\nu\mu}, \pi_{\mu\lambda}$ are projections, then $\pi_{\mu\lambda}\pi_{\nu\mu}$ is a projection of (K_ν, L_ν) into (K_λ, L_λ). By (2.16),

$$\pi^*_{\nu\lambda} = (\pi_{\mu\lambda}\pi_{\nu\mu})^* = \pi^*_{\nu\mu}\pi^*_{\mu\lambda}.$$

Hence $\{H^p; \pi^*; \Lambda(X,A), >\}$ is a direct system. The limit-group

$$\check{H}^p(X,A) = \varinjlim \{H^p; \pi^*; \Lambda(X,A), >\}$$

is the p-th unrestricted Čech cohomology group of X mod A. The natural homomorphism of $H^p(K_\lambda, L_\lambda)$, $\lambda \in \Lambda(X,A)$, into $\check{H}^p(X,A)$ will be denoted by η_λ. If $A = \emptyset$, $\check{H}^p(X,\emptyset)$ will be designated by $\check{H}^p(X)$.

Let (X,A) and (Y,B) be pairs. By a function $f: (X,A) \longrightarrow (Y,B)$ we mean a function f from X to Y such that $f(A) \subset B$. A mapping $f: (X,A) \longrightarrow (Y,B)$ is a function from (X,A) to (Y,B) with $f: X \longrightarrow Y$ continuous. Let $f: (X,A) \longrightarrow (Y,B)$ be a mapping. Given any $\sigma \in \Lambda(Y,B)$ let $f^{-1}\sigma = \{f^{-1}\sigma_0; \sigma_1, \sigma_2\}$; then $f^{-1}\sigma \in \Lambda(X,A)$. By $\lambda \succ \sigma$, $\lambda \in \Lambda(X,A)$, $\sigma \in \Lambda(Y,B)$ we mean that $\lambda > f^{-1}\sigma$. If $\lambda \succ \sigma$, there is a function $f_{\lambda\sigma}: S(K_\lambda) \longrightarrow S(K_\sigma)$ such that (i) for each $u \in S(K_\lambda)$, $\lambda_0(u) \subset f^{-1}(\sigma_0(f_{\lambda\sigma}(u)))$, or $f\lambda_0(u) \subset \sigma_0 f_{\lambda\sigma}(u)$, and (ii) $f_{\lambda\sigma}(S(L_\lambda)) \subset S(L_\sigma)$. Just as $\pi_{\mu\lambda}$, $f_{\lambda\sigma}$ is a simplicial mapping, called an f-projection, of (K_λ, L_λ) into (K_σ, L_σ). Using (2.18), we can easily show that all the f-projections of (K_λ, L_λ) into (K_σ, L_σ) induce the same homomorphism

$$f^*_{\lambda\sigma}: \; H^p(K_\sigma, L_\sigma) \longrightarrow H^p(K_\lambda, L_\lambda).$$

Let $F = \{f^*_{\lambda\sigma} \mid \lambda \in \Lambda(X,A),\ \sigma \in \Lambda(Y,B) \text{ and } \lambda \succ \sigma\}$. We claim

(4.1) F _induces a unique homomorphism_ $\check{f}: \check{H}^p(Y,B) \longrightarrow \check{H}^p(X,A)$ _such_ $\eta_\lambda f^*_{\lambda\sigma} = \check{f}\eta_\sigma$ _for all_ $\lambda \succ \sigma$, _where_ η_λ: $H^p(K_\lambda, L_\lambda) \longrightarrow \check{H}^p(X,A)$ _and_ $\eta_\sigma: H^p(K_\sigma, L_\sigma) \longrightarrow H^p(Y,B)$ _are natural homomorphisms_.

Proof. It is sufficient to show the conditions (i), (ii) of (3.4). Given any $\sigma \in \Lambda(Y,B)$, $f^{-1}\sigma \in \Lambda(X,A)$ and $f^{-1}\sigma \succ \sigma$, proving (i). If $\mu > \lambda \succ \sigma$ and $\pi_{\mu\lambda}$ is a projection of (K_μ, L_μ) into (K_λ, L_λ) and $f_{\lambda\sigma}$ is an f-projection of (K_λ, L_λ) into (K_σ, L_σ), then for each $v \in S(K_\mu)$, $\mu_o(v) \subset \lambda_o(\pi_{\mu\lambda}(v)) \subset f^{-1}(\sigma_o(f_{\lambda\sigma}(\pi_{\mu\lambda}(v))))$ and $f_{\lambda\sigma}\pi_{\mu\lambda}(S(L_\mu)) \subset f_{\lambda\sigma}(S(L_\lambda)) \subset S(L_\sigma)$. Therefore $\mu \succ \sigma$ and $f_{\lambda\sigma}\pi_{\mu\lambda}$ is an f-projection of (K_μ, L_μ) into (K_σ, L_σ). Hence

$$f^*_{\mu\sigma} = (f_{\lambda\sigma}\pi_{\mu\lambda})^* = \pi^*_{\mu\lambda} f^*_{\lambda\sigma}.$$

Similarly, if $\lambda \succ \sigma > \rho$, then $\lambda \succ \rho$ and $f^{*}_{\lambda\rho} = f^{*}_{\lambda\sigma}\pi^{*}_{\sigma\rho}$.
Hence (ii) is proved. Q.E.D.

Let (X,A) be a pair and let i be the injection of $(A,\emptyset)$ into $(X,\emptyset)$. Given any $\lambda \in \Lambda(X,A)$ we denote $\tilde{\lambda}_0 = i^{-1}(\lambda_0|\lambda_2)$ and $\bar{\lambda} = \{\tilde{\lambda}_0; \lambda_2, \emptyset\}$; then $\bar{\lambda} \in \Lambda(A,\emptyset)$. We can easily see that $S(L_\lambda) = S(K_{\bar{\lambda}})$ and the injection $\theta_{\lambda\bar{\lambda}}: S(L_\lambda) \longrightarrow S(K_{\bar{\lambda}})$ is a simplicial mapping of L_λ into $K_{\bar{\lambda}}$. Let $\lambda \in \Lambda(X,A)$ and $\sigma \in \Lambda(A,\emptyset)$. By $\lambda \vdash \sigma$ we mean $\bar{\lambda} > \sigma$. Let $\lambda \vdash \sigma$ and let $\delta_\lambda: H^p(L_\lambda) \longrightarrow H^{p+1}(K_\lambda, L_\lambda)$ be the coboundary operator defined by (2.14). Then

$$\delta_{\lambda\sigma} = \delta_\lambda \theta^{*}_{\lambda\bar{\lambda}} \pi^{*}_{\bar{\lambda}\sigma} \tag{4.2}$$

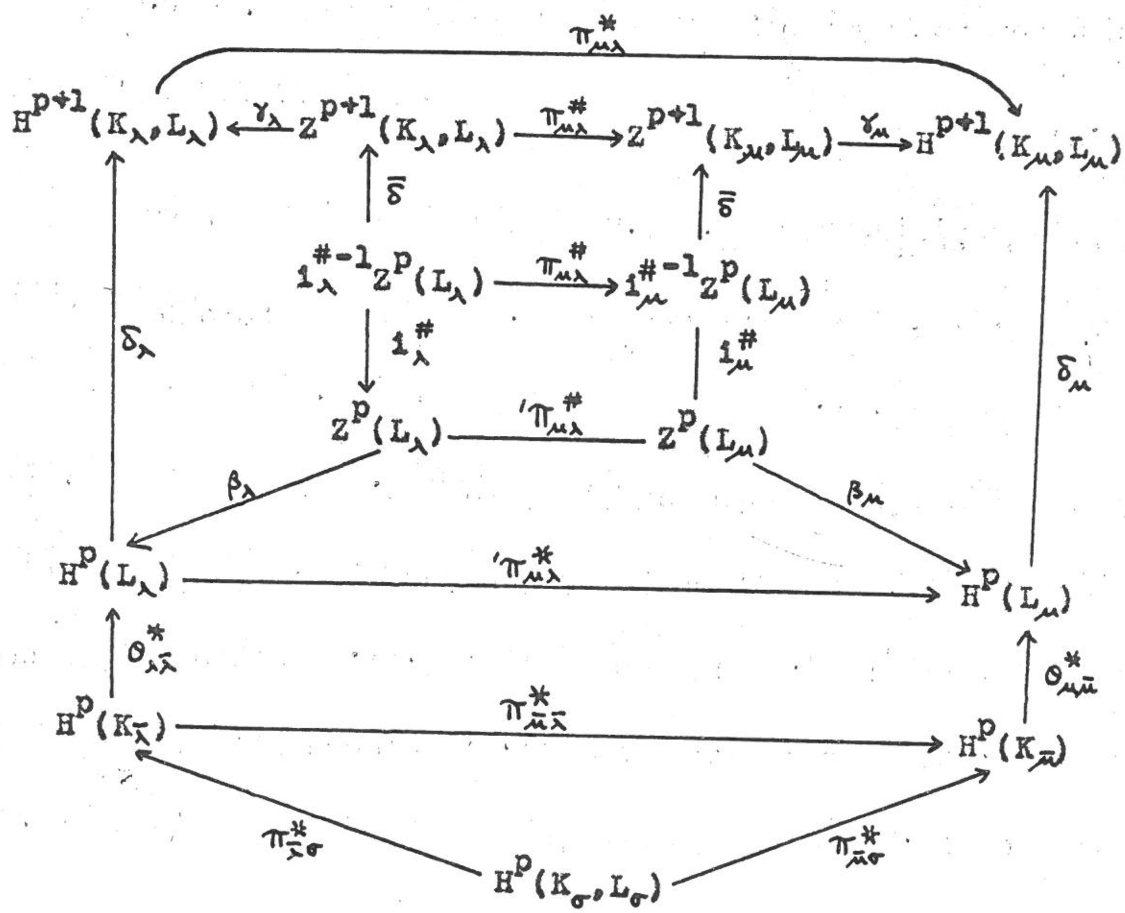
is a homomorphism of $H^p(K_\sigma, L_\sigma)$ into $H^{p+1}(K_\lambda, L_\lambda)$. Let $\Delta = \{\delta_{\lambda\sigma} \mid \lambda \in \Lambda(X,A),\ \sigma \in \Lambda(A,\emptyset) \text{ and } \lambda \vdash \sigma\}$. We assert

(4.3) Δ <u>induces a unique homomorphism</u> $\check{\delta}: \check{H}^p(A)$ $\check{H}^{p+1}(X,A)$ <u>such that</u> $\eta_\lambda\delta_{\lambda\sigma} = \check{\delta}\eta_\sigma$ <u>for all</u> $\lambda \vdash \sigma$, <u>where</u> η_λ: $H^p(K_\lambda, L_\lambda) \longrightarrow \check{H}^p(X,A)$ <u>and</u> $\eta_\sigma: H^{p+1}(K_\sigma, L_\sigma) \longrightarrow \check{H}^{p+1}(A)$ <u>are natural homomorphisms</u>.

Proof. We have only to show the conditions (i) and

and (ii) of (3.4). Let $\sigma \in \Lambda(A,\emptyset)$. Let u_o be an element not contained in σ_1 and let $\lambda_2 = \sigma_1$ and $\lambda_1 = \lambda_2 \cup \{u_o\}$. Define a function $\lambda_o : \lambda_1 \longrightarrow Q(X)$ (the collection of all the open subsets of X) such that $\lambda_o(u_o) = X$ and, for each $u \in \lambda_2$, $\lambda_o(u) = A \cap \sigma_o(u)$. Then $\lambda = \{\lambda_o;\lambda_1,\lambda_2\} \in \Lambda(X,A)$ and $\lambda \vdash \sigma$. Hence (i) is proved.

If $\lambda \vdash \sigma > \rho$, then $\bar{\lambda} > \sigma > \rho$ and hence $\bar{\lambda} > \rho$, or $\lambda \vdash \rho$. By (4.2) we have $\delta_{\lambda\sigma}\pi^*_{\sigma\rho} = \delta_\lambda \theta^*_{\lambda\bar{\lambda}}\pi^*_{\bar{\lambda}\sigma}\pi^*_{\sigma\rho} = \delta_\lambda \theta^*_{\lambda\bar{\lambda}}\pi^*_{\bar{\lambda}\rho} = \delta_{\lambda\rho}$.



If $\mu > \lambda \vdash \sigma$, then $\bar{\mu} > \bar{\lambda} > \sigma$ and hence $\bar{\mu} > \sigma$, or $\mu \vdash \sigma$.

Let $\pi_{\mu\lambda}$ be a projection of (K_μ, L_μ) into (K_λ, L_λ) and let $i_\lambda: L_\lambda \longrightarrow K_\lambda$ and $i_\mu: L_\mu \longrightarrow K_\mu$ be the injections. Define $'\pi_{\mu\lambda} = \pi_{\mu\lambda}|L_\mu$. Then $i_\mu{}'\pi_{\mu\lambda} = \pi_{\mu\lambda} i_\mu$ and hence $'\pi^{\#}_{\mu\lambda} i^{\#}_\lambda = i^{\#}_\mu \pi^{\#}_{\mu\lambda}$. Clearly $'\pi^{\#}_{\mu\lambda} Z^p(L_\lambda) \subset Z^p(L_\mu)$ and $\pi^{\#}_{\mu\lambda} Z^{p+1}(K_\lambda, L_\lambda) \subset Z^{p+1}(K_\mu, L_\mu)$. Since $i^{\#}_\mu \pi^{\#}_{\mu\lambda} i^{\#-1}_\lambda Z^p(L_\lambda) = {}'\pi^{\#}_{\mu\lambda} i^{\#}_\lambda i^{\#-1}_\lambda Z^p(L\ ) = {}'\pi^{\#}_{\mu\lambda} Z^p(L_\lambda) \subset Z^p(L_\mu)$, $\pi^{\#}_{\mu\lambda} i^{\#-1}_\lambda Z^p(L_\lambda) \subset i^{\#-1}_\mu Z^p(L_\mu)$. Moreover, $\bar{\delta}\pi^{\#}_{\mu\lambda} = \pi^{\#}_{\mu\lambda}\bar{\delta}$. It follows by (2.14) that $\pi^{*}_{\mu\lambda}\delta_\lambda\beta_\lambda i^{\#}_\lambda = \pi^{*}_{\mu\lambda}\gamma_\lambda\bar{\delta} = \gamma_\mu\pi^{\#}_{\mu\lambda}\bar{\delta} = \gamma_\mu\bar{\delta}\pi^{\#}_{\mu\lambda} = \delta_\mu\beta_\mu i^{\#}_\mu \pi^{\#}_{\mu\lambda} = \delta_\mu\beta_\mu{}'\pi^{\#}_{\mu\lambda} i^{\#}_\lambda = \delta_\mu{}'\pi^{*}_{\mu\lambda}\beta_\lambda i^{\#}_\lambda$ on $i^{\#-1}_\lambda Z^p(L_\lambda)$, or $\pi^{*}_{\mu\lambda}\delta_\lambda = \delta_\mu{}'\pi^{*}_{\mu\lambda}$. $'\pi_{\mu\lambda}$ induces a projection $\pi_{\bar{\mu}\bar{\lambda}}$: $K_{\bar{\mu}} \longrightarrow K_{\bar{\lambda}}$ such that $\pi_{\bar{\mu}\bar{\lambda}}\theta_{\mu\bar{\mu}} = \theta_{\lambda\bar{\lambda}}{}'\pi_{\mu\lambda}$. Hence $\theta^{*}_{\mu\bar{\mu}}\pi^{*}_{\bar{\mu}\bar{\lambda}} = {}'\pi^{*}_{\mu\lambda}\theta^{*}_{\lambda\bar{\lambda}}$. Since $\pi_{\bar{\mu}\bar{\lambda}}$, $\pi_{\bar{\mu}\sigma}$, $\pi_{\bar{\lambda}\sigma}$ are projections, $\pi^{*}_{\bar{\mu}\bar{\lambda}}\pi^{*}_{\bar{\lambda}\sigma} = \pi^{*}_{\bar{\mu}\sigma}$. Hence $\pi^{*}_{\mu\lambda}\delta_{\lambda\sigma} = \pi^{*}_{\mu\lambda}\delta_\lambda\theta^{*}_{\lambda\bar{\lambda}}\pi^{*}_{\bar{\lambda}\sigma} = \delta_\mu{}'\pi^{*}_{\mu\lambda}\theta^{*}_{\lambda\bar{\lambda}}\pi^{*}_{\bar{\lambda}\sigma} = \delta_\mu\theta^{*}_{\mu\bar{\mu}}\pi^{*}_{\bar{\mu}\bar{\lambda}}\pi^{*}_{\bar{\lambda}\sigma} = \delta_\mu\theta^{*}_{\mu\bar{\mu}}\pi^{*}_{\bar{\mu}\sigma} = \delta_{\mu\sigma}$. The condition (ii) is proved. Q.E.D.

Remark 1. The system $(H^p, \check{f}, \check{\delta})$ is the <u>unrestricted Cech cohomology theory</u> for general spaces. If we replace the ordered cohomology theory (H^p, j^{*}, δ) by the oriented cohomology theory $(\tilde{H}^p, \tilde{j}^{\#}, \tilde{\delta})$ (see the Remark of Section 2), then we can establish a new cohomology theory for general spaces, i.e., the Čech cohomology theory based on infinite coverings in Dowker [2]. Using the remark of Section 2 $(H^p; j^*, \delta)$ and $(\tilde{H}^p, \tilde{j}^*, \tilde{\delta})$ are equivalent; hence, by Dowker's result, the unrestricted Čech cohomology theory $(\check{H}^p, \check{f}, \check{\delta})$ for general spaces satisfies the Eilenberg-Steenrod axioms. The following (4.4) is a fundamental lemma used to prove the homotopy axiom and (4.5) is the exactness axiom.

(4.4) _Let_ (X,A) _be a pair and let_ I _be the closed interval from_ 0 _to_ 1 _with the usual topology. If_ $h_i: (X,A) \longrightarrow (X\times I, A\times I)$ _is defined by_ $h_i(x) = (x,i)$, $i = 0, 1$, _then_ $\check{h}_0 = \check{h}_1$.

(4.5) _Let_ (X,A) _be a pair and let_ $i: (A,\emptyset) \longrightarrow (X,\emptyset)$ _and_ $j: (X,\emptyset) \longrightarrow (X,A)$ _be the injections. Then the sequence of groups and homomorphisms_

$$\check{H}^0(X,A) \xrightarrow{\check{j}} \check{H}^0(X) \xrightarrow{\check{i}} \cdots \xrightarrow{\check{j}} \check{H}^p(X) \xrightarrow{\check{i}} \check{H}^p(A)$$
$$\xrightarrow{\check{\delta}} H^{p+1}(X,A) \xrightarrow{j} \cdots$$

is exact.

Proof. We prove only that the kernel of $\check{i}: \check{H}^p(X) \longrightarrow$

$\check{H}^p(A)$ is equal to the image of $\check{j}$: $\check{H}^p(X,A) \longrightarrow \check{H}^p(X)$. All the rest can be proved in a similar way.

Let $\{\Lambda, >\} = \{\Lambda(X,A), >\}$ and define $t_1 : \Lambda \longrightarrow \Lambda(X,A)$, $t_2 : \Lambda \longrightarrow \Lambda(X,\emptyset)$, $t_3 : \Lambda \longrightarrow \Lambda(A,\emptyset)$ respectively by, for $\lambda = \{\lambda_o ; \lambda_1, \lambda_2\} \in \Lambda$, ...

$$t_1(\lambda) = \{\lambda_o ; \lambda_1, \lambda_2\}, \qquad t_2(\lambda) = \{\lambda_o ; \lambda_1, \emptyset\},$$
$$t_3(\lambda) = \{\tilde{\lambda}_o ; \lambda_2, \emptyset\}.$$

It is easily seen that, if $\mu > \lambda$, $\lambda, \mu \in \Lambda$, then $t_i(\ ) > t_i(\lambda)$, $i = 1, 2, 3$, proving (i) of (3.7). $t_1(\Lambda) = \Lambda(X,A)$. Given any $\mu = \{\mu_o ; \mu_1, \mu_2\} \in \Lambda(X,\emptyset)$ let $\lambda_1 = \mu_1$, $\lambda_2 = \{v \mid v \in \mu_1 \text{ and } \mu_o(v) \cap A \neq \emptyset\}$ and define $\lambda_o : \lambda_1 \longrightarrow Q(X)$ by $\lambda_o = \mu_o$. Then $\lambda = \{\lambda_o ; \lambda_1, \lambda_2\} \in \Lambda$ and $t_2(\lambda) = \{\lambda_o ; \lambda_1, \emptyset\} > \{\mu_o ; \mu_1, \mu_2\}$. Hence $t_2(\Lambda)$ is cofinal in $\{\Lambda(X,\emptyset), >\}$. Similarly, $t_3(\Lambda)$ is cofinal in $\{\Lambda(A,\emptyset), >\}$. (ii) of (3.7) is proved. Fix a $\lambda \in \Lambda$; it is clear that the nerves of $t_1(\lambda)$, $t_2(\lambda)$, $t_3(\lambda)$ are (K_λ, L_λ), $(K_\lambda, \emptyset)$, $(L_\lambda, \emptyset)$. Let $i_\lambda : (L_\lambda, \emptyset) \longrightarrow (K_\lambda, \emptyset)$ and $j_\lambda : (K_\lambda, \emptyset) \longrightarrow (K_\lambda, L_\lambda)$ be the injections, then, by (2.19), the kernel of i^*_λ: $H^p(K_\lambda) \longrightarrow H^p(L_\lambda)$ is equal to the image of j^*_λ: $H^p(K_\lambda, L_\lambda) \longrightarrow$

$H^p(K_\lambda)$. Let $\mu > \lambda$, $\lambda, \mu \in \Lambda$. Then, by (2.16) and (2.17), the commutativity relation holds in each square of the following diagram:

$$\begin{array}{ccccc}
H^p(K_\mu, L_\mu) & \xrightarrow{j_\mu^*} & H^p(K_\mu) & \xrightarrow{i_\mu^*} & H^p(L_\mu) \\
\uparrow{\scriptstyle \pi^*_{t_1(\mu)t_1(\lambda)}} & & \uparrow{\scriptstyle \pi^*_{t_2(\mu)t_2(\lambda)}} & & \uparrow{\scriptstyle \pi^*_{t_3(\mu)t_3(\lambda)}} \\
H^p(K_\lambda, L_\lambda) & \xrightarrow{j_\lambda^*} & H^p(K_\lambda) & \xrightarrow{i_\lambda^*} & H^p(L_\lambda)
\end{array}$$

Hence (iii) of (3.7) holds. It follows by (3.7) that the kernel of i is equal to the image of j. Q.E.D.

Remark 2. Let (X,A) be a pair. A covering $\lambda = \{\lambda_0; \lambda_1, \lambda_2\}$ of (X,A) is said to be finite if and only if λ_1 is finite. Denote by $\mathring{\Lambda}(X,A)$ the set of all the finite coverings of (X,A). Then $\mathring{\Lambda}(X,A)$ is a subset of $\Lambda(X,A)$ and $\{\mathring{\Lambda}(X,A), >\}$ is directed. If we replace $\Lambda(X,A)$ by $\mathring{\Lambda}(X,A)$ in the preceding discussion, then we can develop another cohomology theory $(\mathring{H}^p, \mathring{f}, \mathring{\delta})$ for general spaces, that is, the restricted Čech cohomology theory. Usually the restricted Čech cohomology theory is established by using the oriented cohomology theory of simplicial complexes. But it is equivalent to $(\mathring{H}^p, \mathring{f}, \mathring{\delta})$ because of the Remark of Section 2.

(4.6) <u>For compact Hausdorff spaces the unrestricted Čech cohomology theory is equivalent to the restricted Čech cohomology theory.</u>

Proof. Let (X,A) be a pair. Then

$$\check{H}^p(X,A) = \varinjlim \{H^p; \pi^*; \Lambda(X,A), >\},$$

$$\mathring{H}^p(X,A) = \varinjlim \{H^p; \pi^*; \mathring{\Lambda}(X,A), >\}.$$

By (3.6), there is a homomorphism $\chi: \mathring{H}^p(X,A) \longrightarrow \check{H}^p(X,A)$ such that for any $\lambda \in \mathring{\Lambda}(X,A)$, $\eta_\lambda = \chi\mathring{\eta}_\lambda$, where $\eta_\lambda: H^p(K_\lambda, L_\lambda) \longrightarrow \check{H}^p(X,A)$ and $\mathring{\eta}_\lambda : H^p(K_\lambda, L_\lambda) \longrightarrow \mathring{H}^p(X,A)$ are natural homomorphisms. Let $f: (X,A) \longrightarrow (Y,B)$ be a mapping; we assert that $\check{f}\chi = \chi\mathring{f}$. Given any $e \in H^p(Y,B)$ there is, by (3.2) some $\sigma \in \mathring{\Lambda}(Y,B)$ and some $e_\sigma \in H^p(K_\sigma, L_\sigma)$ such that $e = \eta_\sigma e_\sigma$. Let $\lambda = f^{-1}\sigma = \{f^{-1}\sigma_0; \sigma_1, \sigma_2\}$; then $\lambda \in \mathring{\Lambda}(X,A)$ and $\lambda \succ \sigma$. By (3.6) and (4.1), we have $\chi\mathring{f}e = \chi\mathring{f}\mathring{\eta}_\sigma e_\sigma = \chi\mathring{\eta}_\lambda f^*_{\lambda\sigma} e_\sigma = \eta_\lambda f^*_{\lambda\sigma} e_\sigma = \check{f}\eta_\sigma e_\sigma = \check{f}\chi\mathring{\eta}_\sigma e_\sigma = \check{f}\chi e$. Hence $\chi\mathring{f} = \check{f}\chi$. Similarly we have $\chi\mathring{\delta} = \check{\delta}\chi$.

Now let (X,A) be a pair of compact Hausdorff spaces, that is, X is compact Hausdorff and A, with the relative topology, is also compact Hausdorff. Then A is closed in X and $\mathring{\Lambda}(X,A)$ is cofinal in $\{\Lambda(X,A),>\}$. By (3.6), χ: $\mathring{H}^p(X,A) \approx \check{H}^p(X,A)$. Hence (4.6) is proved. Q.E.D.

5. Alexander-Kolmogoroff Cohomology Theory.

Let X be a topological space and let p be a non-negative integer. For each cover $\{\lambda_0;\lambda_1\}$ of X, $\cup\{\lambda_0(u)^{p+1} \mid u \in \lambda_1\}$ is a neighborhood of ΔX^{p+1} in X^{p+1}, denoted by $\{\lambda_0;\lambda_1\}^{(p+1)}$, where ΔX^{p+1} is the diagonal of X^{p+1}. Let A be a subset of X and let i be the injection of A into X. If $\{\lambda_0;\lambda_1\}$ is a cover of A in X, then $\{i^{-1}\lambda_0;\lambda_1\}$ is a cover of A (in itself) and $\cup\{(i^{-1}\lambda_0(u))^{p+1} \mid u \in \lambda_1\}$ is a neighborhood of ΔA^{p+1} (the diagonal of A^{p+1}) in A^{p+1}, denoted by $\{i^{-1}\lambda_0;\lambda_1\}^{(p+1)}$. If $\{\lambda_0;\lambda_1\}$ and $\{\mu_0;\mu_1\}$ are covers of A in X, we define $\lambda_0 \wedge \mu_0 \colon \lambda_1 \times \mu_1 \longrightarrow Q(X)$ ($\lambda_1 \times \mu_1$ is the cartesian product of λ_1 and μ_1; Q(X) is the collection of open subsets of X) by $(\lambda_0 \wedge \mu_0)(u,v) = \lambda_0(u) \cap \mu_0(v)$, $(u,v) \in \lambda_1 \times \mu_1$. Then $\{\lambda_0 \wedge \mu_0; \lambda_1 \times \mu_1\}$ is a cover of A in X.

Let (X,A) be a pair. A p-<u>cochain</u> of X is a function from X^{p+1} to G (G is the coefficient group). The set of all the p-cochains of X is a group $C^p(X)$ with functional addition as its group operation. A p-cochain φ of X is a p-cochain of X mod A if and only if there is a cover $\{\lambda_o;\lambda_1\}$ of A in X such that $\varphi = 0$ on $\{i^{-1}\lambda_o;\lambda_1\}^{(p+1)}$ (i is the injection of A into X). We can easily show that the set of all the p-cochains of X mod A is a subgroup $C^p(X,A)$ of $C^p(X)$.

There is a homomorphism $\bar{\delta} : C^p(X) \longrightarrow C^{p+1}(X)$ defined by

(5.1) $(\bar{\delta}\varphi)(x_o,\ldots,x_{p+1}) = \sum_{i=1}^{p+1}(-1)^i\varphi(x_o,\ldots,\hat{x}_i,\ldots,x_{p+1})$.

The following are immediate:

(5.2) $\bar{\delta}\bar{\delta} = 0$.

(5.3) $\bar{\delta}\, C^p(X,A) \subset C^{p+1}(X,A)$.

Let

$$C_o^p(X,A) = C^p(X,X);$$

$$Z^p(X,A) = \bar{\delta}^{-1}C_o^{p+1}(X,A) \cap C^p(X,A);$$

$$B^p(X,A) = \begin{cases} \bar{\delta}\, C^{p-1}(X,A) + C_o^p(X,A) & \text{if } p > 0, \\ \{0\} & \text{if } p = 0. \end{cases}$$

It follows by (5.3) and (5.2) that $B^p(X,A)$ is a subgroup of $Z^p(X,A)$. $Z^p(X,A)$ is the group of p-cocycles of X mod A and $B^p(X,A)$ is the group of p-coboundaries of X mod A. The factor group

$$H^p(X,A) = Z^p(X,A)/B^p(X,A)$$

is known as the p-th Alexander-Kolmogoroff cohomology group of X mod A and its elements as the p-th cohomology classes of X mod A. If $A = \emptyset$, then $C^p(X,\emptyset) = C^p(X)$. The groups $Z^p(X,\emptyset)$, $B^p(X,\emptyset)$, $H^p(X,\emptyset)$ will be respectively designated by $Z^p(X)$, $B^p(X)$, $H^p(X)$.

If $f: X \longrightarrow Y$ is a function, then there is a homomorphism $f^{\#}: C^p(Y) \longrightarrow C^p(X)$ defined by

$$(5.4) \qquad (f^{\#}\varphi)(x_o,\dots,x_p) = \varphi(f(x_o),\dots,f(x_p))$$

for $\varphi \in C^p(Y)$ and $(x_o,\dots,x_p) \in X^{p+1}$. By (5.1) and (5.4) we can easily verify that $f^{\#} = \bar{\delta} f^{\#}$. Now suppose that (X,A), (Y,B) are pairs and $f: (X,A) \longrightarrow (Y,B)$ is a mapping. Then $f^{\#}C^p(Y,B) \subset C^p(X,A)$ and $f^{\#}C_o^p(Y,B) \subset C_o^p(X,A)$. Therefore $f^{\#}Z^p(Y,B) \subset Z^p(X,A)$ and $f^{\#}B^p(Y,B) \subset B^p(X,A)$. Consequently there is a homomorphism $f^*: H^p(Y,B) \longrightarrow H^p(X,A)$ such that

$$f^{*}\gamma_{1} = \gamma f^{\#} \quad \text{on} \quad Z^{p}(Y,B),$$

where $\gamma : Z^{p}(X,A) \longrightarrow H^{p}(X,A)$ and $\gamma_{1} : Z^{p}(Y,B) \longrightarrow H^{p}(Y,B)$ are natural homomorphisms. f^{*} is the induced homomorphism of the mapping f in the Alexander-Kolmogoroff cohomology theory.

Let (X,A) be a pair and let i be the injection of A into X. As (2.14), there is homomorphism $\delta : H^{p}(A) \longrightarrow H^{p+1}(X,A)$ such that

$$\delta \beta i^{\#} = \gamma \bar{\delta} \quad \text{on} \quad i^{\#-1}Z^{p}(A),$$

where $\beta : Z^{p}(A) \longrightarrow H^{p}(A)$ and $\gamma : Z^{p+1}(X,A) \longrightarrow H^{p+1}(X,A)$ are natural homomorphisms. δ is the coboundary operator in the Alexander-Kolmogoroff cohomology theory.

According to Spanier [7], the Alexander-Kolmogoroff cohomology theory (H^{p}, f^{*}, δ) for general spaces satisfies the Eilenberg-Steenrod axioms except the homotopy axiom. The homotopy axiom was proved by Spanier only when the spaces are compact Hausdorff. This will be generalized in (7.10).

Here we state several results without proof which are needed in Section 7. For more details, see Spanier [7].

(5.5) (Algebraic axiom 1) If $i: (X,A) \longrightarrow (X,A)$ is the identity mapping, then $i^{*}: H^{p}(X,A) \longrightarrow H^{p}(X,A)$ is the identity isomorphism.

(5.6) (Algebraic axiom 2) If $f: (X,A) \longrightarrow (Y,B)$ and $g: (Y,B) \longrightarrow (Z,C)$ are mappings, then $(gf)^{*} = f^{*}g^{*}$.

(5.7) (Exactness axiom) Let $i: (A,\emptyset) \longrightarrow (X,\emptyset)$ and $j: (X,\emptyset) \longrightarrow (X,A)$ be the injections; then the sequence of groups and homomorphisms

$$H^{0}(X,A) \xrightarrow{j^{*}} H^{0}(X) \xrightarrow{i^{*}} H^{0}(A) \xrightarrow{\delta} \cdots \xrightarrow{j^{*}} H^{p}(X)$$

$$\xrightarrow{i^{*}} H^{p}(A) \xrightarrow{\delta} H^{p+1}(X,A) \xrightarrow{j^{*}} \cdots$$

is exact.

(5.8) (Dimension axiom) If X contains a single point, then $H^{p}(X) = \{0\}$ for all $p > 0$ and $H^{0}(X) \approx G$.

(5.9) Let f and g be two functions (not necessarily continuous) from (X,A) to (Y,B) and $\varphi \in Z^{p}(Y,B)$. Let

$\{\sigma_0;\sigma_1\}$ be a cover of Y and $\{\rho_0;\rho_1\}$ a cover of B (in itself) such that $\varphi = 0$ on $\{\rho_0;\rho_1\}^{(p+1)}$ and $\bar{\delta}\varphi = 0$ on $\{\sigma_0;\sigma_1\}^{(p+2)}$. If there is a cover $\{\lambda_0;\lambda_1\}$ of X and a function $p: \lambda_1 \longrightarrow \sigma_1$ such that for each $u \in \lambda_1$, $f(\lambda_0(u)) \cup g(\lambda_0(u)) \subset \sigma_0(p(u))$, and if there is cover $\{\mu_0;\mu_1\}$ of A and a function $q: \mu_1 \longrightarrow \rho_1$ such that for each $v \in \mu_1$, $f(\mu_0(v)) \cup g(\mu_0(v)) \subset \rho_0(q(v))$, then $f^{\#}\varphi$, $g^{\#}\varphi \in Z^p(X,A)$ and $f^{\#}\varphi - g^{\#}\varphi = 0$ or $\bar{\delta}\psi + \psi'$ according to $p = 0$ or $p > 0$, where $\psi \in C^{p-1}(X,A)$ with $\psi = 0$ on $\{\mu_0;\mu_1\}^{(p)}$ and $\psi' \in C_0^p(X,A)$ with $\psi' = 0$ on $\{\lambda_0;\lambda_1\}^{(p+1)}$.

(5.9) is a slight variation of the lemma 9.1 in Spanier [7;p.413]; but their proofs are exactly the same.

6. Full Normality and Some Lemmas.

Let X be a topological space and $\{\lambda_0;\lambda_1\}$ a cover of X. Define a function $\lambda_0^*: \lambda_1 \longrightarrow Q(X)$ by, for each $u \in \lambda_1$,

$$\lambda_0^*(u) = \cup \{ \lambda_0(u_1) \mid u_1 \in \lambda_1 \text{ and } \lambda_0(u_1) \cap \lambda_0(u) \neq \emptyset \}.$$

Then $\{\lambda_0;\lambda_1\}^* = \{\lambda_0^*;\lambda_1\}$ is also a cover of X. Similarly we have $\{\lambda_0;\lambda_1\}^{**} = (\{\lambda_0;\lambda_1\}^*)^* = \{\lambda_0^{**};\lambda_1\}$, where $\lambda_0^{**}: \lambda_1 \longrightarrow Q(X)$ is defined by, for each $u \in \lambda_1$,

$$\lambda_0^{**}(u) = \cup \{\lambda_0^*(u_1) \mid u_1 \in \lambda_1 \text{ and } \lambda_0^*(u_1) \cap \lambda_0^*(u) \neq \emptyset\}.$$

If $\{\lambda_0;\lambda_1\}$ and $\{\mu_0;\mu_1\}$ are covers of X with $\{\mu_0;\mu_1\} > \{\lambda_0;\lambda_1\}$, that is, there is a function $p: \mu_1 \longrightarrow \lambda_1$ such that for each $v \in \mu_1$, $\mu_0(v) \subset \lambda_0(p(v))$, then for each $v \in \mu_1$ we have $\mu_0^*(v) \subset \lambda_0^*(p(v))$ and $\mu_0^{**}(v) \subset \lambda_0^{**}(p(v))$. Therefore $\{\mu_0;\mu_1\}^* > \{\lambda_0;\lambda_1\}^*$ and $\{\mu_0;\mu_1\}^{**} > \{\lambda_0;\lambda_1\}^{**}$. For any cover $\{\lambda_0;\lambda_1\}$ of X we have immediately $\{\lambda_0;\lambda_1\} > \{\lambda_0;\lambda_1\}^* > \{\lambda_0;\lambda_1\}^{**}$.

Let $\{\lambda_0;\lambda_1\}$ and $\{\mu_0;\mu_1\}$ be covers of X. $\{\mu_0;\mu_1\}$ is a <u>refinement</u>, or a *-<u>refinement</u>, or a **-<u>refinement</u> of $\{\lambda_0;\lambda_1\}$ according as $\{\mu_0;\mu_1\} > \{\lambda_0;\lambda_1\}$, or $\{\mu_0;\mu_1\}^* > \{\lambda_0;\lambda_1\}$, or $\{\mu_0;\mu_1\}^{**} > \{\lambda_0;\lambda_1\}$. A topological spaces of which every cover has a *-refinement is <u>fully normal</u>.

According to Tukey [9,p.53] we have

(6.1) <u>Compact Hausdorff spaces and metric spaces are fully normal</u>.

Let (X,A) be a pair and let i be the injection of A into X. For each covering $\lambda = \{\lambda_0;\lambda_1,\lambda_2\}$ of (X,A), $\{\lambda_0;\lambda_1\}$ is a cover of X and hence $\{\lambda_0;\lambda_1\}^*$ and $\{\lambda_0;\lambda_1\}^{**}$ are well-defined. Let $\tilde{\lambda}_0 = i^{-1}(\lambda_0 | \lambda_2)$; then $\{\tilde{\lambda}_0;\lambda_2\}$ is a cover of A (in itself) and hence $\{\tilde{\lambda}_0;\lambda_2\}^*$ and $\{\tilde{\lambda}_0;\lambda_2\}^{**}$ are also well-defined. Let λ, μ be coverings of (X,A). By $\mu > \lambda$ we mean that there is a function $p: \mu_1 \longrightarrow \lambda_1$ such that $p(\mu_2) \subset \lambda_2$ and for each $v \in \mu_1$, $\mu_0(v) \subset \lambda_0(p(v))$. μ is a <u>refinement</u> of λ if and only if $\mu > \lambda$. is a *-<u>refinement</u> of λ if and only if $\mu > \lambda$, $\{\mu_0;\mu_1\}^* > \{\lambda_0;\lambda_1\}$ and $\{\tilde{\mu}_0;\mu_2\}^* > \{\tilde{\lambda}_0;\lambda_2\}$. μ is a **-<u>refinement</u> of λ if and only if $\mu > \lambda$, $\{\mu_0;\mu_1\}^{**} > \{\lambda_0;\lambda_1\}$ and $\{\tilde{\mu}_0;\mu_2\}^{**} > \{\tilde{\lambda}_0;\lambda_2\}$.

A pair (X,A) is <u>fully normal</u> if and only if X is fully normal and A, with the relative topology, is also fully normal.

(6.2) *If (X,A) is fully normal, then every covering of (X,A) has a *-refinement and hence a **-refinement.*

Proof. Let $\lambda = \{\lambda_o;\lambda_1,\lambda_2\}$ be a covering of (X,A). Then $\{\lambda_o;\lambda_1\}$ is a cover of X and it has a *-refinement $\{\sigma_o;\sigma_1\}$. Moreover, $\{\tilde{\lambda}_o;\lambda_2\}$ is a cover of A and it has a *-refinement $\{\rho_o;\rho_1\}$. Define a function $\rho_o'\colon \rho_1 \longrightarrow Q(X)$ such that for each $w \in \rho_1$, $\rho_o(w) = A \cap \rho_o'(w)$. Then $\{\rho_o';\rho_1\}$ is a cover of A in X. Let $\mu_2 = \lambda_2 \times \sigma_1 \times \rho_1$ and $\mu_1 = \sigma_1 \cup (\lambda_2 \times \sigma_1 \times \rho_1)$. Define $\mu_o\colon \mu_1 \longrightarrow Q(X)$ by $\mu_o | \sigma_1 = \sigma_o$ and $\mu_o | (\lambda_2 \times \sigma_1 \times \rho_1) = \lambda_o \wedge \sigma_o \wedge \rho_o'$, that is, for each $v \in \sigma_1$ we have $\mu_o(v) = \sigma_o(v)$ and for each $(u,v,w) \in \lambda_2 \times \sigma_1 \times \rho_1$ we have $\mu_o(u,v,w) = \lambda_o(u) \cap \sigma_o(v) \cap \rho_o'(w)$. Then $\mu = \{\mu_o;\mu_1,\mu_2\}$ is a covering of (X,A) and clearly $\mu > \lambda$, $\{\mu_o;\mu_1\}^* > \{\sigma_o;\sigma_1\}^* > \{\lambda_o;\lambda_1\}$ and $\{\tilde{\mu}_o;\mu_2\}^*$ $\{\rho_o;\rho_1\}^* > \{\tilde{\lambda}_o;\lambda_2\}$. Hence μ is a *-refinement of λ. Using this process, μ has a *-refinement ν. Therefore ν

is a **-refinement of λ.

(6.3) Let (X,A) be fully normal.

(i) For each $\varphi \in Z^p(X,A)$ there is a covering λ of (X,A) such that $\varphi = 0$ on $(\{\tilde{\lambda}_o;\lambda_2\}^{**})^{(p+1)}$ and $\bar{\delta}\varphi = 0$ on $(\{\lambda_o;\lambda_1\}^{**})^{(p+2)}$.

(ii) For each $\varphi \in B^p(X,A)$, $p > 0$, there is a covering λ of (X,A) and a $\psi \in C^{p-1}(X,A)$ such that $\psi = 0$ on $(\{\tilde{\lambda}_o;\lambda_2\}^{**})^{(p)}$ and $\varphi = \bar{\delta}\psi$ on $(\{\lambda_o;\lambda_1\}^{**})^{(p+1)}$. Therefore $\varphi = 0$ on $(\{\tilde{\lambda}_o;\lambda_2\}^{**})^{(p+1)}$ and $\bar{\delta}\varphi = 0$ on $(\{\lambda_o;\lambda_1\}^{**})^{(p+2)}$.

Proof. (i) Since $\varphi \in Z^p(X,A)$, there is a cover $\{\sigma_o;\sigma_1\}$ of X and a cover $\{\rho_o;\rho_1\}$ of A in X such that $\varphi = 0$ on $\{i^{-1}\rho_o;\rho_1\}^{(p+1)}$ and $\bar{\delta}\varphi = 0$ on $\{\sigma_o;\sigma_1\}^{(p+2)}$, where i is the injection of A into X. Let $\mu_2 = \sigma_1 \times \rho_1$, $\mu_1 = \sigma_1 \cup (\sigma_1 \times \rho_1)$ and define $\mu_o: \mu_1 \longrightarrow Q(X)$ by $\mu_o | \sigma_1 = \sigma_o$ and $\mu_o | (\sigma_1 \times \rho_1) = \sigma_o \wedge \rho_o$. Then $\mu = \{\mu_o;\mu_1,\mu_2\}$ is a covering of (X,A) such that $\varphi = 0$ on $\{\tilde{\mu}_o;\mu_2\}^{(p+1)}$ and $\bar{\delta}\varphi = 0$ on $\{\mu_o;\mu_1\}^{(p+2)}$. By (6.2), μ has a **-refinement λ; λ behaves as required.

(ii) Since $\varphi \in B^p(X,A)$, $p > 0$, there is a $\psi \in C^{p-1}(X,A)$ and a $\psi' \in C_o^p(X,A)$ such that $\varphi = \overline{\delta}\psi + \psi'$. Let $\{\sigma_o;\sigma_1\}$ be a cover of X such that $\psi' = 0$ on $\{\sigma_o;\sigma_1\}^{(p+1)}$, and let $\{\rho_o;\rho_1\}$ be a cover of A in X such that $\psi = 0$ on $\{i^{-1}\rho_o;\rho_1\}^{(p)}$. Construct λ as in (i). Then $\psi = 0$ on $(\{\tilde{\lambda}_o;\lambda_2\}^{**})^{(p)}$ and $\varphi = \overline{\delta}\psi$ on $(\{\lambda_o;\lambda_1\}^{**})^{(p+1)}$. Since $(\{\tilde{\lambda}_o;\lambda_2\}^{**})^{(p+1)} \subset (\{\lambda_o;\lambda_1\}^{**})^{(p+1)}$, $\varphi = \overline{\delta}\psi = 0$ on $(\{\tilde{\lambda}_o;\lambda_2\}^{**})^{(p+1)}$. Moreover, it is clear that $\overline{\delta}\varphi = \overline{\delta}\overline{\delta}\psi = 0$ on $(\{\lambda_o;\lambda_1\}^{**})^{(p+2)}$. Q.E.D.

Let (X,A) be a pair. A covering $\lambda = \{\lambda_o;\lambda_1,\lambda_2\}$ is <u>canonical</u> if and only if the following conditions hold:

(i) $u \in \lambda_2$ implies $A \cap \lambda_o(u) \neq \emptyset$.

(ii) $u \in \lambda_1 - \lambda_2$ implies $\lambda_o(u) - A \neq \emptyset$.

(iii) There exists a 1-1 function s_λ from $S(K_\lambda)$ to X such that for each $u \in S(K_\lambda)$, $s_\lambda(u)$ is contained in $A \cap \lambda_o(u)$ or $\lambda_o(u) - A$ according to $u \in \lambda_2$ or $u \in \lambda_1 - \lambda_2$.

Such a function s_λ is a canonical function for λ. By (i) and (ii) we have $S(K_\lambda) = \lambda_1$ and $S(L_\lambda) = \lambda_2$. Moreover, if $A = \emptyset$, then $\lambda_2 = \emptyset$.

(6.4) Every covering of (X,A) has a canonical refinement.

Proof. Let $\lambda = \{\lambda_o; \lambda_1, \lambda_2\}$ be a covering of (X,A). Let $\mu_1 = X$, $\mu_2 = A$ and define a function $p\colon \mu_1 \longrightarrow \lambda_1$ such that $p(\mu_2) \subset \lambda_2$ and for each $v \in \mu_1$, $v \in \lambda_o(p(v))$. The existence of such a function p is obvious. Let $\mu_o = \lambda_o p$; then $\mu = \{\mu_o; \mu_1, \mu_2\}$ is a covering of (X,A) refining λ. If $v \in \mu_2$, then $v \in A \cap \mu_o(v)$ and hence $A \cap \mu_o(v) \neq \emptyset$. If $v \in \mu_1 - \mu_2$, then $v \in (X-A) \cap \mu_o(v) = \mu_o(v) - A$ and hence $\mu_o(v) - A \neq \emptyset$. Define $s_\mu : \mu_1 \longrightarrow X$ by $s_\mu(v) = v$, $v \in \mu_1$. Then s_μ is a canonical function for μ and μ is a canonical covering of (X,A). Q.E.D.

Let X be a topological space and K a simplicial complex. If s is a function from S(K) to X, then s induces a homomorphism $s^\#\colon C^p(X) \longrightarrow C^p(K)$ defined by

$$(s^\#\varphi)(a_o,\ldots,a_p) = \varphi(s(a_o),\ldots,s(a_p))$$

for each $\varphi \in C^p(X)$ and each $(a_o,\ldots,a_p) \in K(p)$. Note that $s^\#\bar{\delta} = \bar{\delta}s^\#$.

If t is a function from X to $S(K)$, then t induces a homomorphism $t^{\#}: C^p(K) \longrightarrow C^p(X)$ defined by

$$(t^{\#}\varphi)(x_o,\ldots,x_p) = \begin{cases} \varphi(t(x_o),\ldots,t(x_p)) & \text{if } (t(x_o),\ldots,t(x_p)) \in K(p), \\ 0 & \text{otherwise} \end{cases}$$

for each $\varphi \in C^p(K)$ and each $(x_o,\ldots,x_p) \in X^{p+1}$. In general $t^{\#}\bar{\delta} \neq \bar{\delta}t^{\#}$; but if $(x_o,\ldots,x_{p+1}) \in X^{p+2}$ such that $(t(x_o),\ldots,t(x_{p+1})) \in K(p+1)$, then $(t^{\#}\bar{\delta}\varphi)(x_o,\ldots,x_{p+1}) = (\bar{\delta}t^{\#}\varphi)(x_o,\ldots,x_{p+1})$ for every $\varphi \in C^p(K)$.

(6.5) Let λ be a canonical covering of (X,A) and s_λ a canonical function for λ.

(i) If $\varphi, \varphi' \in C^p(X)$ such that $\varphi = \varphi'$ on $(\{\lambda_o;\lambda_1\}^*)^{(p+1)}$, then $s_\lambda^{\#}\varphi = s_\lambda^{\#}\varphi'$.

(ii) If $\varphi \in C^p(X,A)$ such that $\varphi = 0$ on $(\{\tilde{\lambda}_o;\lambda_2\}^*)^{(p+1)}$, then $s_\lambda^{\#}\varphi \in C^p(K_\lambda,L_\lambda)$.

(iii) If $\varphi \in Z^p(X,A)$ such that $\varphi = 0$ on $(\{\tilde{\lambda}_o;\lambda_2\}^*)^{(p+1)}$ and $\bar{\delta}\varphi = 0$ on $(\{\lambda_o;\lambda_1\}^*)^{(p+2)}$, then $s_\lambda^{\#}\varphi \in Z^p(K_\lambda,L_\lambda)$.

Proof. (i) If $(u_o,\ldots,u_p) \in K(p)$, then $\lambda_o(u_o) \cap$

$\ldots \cap \lambda_o(u_p) \neq \emptyset$. There $s_\lambda(u_i) \in \lambda_o(u_i) \subset \lambda_o^*(u_o)$ for all i and thus $(s_\lambda(u_o),\ldots,s_\lambda(u_p)) \in \lambda_o^*(u_o)^{p+1} \subset (\{\lambda_o;\lambda_1\}^*)^{(p+1)}$. Hence $(s_\lambda^\# \varphi)(u_o,\ldots,u_p) = \varphi(s_\lambda(u_o),\ldots,s_\lambda(u_p)) = \varphi'(s_\lambda(u_o),\ldots,s_\lambda(u_p)) = (s_\lambda^\# \varphi')(u_o,\ldots,u_p)$.

(ii) If $(u_o,\ldots,u_p) \in L_\lambda(p)$, then $u_i \in \lambda_2$ for all i and $A \cap \lambda_o(u_o) \cap \ldots \cap \lambda_o(u_p) \neq \emptyset$. Therefore $s_\lambda(u_i) \in A \cap \lambda_o(u_i) = \tilde{\lambda}_o(u_i) \subset \tilde{\lambda}_o^*(u_o)$ for all i and thus $(s_\lambda(u_o),\ldots,s_\lambda(u_p)) \in \tilde{\lambda}_o^*(u_o)^{p+1} \subset (\{\tilde{\lambda}_o;\lambda_2\}^*)^{(p+1)}$. Hence $(s_\lambda^\# \varphi)(u_o,\ldots,u_p) = \varphi(s_\lambda(u_o),\ldots,s_\lambda(u_p)) = 0$.

(iii) By (ii), $s_\lambda^\# \varphi \in C^p(K_\lambda,L_\lambda)$. Since $\bar{\delta}\varphi = 0$ on $(\{\lambda_o;\lambda_1\}^*)^{(p+2)}$, it follows by (i) that $s_\lambda^\# \bar{\delta}\varphi = 0$, or $\bar{\delta} s_\lambda^\# \varphi = 0$. Hence $s_\lambda^\# \varphi \in Z^p(K_\lambda,L_\lambda)$. Q.E.D.

(6.6) <u>Let</u> $\lambda = \{\lambda_o;\lambda_1,\lambda_2\}$ <u>be a covering of</u> $(X,\emptyset)$ <u>and let</u> $\{\mu_o;\mu_1\}$ <u>be a cover of</u> X <u>such that</u> $\{\mu_o;\mu_1\}^* > \{\lambda_o;\lambda_1\}$. <u>Let</u> t_λ <u>be a function from</u> X <u>to</u> $S(K_\lambda)$ <u>such that whenever</u> $x \in X$, $x \in \mu_o(v) \subset \mu_o^*(v) \subset \lambda_o(t_\lambda(x))$ <u>for some</u> $v \in \mu_1$.

(i) <u>If</u> $\varphi_\lambda \in C^p(K_\lambda)$, <u>then</u> $\bar{\delta} t_\lambda^\# \varphi_\lambda = t_\lambda^\# \bar{\delta}\varphi_\lambda$ <u>on</u> $\{\mu_o;\mu_1\}^{(p+2)}$.

(ii) <u>If</u> $\varphi_\lambda \in Z^p(K_\lambda)$, <u>then</u> $t_\lambda^\# \varphi_\lambda \in Z^p(X)$ <u>with</u> $\bar{\delta} t_\lambda^\# \varphi_\lambda = 0$

on $\{\mu_o;\mu_1\}^{(p+2)}$.

Proof. Let $(x_o,\ldots,x_{p+1}) \in \{\mu_o;\mu_1\}^{(p+2)}$, say $(x_o,\ldots,x_{p+1}) \in \mu_o(v)^{p+2}$ with $v \in \mu_1$. Since $x_i \in \mu_o(v_i)$ $\mu_o^*(v_i) \subset \lambda_o(t_\lambda(x_i))$ for some $v_i \in \mu_1$, $\mu_o(v) \subset \mu_o^*(v_i) \subset \lambda_o(t_\lambda(x_i))$, $i = 0,\ldots,p+1$. Therefore $\lambda_o(t_\lambda(x_o)) \cap \ldots \cap \lambda_o(t_\lambda(x_{p+1})) \neq \emptyset$, that is, $(t_\lambda(x_o),\ldots,t_\lambda(x_{p+1})) \in K_\lambda(p+1)$. Hence $(\bar{\delta}t_\lambda^{\#}\varphi_\lambda)(x_o,\ldots,x_{p+1}) = (t_\lambda^{\#}\bar{\delta}\varphi_\lambda)(x_o,\ldots,x_{p+1})$, proving (i).

If $\varphi_\lambda \in Z^p(K_\lambda)$, then $\bar{\delta}\varphi_\lambda = 0$. It follows that $\bar{\delta}t_\lambda^{\#}\varphi_\lambda = t_\lambda^{\#}\bar{\delta}\varphi_\lambda = 0$ on $\{\mu_o;\mu_1\}^{(p+2)}$. Hence $t_\lambda^{\#}\varphi_\lambda \in Z^p(X)$, proving (ii).

7. Main Theorem.

Let (X,A) be fully normal. Given any $\varphi \in Z^p(X,A)$ there is, by (6.3),(i) and (6.4), a canonical covering of (X,A) such that $\varphi = 0$ on $(\{\tilde{\lambda}_0;\lambda_2\}^{**})^{(p+1)}$ and $\bar{\delta}\varphi = 0$ on $(\{\lambda_0;\lambda_1\}^{**})^{(p+2)}$. Since $(\{\tilde{\lambda}_0;\lambda_2\}^{*})^{(p+1)} \subset (\{\tilde{\lambda}_0;\lambda_2\}^{**})^{(p+1)}$ and $(\{\lambda_0;\lambda_1\}^{*})^{(p+2)} \subset (\{\lambda_0;\lambda_1\}^{**})^{(p+2)}$, it follows by (6.5),(iii), that $s_\lambda^{\#}\varphi \in Z^p(K_\lambda,L_\lambda)$ and hence $\eta_\lambda\gamma_\lambda s_\lambda^{\#}\varphi \in \check{H}^p(X,A)$, where s_λ is a canonical function for λ and γ_λ: $Z^p(K_\lambda,L_\lambda) \longrightarrow H^p(K_\lambda,L_\lambda)$ and $\eta_\lambda : H^p(K_\lambda,L_\lambda) \longrightarrow \check{H}^p(X,A)$ are natural homomorphisms.

Let λ and μ be canonical coverings of (X,A) such that $\varphi = 0$ on $(\{\tilde{\lambda}_0;\lambda_2\}^{**})^{(p+1)} \cup (\{\tilde{\mu}_0;\mu_2\}^{**})^{(p+1)}$ and $\bar{\delta}\varphi = 0$ on $(\{\lambda_0;\lambda_1\}^{**})^{(p+2)} \cup (\{\mu_0;\mu_1\}^{**})^{(p+2)}$, then $\eta_\lambda\gamma_\lambda s_\lambda^{\#}\varphi$ and $\eta_\mu\gamma_\mu s_\mu^{\#}\varphi$ are elements of $\check{H}^p(X,A)$. By (6.4) and the directedness of $\{\Lambda(X,A),>\}$, there is a canonical covering ν of (X,A) such that $\nu > \lambda$ and $\nu > \mu$. Then $\varphi = 0$ on $(\{\tilde{\nu}_0;\nu_2\}^{**})^{(p+1)}$ and $\bar{\delta}\varphi = 0$ on $(\{\nu_0;\nu_1\}^{**})^{(p+2)}$ and $\eta_\nu\gamma_\nu s_\nu^{\#}\varphi$ is an element of $H^p(X,A)$. If we can show that $\eta_\nu\gamma_\nu s_\nu^{\#}\varphi = \eta_\lambda\gamma_\lambda s_\lambda^{\#}\varphi$ and similarly $\eta_\nu\gamma_\nu s_\nu^{\#}\varphi = \eta_\mu\gamma_\mu s_\mu^{\#}\varphi$, then

$\eta_\lambda \gamma_\lambda s_\lambda^\# \varphi = \eta_\mu \gamma_\mu s_\mu^\# \varphi$. Hence a function $\varkappa: Z^p(X,A) \longrightarrow \check{H}^p(X,A)$ is defined by

(7.1) $\qquad \varkappa\varphi = \eta_\lambda \gamma_\lambda s_\lambda^\# \varphi \quad$ for each $\varphi \in Z^p(X,A)$,

where λ is a canonical covering of (X,A) such that $\varphi = 0$ on $(\{\tilde{\lambda}_0;\lambda_2\}^{**})^{(p+1)}$ and $\bar{\delta}\varphi = 0$ on $(\{\lambda_0;\lambda_1\}^{**})^{(p+2)}$.

Let $\pi_{\nu\lambda}: (K_\nu, L_\nu) \longrightarrow (K_\lambda, L_\lambda)$ be a projection. Define a function $g: X \longrightarrow X$ as follows:

(i) If $x \in X - s_\nu(\nu_1)$, then $g(x) = x$.

(ii) If $x \in s(\nu_1)$, there is a unique $v \in \nu_1$ such that $x = s_\nu(v)$ (for s_ν is 1-1). Then $g(x) = s_\nu \pi_{\nu\lambda}(v)$.

$g(A) \subset A$. In fact, let $x \in A$. If $x \notin s_\nu(\nu_1)$, then $g(x) = x \in A$. If $x = s_\nu(v)$ for some $v \in \nu_1$, then by the definition of a canonical function, $v \in \nu_2$. Therefore $\pi_{\nu\lambda}(v) \in \lambda_2$ and $g(x) = s_\lambda \pi_{\nu\lambda}(v) \in A$. Hence g is a function of (X,A) into (X,A). Since $\nu > \lambda$, there is function $p: \nu_1 \longrightarrow \lambda_1$ such that $p(\nu_2) \subset \lambda_2$ and for each $v \in \nu_1$, $\nu_0(v) \subset \lambda_0(p(v))$. Fix a $v \in \nu_1$ and let $x \in \nu_0(v)$. If $x \notin s_\nu(\nu_1)$, then $g(x) = x \in \nu_0(v) \subset \lambda_0(p(v)) \subset \lambda_0^*(p(v))$. If $x = s_\nu(v')$ for some $v' \in \nu_1$, then $x \in \nu_0(v') \subset \lambda_0(\pi_{\nu\lambda}(v'))$ and $g(x) = s_\lambda \pi_{\nu\lambda}(v')$

$\in \lambda_o(\pi_{\nu\lambda}(v'))$. It follows that $\lambda_o(p(v)) \cap \lambda_o(\pi_{\nu\lambda}(v'))$ $\neq \emptyset$ and $g(x) \in \lambda_o^*(p(v))$. Hence for each $v \in \nu_1$, $g(\nu_o(v))$ $\subset \lambda_o^*(p(v))$ and $g(\nu_o^*(v)) \subset \lambda_o^{**}(p(v))$. Similarly, for each $v \in \nu_2$, $g(\tilde{\nu}_o(v)) \subset \tilde{\lambda}_o^*(p(v))$ and $g(\tilde{\nu}_o^*(v)) \subset \tilde{\lambda}_o^{**}(p(v))$. By (5.9), $g^{\#}\varphi \in Z^p(X,A)$. Moreover, if $p = 0$, then $\varphi - g^{\#}\varphi = 0$ and hence $\gamma_\nu s_\nu^{\#}\varphi = \gamma_\nu s_\nu^{\#} g^{\#}\varphi$. If $p > 0$, there is some $\psi \in C^{p-1}(X,A)$ such that $\psi = 0$ on $(\{\tilde{\nu}_o;\nu_2\}^*)^{(p)}$ and $\varphi - g^{\#}\varphi = \bar{\delta}\psi$ on $(\{\nu_o;\nu_1\}^*)^{(p+1)}$. By (6.5), (i) and (ii), $s_\nu^{\#}\varphi - s_\nu^{\#}g^{\#}\varphi$ $= s_\nu^{\#}\bar{\delta}\psi = \bar{\delta}s_\nu^{\#}\psi$ and $s_\nu^{\#}\psi \in C^{p-1}(K_\nu,L_\nu)$. Therefore $s_\nu^{\#}\varphi -$ $s_\nu^{\#}g^{\#}\varphi \in B^p(K_\nu,L_\nu)$ and $\gamma_\nu s_\nu^{\#}\varphi = \gamma_\nu s_\nu^{\#}g^{\#}\varphi$. Since $gs_\nu = s_\lambda\pi_{\nu\lambda}$, it follows by the definitions of $g^{\#}$, $s_\nu^{\#}$, $s_\lambda^{\#}$, $\pi_{\nu\lambda}^{\#}$ that $s_\nu^{\#}g^{\#} = \pi_{\nu\lambda}^{\#}s_\lambda^{\#}$. Hence $\gamma_\nu s_\nu^{\#}\varphi = \gamma_\nu \pi_{\nu\lambda}^{\#} s_\lambda^{\#}\varphi = \pi_{\nu\lambda}^{*}\gamma_\lambda s_\lambda^{\#}\varphi$, or

(7.2) $$\gamma_\nu s_\nu^{\#}\varphi = \pi_{\nu\lambda}^{*}\gamma_\lambda s_\lambda^{\#}\varphi.$$

Using (3.1), $\eta_\nu\gamma_\nu s_\nu^{\#}\varphi = \eta_\nu\pi_{\nu\lambda}^{*}\gamma_\lambda s_\lambda^{\#}\varphi = \eta_\lambda\gamma_\lambda s_\lambda^{\#}\varphi$, proving our assertion.

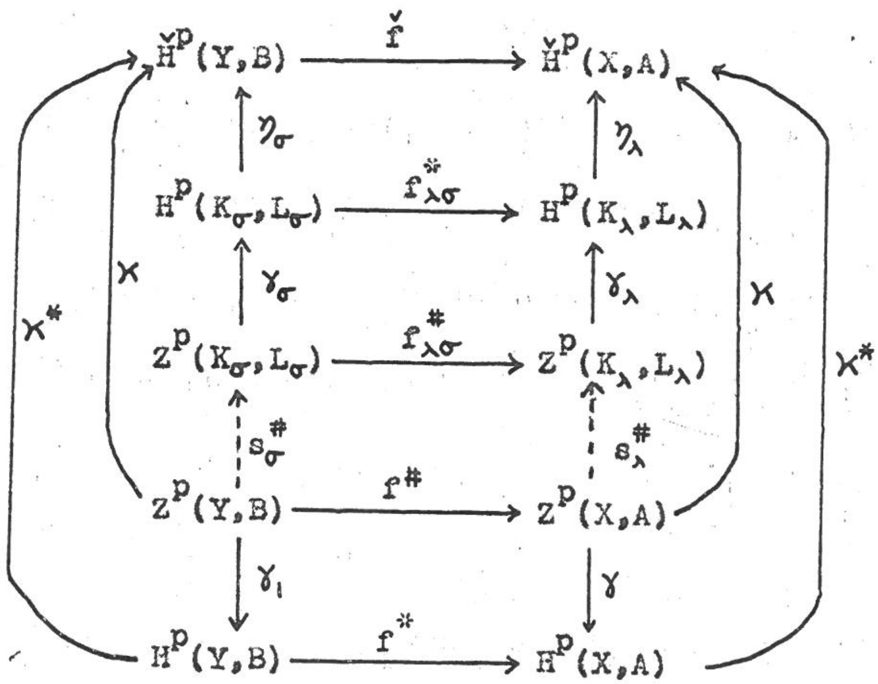
(7.3) $\varkappa: Z^p(X,A) \longrightarrow \check{H}^p(X,A)$ <u>defined by (7.1), is a homomorphism and</u> $\varkappa B^p(X,A) = \{0\}$. <u>Hence</u> $\varkappa$ <u>induces a homomorphism</u> $\varkappa^*: H^p(X,A) \longrightarrow \check{H}^p(X,A)$ <u>such that</u> $\varkappa = \varkappa^*\gamma$, <u>where</u> $\gamma: Z^p(X,A) \longrightarrow H^p(X,A)$ <u>is the natural homomorphism</u>.

Proof. Let $\varphi, \varphi' \in Z^p(X,A)$. Then $\varkappa\varphi = \eta_\lambda \gamma_\lambda s_\lambda^\# \varphi$ and $\varkappa\varphi' = \eta_\mu \gamma_\mu s_\mu^\# \varphi'$ for some canonical coverings λ and μ of (X,A) (see (7.1)). By (6.4) and the directedness of $\{\Lambda(X,A), >\}$, there is a canonical covering ν of (X,A) such that $\nu > \lambda$ and $\nu > \mu$. Therefore $\varkappa\varphi = \eta_\nu \gamma_\nu s_\nu^\# \varphi$ and $\varkappa\varphi' = \eta_\nu \gamma_\nu s_\nu^\# \varphi'$. By the definition of $\varkappa$, φ and φ' vanish on $(\{\tilde{\nu}_0;\nu_2\}^{**})^{(p+1)}$ and $\bar{\delta}\varphi$ and $\bar{\delta}\varphi'$ vanish on $(\{\nu_0;\nu_1\}^{**})^{(p+2)}$; it follows that $\varphi + \varphi' = 0$ on $(\{\tilde{\nu}_0;\nu_2\}^{**})^{(p+1)}$ and $\bar{\delta}(\varphi + \varphi') = 0$ on $(\{\nu_0;\nu_1\}^{**})^{(p+2)}$. Hence $\varkappa(\varphi + \varphi') = \eta_\nu \gamma_\nu s_\nu^\#(\varphi + \varphi') = \eta_\nu \gamma_\nu s_\nu^\# \varphi + \eta_\nu \gamma_\nu s_\nu^\# \varphi' = \varkappa\varphi + \varkappa\varphi'$.

$\varkappa B^p(X,A) = \{0\}$ is trivial when $p = 0$. Therefore we may assume that $p > 0$. Let $\varphi \in B^p(X,A)$. By (6.3), (ii) and (6.4), there is a canonical covering λ of (X,A) and a $\psi \in C^{p-1}(X,A)$ such that $\psi = 0$ on $(\{\tilde{\lambda}_0;\lambda_2\}^{**})^{(p)}$ and $\varphi = \bar{\delta}\psi$ on $(\{\lambda_0;\lambda_1\}^{**})^{(p+1)}$. We infer by (6.5), (i) and (ii), that $s_\lambda^\# \varphi = s_\lambda^\# \bar{\delta}\psi = \bar{\delta} s_\lambda^\# \psi \in B^p(K_\lambda, L_\lambda)$. By (6.3), (ii), $\varphi = 0$ on $(\{\tilde{\lambda}_0;\lambda_2\}^{**})^{(p+1)}$ and $\bar{\delta}\varphi = 0$ on $(\{\lambda_0;\lambda_1\}^{**})^{(p+2)}$. Hence $\varkappa\varphi = \eta_\lambda \gamma_\lambda s_\lambda^\# \varphi = \eta_\lambda \gamma_\lambda \bar{\delta} s_\lambda^\# \psi = 0$. Q.E.D.

(7.4) *If* $f\colon (X,A) \longrightarrow (Y,B)$ *is a mapping, then* $\check{f}\varkappa^* = \varkappa^* f^*$.

Proof. The proof of (7.4) is similar to the justification of (7.1). Fix a $\varphi \in Z^p(Y,B)$ and let σ be a canonical covering of (Y,B) such that $\varkappa\varphi = \eta_\sigma \gamma_\sigma s_\sigma^\# \varphi$. Since $f^\# \varphi \in Z^p(X,A)$ and $f^{-1}\sigma = \{f^{-1}\sigma_0; \sigma_1, \sigma_2\}$ is a covering of (X,A), we can easily show that there is a canonical covering λ of (X,A) such that $\lambda > f^{-1}\sigma$ and $\varkappa f^\# \varphi = \eta_\lambda \gamma_\lambda s_\lambda^\# f^\# \varphi$. Our assertion will follows from the commutativity as shown in the following diagram:



The diagram is dependent on φ; $Z^p(Y,B) \overset{s_\sigma^\#}{\dashrightarrow} Z^p(K_\sigma, L_\sigma)$ indicates $\varphi \in Z^p(Y,B)$ and $s_\sigma^\# \varphi \in Z^p(K_\sigma, L_\sigma)$ and

$Z^p(X,A) \xrightarrow{s_\lambda^\#} Z^p(K_\lambda,L_\lambda)$ indicates $f^\#\varphi \in Z^p(X,A)$ and $s_\lambda^\# f^\# \varphi \in Z^p(K_\lambda,L_\lambda)$.

Let $f_{\lambda\sigma}$ be an f-projection of (K_λ,L_λ) into (K_σ,L_σ). Define $g\colon X \longrightarrow Y$ as follows:

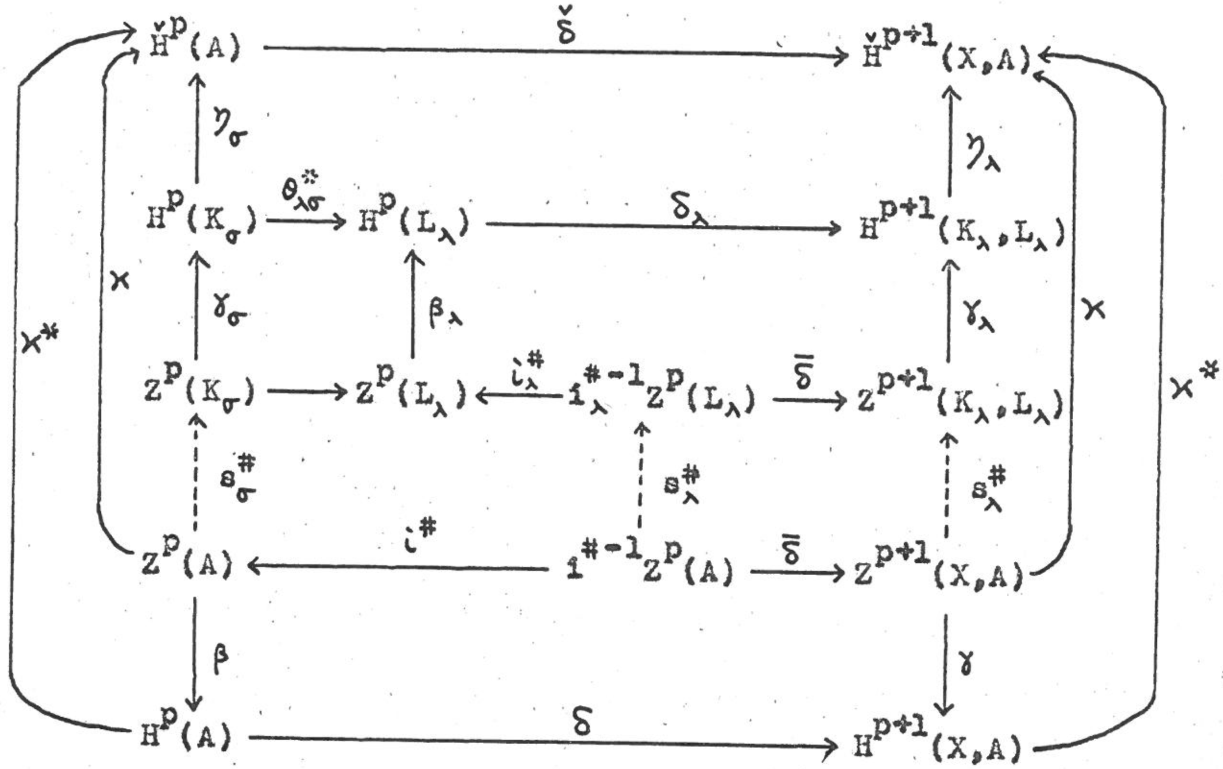
(i) If $x \in X - s_\lambda(\lambda_1)$, then $g(x) = f(x)$.

(ii) If $x \in s_\lambda(\lambda_1)$, there is a unique $u \in \lambda_1$ such that $x = s_\lambda(u)$ (for s_λ is 1-1). Then $g(x) = s_\sigma f_{\lambda\sigma}(u)$.

Then $g(A) \subset B$ and g is a function of (X,A) into (Y,B). Since $\lambda > f^{-1}\sigma$, there is a function $p\colon \lambda_1 \longrightarrow \sigma_1$ such that $p(\lambda_2) \subset \sigma_2$ and for each $u \in \lambda_1$, $\lambda_o(u) \subset f^{-1}\sigma_o(p(u))$ or $f\lambda_o(u) \subset \sigma_o(p(u))$. By the definition of g we can easily show that for each $u \in \lambda_1$, $g(\lambda_o(u)) \subset \sigma_o^*(p(u))$. Therefore $f(\lambda_o(u)) \cup g(\lambda_o(u)) \subset \sigma_o^*(p(u))$ and hence $f(\lambda_o^*(u)) \cup g(\lambda_o^*(u)) \subset \sigma_o^{**}(p(u))$. Similarly, for each $u \in \lambda_2$, $f(\tilde{\lambda}_o^*(u)) \cup g(\tilde{\lambda}_o^*(u)) \subset \tilde{\sigma}_o^{**}(p(u))$. By (5.9), $g^\#\varphi \in Z^p(X,A)$. Moreover, if $p = 0$, then $f^\#\varphi - g^\#\varphi = 0$ and $\gamma_\lambda s_\lambda^\# f^\#\varphi = \gamma_\lambda s_\lambda^\# g^\#\varphi$. If $p > 0$, there is some $\psi \in C^{p-1}(X,A)$ such that $\psi = 0$ on $(\{\tilde{\lambda}_o;\lambda_2\}^*)^{(p)}$ and $f^\#\varphi - g^\#\varphi = \bar{\delta}\psi$ on $(\{\lambda_o;\lambda_1\}^*)^{(p+1)}$. By (6.5), $s_\lambda^\# f^\#\varphi - s_\lambda^\# g^\#\varphi = s_\lambda^\#\bar{\delta}\psi = \bar{\delta} s_\lambda^\#\psi \in B^p(K_\lambda,L_\lambda)$ and $\gamma_\lambda s_\lambda^\# f^\#\varphi = \gamma_\lambda s_\lambda^\# g^\#\varphi$. Since $gs_\lambda = s_\sigma f_{\lambda\sigma}$, $s_\lambda^\# g^\# = f_{\lambda\sigma}^\# s_\sigma^\#$. We infer that $\gamma_\lambda s_\lambda^\# f^\# \varphi = \gamma_\lambda f_{\lambda\sigma}^\# s_\sigma^\# \varphi = f_{\lambda\sigma}^* \gamma_\sigma s_\sigma^\# \varphi$. By (4.1), $\eta_\lambda \gamma_\lambda s_\lambda^\# f^\# \varphi = \eta_\lambda f_{\lambda\sigma}^* \gamma_\sigma s_\sigma^\# \varphi = \check{f} \eta_\sigma \gamma_\sigma s_\sigma^\# \varphi$, or $\varkappa^* f^*(\gamma_1 \varphi) = \check{f} \check{\varkappa}^*(\gamma_1 \varphi)$. The proof is completed. Q.E.D.

(7.5) $\check{\delta} \varkappa^* = \varkappa^* \delta$.

Proof. Since $H^p(A) = \beta i^\# i^{\#-1} Z^p(A)$, where $i: A \longrightarrow X$ is the injection and $\beta: Z^p(A) \longrightarrow H^p(A)$ is the natural homomorphism, it is sufficient to show that for each $\varphi \in i^{\#-1} Z^p(A)$, $\check{\delta} \varkappa^*(\beta i^\# \varphi) = \varkappa^* \delta(\beta i^\# \varphi)$. Fix a $\varphi \in i^{\#-1} Z^p(A)$, and let σ be a canonical covering of $(A, \emptyset)$ such that $\varkappa i^\# \varphi = \eta_\sigma \gamma_\sigma s_\sigma^\# \varphi$. Let u_o be an element not contained in σ_1. Let $\lambda_2 = \sigma_1$ and let $\lambda_1 = \lambda_2$ or $\lambda_2 \cup \{u_o\}$ according as $A = X$ or $A \neq X$. Define $\lambda_o: \lambda_1 \longrightarrow Q(X)$ such that for each $u \in \sigma_1$, $\sigma_o(u) = A \cap \lambda_o(u)$ and in the case $A \neq X$ we have $\lambda_o(u_o) = X$. Then $\lambda = \{\lambda_o; \lambda_1, \lambda_2\}$ is a canonical covering of (X, A) and a canonical function s_λ for λ is defined as follows: For each $u \in \sigma_1$, $s_\lambda(u) = s_\sigma(u)$ and $s_\lambda(u_o) \in X - A$ in the case $A \neq X$. Our assertion follows from the commutativity as shown in the following diagram:



The diagram depends on φ. $Z^p(A) \overset{s^\#_\sigma}{\dashrightarrow} Z^p(K\ )$ indicates $i^\#\varphi \in Z^p(A)$ and $s^\#_\sigma i^\#\varphi \in Z^p(K_\sigma)$, $i^{\#-1}Z^p(A) \overset{s^\#_\lambda}{\dashrightarrow} i_\lambda^{\#-1}Z^p(L_\lambda)$ indicates $\varphi \in i^{\#-1}Z^p(A)$ and $s^\#_\lambda\varphi \in i_\lambda^{\#-1}Z^p(L_\lambda)$ and $Z^{p+1}(X,A) \overset{s^\#_\lambda}{\dashrightarrow} Z^{p+1}(K_\lambda,L_\lambda)$ indicates $\bar{\delta}\varphi \in Z^{p+1}(X,A)$ and $s^\#_\lambda\bar{\delta}\varphi \in Z^{p+1}(K_\lambda,L_\lambda)$. All these indications will be justified.

Since $\chi i^\#\varphi = \eta_\sigma\gamma_\sigma s^\#_\sigma i^\#\varphi$, $s^\#_\sigma i^\#\varphi \in Z^p(K_\sigma)$ and $\bar{\delta} i^\#\varphi = 0$ on $(\{\sigma_0;\sigma_1\}^{**})^{(p+2)}$. By the definition of λ, $\{\tilde{\lambda}_0;\lambda_2\} = \{\sigma_0;\sigma_1\}$ and hence $(\{\tilde{\lambda}_0;\lambda_2\}^{**})^{(p+2)} = (\{\sigma_0;\sigma_1\}^{**})^{(p+2)}$. Then on $(\{\tilde{\lambda}_0;\lambda_2\}^{**})^{(p+2)}$, $\bar{\delta}\varphi = i^\#\bar{\delta}\varphi = \bar{\delta} i^\#\varphi = 0$.

Moreover, $\bar{\delta}(\bar{\delta}\varphi) = 0$ on $(\{\lambda_0;\lambda_1\}^{**})^{(p+3)}$. Hence $s_\lambda^{\#}\bar{\delta}\varphi \in Z^{p+1}(K_\lambda, L_\lambda)$ and $\kappa\bar{\delta}\varphi = \eta_\lambda\gamma_\lambda s_\lambda^{\#}\bar{\delta}\varphi$. Let $i_\lambda: L_\lambda \longrightarrow K_\lambda$ be the injection and define $\theta_{\lambda\sigma}: L_\lambda \longrightarrow K_\sigma$ by $\theta_{\lambda\sigma}(u) = u$ for each $u \in S(L_\lambda)$ $(= \lambda_2 = \sigma_1 = S(K_\sigma))$. Then, by (2.14), (4.2) and (4.3), $\delta_\lambda\beta_\lambda i_\lambda^{\#} = \gamma_\lambda\bar{\delta}$ on $i_\lambda^{\#-1}Z^p(L_\lambda)$, $\delta_{\lambda\sigma} = \delta_\lambda\theta_{\lambda\sigma}^{*}$ and $\check{\delta}\eta_\sigma = \eta_\lambda\delta_{\lambda\sigma}$. $is_\sigma\theta_{\lambda\sigma} = s_\lambda i_\lambda$, since for any $u \in \lambda_2$ we have $is_\sigma\theta_{\lambda\sigma}(u) = is_\sigma(u) = s_\sigma(u) = s_\lambda(u) = s_\lambda i_\lambda(u)$. Therefore $\theta_{\lambda\sigma}^{\#}s_\sigma^{\#}i^{\#} = i_\lambda^{\#}s_\lambda^{\#}$. Since $s_\sigma^{\#}i^{\#}\varphi \in Z^p(K_\sigma)$, $i_\lambda^{\#}s_\lambda^{\#}\varphi = \theta_{\lambda\sigma}^{\#}s_\sigma^{\#}i^{\#}\varphi \in Z^p(L_\lambda)$. Hence $s_\lambda^{\#}\varphi \in i_\lambda^{\#-1}Z^p(L_\lambda)$, and $\delta_\lambda\beta_\lambda i_\lambda^{\#}s_\lambda^{\#}\varphi = \gamma_\lambda\bar{\delta}s_\lambda^{\#}\varphi$. Consequently $\kappa^{*}\delta(\beta i^{\#}\varphi) = \kappa^{*}\gamma\bar{\delta}\varphi = \kappa\bar{\delta}\varphi = \eta_\lambda\gamma_\lambda s_\lambda^{\#}\bar{\delta}\varphi = \eta_\lambda\gamma_\lambda\bar{\delta}s_\lambda^{\#}\varphi$ $= \eta_\lambda\delta_\lambda\beta_\lambda i_\lambda^{\#}s_\lambda^{\#}\varphi = \eta_\lambda\delta_\lambda\beta_\lambda\theta_{\lambda\sigma}^{\#}s_\sigma^{\#}i^{\#}\varphi = \eta_\lambda\delta_\lambda\theta_{\lambda\sigma}^{*}\gamma_\sigma s_\sigma^{\#}i^{\#}\varphi =$ $\eta_\lambda\delta_{\lambda\sigma}\gamma_\sigma s_\sigma^{\#}i^{\#}\varphi = \check{\delta}\eta_\sigma\gamma_\sigma s_\sigma^{\#}i^{\#}\varphi = \check{\delta}\kappa i^{\#}\varphi = \check{\delta}\kappa^{*}(\beta i^{\#}\varphi)$. Q.E.D.

(7.6) $\kappa^{*}: H^p(X) \approx \check{H}^p(X)$.

Proof. (i) κ^{*} is 1-1. It is sufficient to show that the kernel of $\kappa: Z^p(X) \longrightarrow H^p(X)$ is contained in $B^p(X)$. Fix a φ in the kernel of κ and let λ be a canonical covering of $(X,\varnothing)$ such that $\kappa\varphi = \eta_\lambda\gamma_\lambda s_\lambda^{\#}\varphi$. Since $\eta_\lambda\gamma_\lambda s_\lambda^{\#}\varphi = \kappa\varphi = 0$, it follows by (3.3) and (6.4) that there is a canonical covering μ of $(X,\varnothing)$ such that $\mu > \lambda$ and $\pi_{\mu\lambda}^{*}\gamma_\lambda s_\lambda^{\#}\varphi$

$= 0$. That is, by (7.2), $\gamma_\mu s_\mu^{\#}\varphi = 0$, or $s_\mu^{\#}\varphi \in B^p(K_\mu)$. Let $\{\nu_o;\nu_1\}$ be a *-refinement of $\{\mu_o;\mu_1\}$ and define $t_\mu: X \longrightarrow S(K_\mu)$ such that whenever $x \in X$, $x \in \nu_o(w) \subset \nu_o^*(w) \subset \mu_o(t_\mu(x))$ for some $w \in \nu_1$. If $p = 0$, then $s_\mu^{\#}\varphi = 0$. It follows that $t_\mu^{\#}s_\mu^{\#}\varphi = 0$. If $p > 0$, then there is some $\psi_\mu \in C^{p-1}(K_\mu)$ such that $s_\mu^{\#}\varphi = \bar{\delta}\psi_\mu$. By (6.6), (i), $t_\mu^{\#}s_\mu^{\#}\varphi = t_\mu^{\#}\bar{\delta}\psi_\mu = \bar{\delta}t_\mu^{\#}\psi_\mu$ on $\{\nu_o;\nu_1\}^{(p+1)}$. Hence $t_\mu^{\#}s_\mu^{\#}\varphi \in B^p(X)$. By the definitions of $s_\mu^{\#}$ and $t_\mu^{\#}$ it is easily seen that $t_\mu^{\#}s_\mu^{\#}\varphi = (s_\mu t_\mu)^{\#}\varphi$ on $\{\nu_o;\nu_1\}^{(p+1)}$. Hence $(s_\mu t_\mu)^{\#}\varphi \in B^p(X)$. Since $\{\nu_o;\nu_1\} > \{\mu_o;\mu_1\}$, there is a function $p: \nu_1 \longrightarrow \mu_1$ such that for each $w \in \nu_1$, $\nu_o(w) \subset \mu_o(p(w))$. Fix a $w \in \nu_1$. If $x \in \nu_o(w)$, then $x \in \mu_o(p(w)) \cap \mu_o(t_\mu(x))$ and so $s_\mu t_\mu(x) \in \mu_o(t_\mu(x)) \subset \mu_o^*(p(w))$. Therefore for each $w \in \nu_1$, $s_\mu t_\mu(\nu_o(w)) \subset \mu_o^*(p(w))$. Since $\bar{\delta}\varphi = 0$ on $(\{\lambda_o;\lambda_1\}^{**})^{(p+2)} \supset (\{\mu_o;\mu_1\}^{**})^{(p+2)} \supset (\{\mu_o;\mu_1\}^{*})^{(p+2)}$, it follows by (5.9) that $\varphi - (s_\mu t_\mu)^{\#}\varphi \in B^p(X)$. Hence $\varphi \in B^p(X)$.

(ii) χ^* is onto. We have only to show that $\chi: Z^p(X) \longrightarrow \check{H}^p(X)$ is onto. Given any $e \in \check{H}^p(X)$ there is, by (3.2), a covering μ of $(X,\emptyset)$ and a $\varphi_\mu \in Z^p(K_\mu, L_\mu)$ such that $e = \eta_\mu \gamma_\mu \varphi_\mu$. Let $\{\nu_0;\nu_1\}$ be a cover of X such that $\{\nu_0;\nu_1\}^* > \{\mu_0;\mu_1\}$ and define t_μ as in (i). Let $\varphi = t_\mu^\# \varphi_\mu$; then, by (6.6), (ii), $\varphi \in Z^p(X)$ and $\bar{\delta}\varphi = 0$ on $\{\nu_0;\nu_1\}^{(p+2)}$. By (6.2) and (6.4), there is a canonical covering λ of $(X,\emptyset)$ such that $\{\lambda_0;\lambda_1\}^{**} > \{\nu_0;\nu_1\}$. Then $\bar{\delta}\varphi = 0$ on $(\{\lambda_0;\lambda_1\}^{**})^{(p+2)}$ and hence $\varkappa\varphi = \eta_\lambda\gamma_\lambda s_\lambda^\#\varphi = \eta_\lambda\gamma_\lambda s_\lambda^\# t_\mu^\# \varphi_\mu$. Fix a $u \in \lambda_1$. By the definition of t_μ, $s_\lambda(u) \in \nu_0(w) \subset \nu_0^*(w) \subset \mu_0(t_\mu s_\lambda(u))$ for some $w \in \nu_1$. Since $\{\lambda_0;\lambda_1\} > \{\nu_0;\nu_1\}$, there is some $w' \in \nu_1$ such that $s_\lambda(u) \in \lambda_0(u) \subset \nu_0(w')$. Therefore $\lambda_0(u) \subset \nu_0(w') \subset \nu_0^*(w) \subset \mu_0(t_\mu s_\lambda(u))$. This proves that $t_\mu s_\lambda: K_\lambda \longrightarrow K_\mu$ is a projection. Hence $(t_\mu s_\lambda)^\# = s_\lambda^\# t_\mu^\#$ and $\varkappa\varphi = \eta_\lambda\gamma_\lambda(t_\mu s_\lambda)^\#\varphi = \eta_\lambda \pi_{\lambda\mu}^* \gamma_\mu\varphi_\mu = \eta_\mu\gamma_\mu\varphi_\mu = e$. Q.E.D.

(7.7) $\varkappa^*: H^p(X,A) \approx \check{H}^p(X,A)$.

Proof. In order to make the argument a little clearer, we denote by $\varkappa^*_{(X,A)}$, $\varkappa^*_X$ and $\varkappa^*_A$ the respective homomorphism from $H^p(X,A)$ to $\check{H}^p(X,A)$, from $H^p(X)$ to $\check{H}^p(X)$ and from $H^p(A)$ to $\check{H}^p(A)$. By (7.4) and (7.5) the commutativity holds in every square of the following diagram:

$$\begin{array}{ccccccccc}
\check{H}^{p-1}(X) & \xrightarrow{\check{i}} & \check{H}^{p-1}(A) & \xrightarrow{\check{\delta}} & \check{H}^p(X,A) & \xrightarrow{\check{j}} & \check{H}^p(X) & \xrightarrow{\check{i}} & H^p(A) \\
\uparrow \kappa^*_X & & \uparrow \kappa^*_A & & \uparrow \kappa^*_{(X,A)} & & \uparrow \kappa^*_X & & \uparrow \kappa^*_A \\
H^{p-1}(X) & \xrightarrow{i^*} & H^{p-1}(A) & \xrightarrow{\delta} & H^p(X,A) & \xrightarrow{j^*} & H^p(X) & \xrightarrow{i^*} & H^p(A)
\end{array}$$

By (7.6), κ^*_X and κ^*_A are isomorphisms onto. Hence $i^*\kappa_X^{*-1} = \kappa_A^{*-1}\check{i}$, $j^* = \kappa_X^{*-1}\check{j}\kappa^*_{(X,A)}$ and $\check{\delta} = \kappa^*_{(X,A)}\delta\kappa_A^{*-1}$.

(i) $\kappa^*_{(X,A)}$ is 1-1. Fix $e \in H^p(X,A)$ such that $\kappa^*_{(X,A)}e = 0$. Then $j^*e = \kappa_X^{*-1}\check{j}\kappa^*_{(X,A)}e = 0$. If $p = 0$, then, by (5.7), $e = 0$. If $p > 0$, then there is, by (5.7), some $e' \in H^{p-1}(A)$ such that $\delta e' = e$. Since $\check{\delta}\kappa^*_A e' = \kappa^*_{(X,A)}\delta e' = \kappa^*_{(X,A)}e = 0$, there is, by (4.5), some $e'' \in \check{H}^{p-1}(X)$ such that $\check{i}e'' = \kappa^*_A e'$. Let $e''' = \kappa_X^{*-1}e''$; then $i^*e''' = i^*\kappa_X^{*-1}e'' = \kappa_A^{*-1}ie'' = e'$. Hence $e = \delta e' = \delta i^*e'''$ and, by (5.7), $e = 0$.

(ii) $\kappa^*_{(X,A)}$ is onto. Fix $e \in \check{H}^p(X,A)$. By (4.5), $\check{i}\check{j}e = 0$. Then $i^*\kappa_X^{*-1}\check{j}e = \kappa_A^{*-1}\check{i}\check{j}e = 0$. Using (5.7), there is some $e' \in H^p(X,A)$ such that $j^*e' = \kappa_X^{*-1}\check{j}e$. Therefore $\check{j}e = \kappa^*_X j^*e' = \check{j}\kappa^*_{(X,A)}e'$, or $\check{j}(e - \kappa^*_{(X,A)}e') = 0$. If $p = 0$, then, by (4.5), $e - \kappa^*_{(X,A)}e' = 0$, or $e = \kappa^*_{(X,A)}e' \in$

$\varkappa^*_{(X,A)} H^p(X,A)$. If $p > 0$, there is, by (4.5), some $e'' \in \check{H}^{p-1}(A)$ such that $\check{\delta} e'' = e - \varkappa^*_{(X,A)} e'$. Let $e''' = e' + \delta \varkappa_A^{*-1} e''$; then $\varkappa^*_{(X,A)} e''' = \varkappa^*_{(X,A)} e' + \varkappa^*_{(X,A)} \delta \varkappa_A^{*-1} e'' = \varkappa^*_{(X,A)} e' + \check{\delta} e'' = e$, or $e \in \varkappa^*_{(X,A)} H^p(X,A)$. Q.E.D.

Combining (7.4), (7.5) and (7.7), we obtain the following main theorem:

(7.8) For fully normal spaces the Alexander-Kolmogoroff cohomology theory is equivalent to the unrestricted Čech cohomology theory.

From (6.1), (4.6) and (7.8) it follows that

(7.9) For compact Hausdorff spaces the Alexander-Kolmogoroff cohomology theory is equivalent to the restricted Čech cohomology theory.

Let (X,A) be a pair and I the closed interval from 0 to 1 with the usual topology. If $(X \times I, A \times I)$ is fully normal, then so is (X,A). By (4.4), (7.4) and (7.8) it follows that

(7.10) *Let* (X,A) *be a pair such that* $(X\times I, A\times I)$ *is fully normal. If* $h_i: (X,A) \longrightarrow (X\times I, A\times I)$ *is defined by* $h_i(x) = (x,i)$, $i = 0, 1$, *then* $h_0^* = h_1^*$.

In the Appendix we shall give a direct proof of (7.10). The idea of our proof is essentially the same as that of Dowker's [2].

As a consequence of (7.10), we have the following homotopy theorem which is a generalization of one stated by Spanier [7,p.416]. Two mappings f_0 and f_1 from (X,A) to (Y,B) are *homotopic* if and only if there exists a mapping $f: (X\times I, A\times I) \longrightarrow (Y,B)$ such that for each $x \in X$, $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$.

(7.11) *Let* (X,A) *be a pair such that* $(X\times I, A\times I)$ *is fully normal and let* (Y,B) *be an arbitrary pair. If* f_0 *and* f_1 *are homotopic mappings from* (X,A) *to* (Y,B), *then* $f_0^* = f_1^*$

As an application of (7.11) we prove

(7.12) *If* X *is a non-null convex subset of a linear*

<u>metric space, then</u> $H^o(X) \approx G$ <u>and</u> $H^p(X) = \{0\}$ <u>for all</u> $p > 0$.

Proof. Since X is convex, it is connected. Therefore it can be easily shown that $\varphi \in Z^o(X)$ if and only if it is a constant function. Hence $H^o(X) = Z^o(X) \approx G$.

Let $i: X \longrightarrow X$ be the identity mapping. Let $a \in X$ and define $g: X \longrightarrow X$ by $g(x) = a$ for all $x \in X$. Since X is convex, $F(x,t) = ta + (1 - t)x$ defines a mapping F on $X \times I$ to X. For each $x \in X$ we have $F(x,0) = x = i(x)$ and $F(x,1) = a = g(x)$; therefore i and g are homotopic. By hypothesis, X is metric; then $X \times I$ is metric and hence, by(6.1), fully normal. It follows by (7.11) that $i^* = g^*$. Let $Y = \{a\}$ and let $g_1: X \longrightarrow Y$ be the trivial function and $g_2: Y \longrightarrow X$ the injection. Then $g = g_2 g_1$ and by (5.6), $g^* = g_1^* g_2^*$. Since, by (5.8), $H^p(Y) = \{0\}$ for all $p > 0$, and since, by (5.5), i^* is the identity isomorphism, it follows that for all $p > 0$, $H^p(X) = i^* H^p(X) = g^* H^p(X) = g_1^* g_2^* H^p(X) = g_1^* H^p(Y) = \{0\}$. Q.E.D.

Let R^n be an n-dimensional Euclidean space. Since R^n is a linear metric space, it follows, by (7.12) that

(7.13) *$H^o(R^n) \approx G$ and $H^p(R^n) = \{0\}$ for all $p > 0$.*

According to Wallace [11], the extension, reduction and hence map excision theorems hold for the Alexander-Kolmogoroff cohomology groups on fully normal spaces. Using (7.8), we have

(7.14) *The extension, reduction and hence map excision theorems hold for the unrestricted Čech cohomology groups over fully normal spaces.*

A cover $\{\lambda_o;\lambda_1\}$ of a topological spaces X is of *order* n if and only if n+1 is the largest number such that there exists a subset F of λ_1 consisting of n+1 distinct elements and satisfying $\cap\{\lambda_o(u) \mid u \in F\} \neq \emptyset$. A topological space X is of *Lebesgue dimension* $\leq n$ if and only if every cover of X has a refinement of order $\leq n$.

(7.15) *If X is a fully normal space of Lebesgue dimension $\leq n$ and A is a closed subset of X, then $H^p(X,A) = \{0\}$ for all $p > n$.*

Proof. It is easily seen that in a fully normal space any closed set with the relative topology is also fully normal. Therefore (X,A) is fully normal. By (7.7), it is sufficient to show that $\check{H}^p(X,A) = \{0\}$ for all $p > n$.

Fix an element e in $\check{H}^p(X,A)$. By (3.2), there is some covering λ of (X,A) and some $e_\lambda \in H^p(K_\lambda, L_\lambda)$ such that $e = \eta_\lambda e_\lambda$, where η_λ is the natural homomorphism of $H^p(K_\lambda, L_\lambda)$ into $\check{H}^p(X,A)$. Let $\mu_1 = \lambda_1$ and define μ_0: $\mu_1 \longrightarrow Q(X)$ such that for each $u \in \lambda_2$, $\mu_0(u) = \lambda_0(u)$ and for each $u \in \lambda_1 - \lambda_2$, $\mu_0(u) = \lambda_0(u) - A$. Then $\{\mu_0;\mu_1\}$ is a cover of X such that for each $u \in \lambda_1$, $\mu_0(u) \subset \lambda_0(u)$. Since X is of Lebesgue dimension $\leq n$, $\{\mu_0;\mu_1\}$ has a refinement $\{\nu_0;\nu_1\}$ of order $\leq n$. Let $\nu_2 = \{w \mid w \in \nu_1$ and $A \cap \nu_0(w) \neq \emptyset\}$. Then $\nu = \{\nu_0;\nu_1,\nu_2\}$ is a covering of (X,A). Since $\{\nu_0;\nu_1\} > \{\mu_0;\mu_1\}$, there is a function p: $\nu_1 \longrightarrow \mu_1$ such that for each $w \in \nu_1$, $\nu_0(w) \subset \mu_0(p(w))$. Therefore for each $w \in \nu_1$, $\nu_0(w) \subset \lambda_0(p(w))$ and for each $w \in \nu_2$, $p(w) \in \lambda_2$, proving that $\nu > \lambda$. Since $\{\nu_0;\nu_1\}$ is of order $\leq n$, every simplex of K_ν contains at most n elements. Therefore the group of oriented p-cochains of K_ν is $\{0\}$ and hence the p-th oriented cohomology group of K_ν

mod L_ν is $\{0\}$ for all $p > n$. By the Remark of Section 2, $H^p(K_\nu, L_\nu) = \{0\}$ and so $\pi^*_{\nu\lambda} e_\lambda = 0$. Using (3.3), $e = \eta_\lambda e_\lambda = 0$. Q.E.D.

Appendix

8. A Theorem on the Alexander-Kolmogoroff Cohomology Theory.

Let I be the additive group of integers. For any set X and any non-negative integer p we denote by $C_p(X)$ the group of all the functions on X^{p+1} to I which are finitely not zero. Given any $\xi = (x_o,\ldots,x_p)$ in X^{p+1} we denote by $\bar{\xi}$ or $[x_o,\ldots,x_p]$ the characteristic function of ξ, that is an element of $C_p(X)$ defined as follows: If $\xi' \in X^{p+1}$, then $\bar{\xi}(\xi')$ is 1 or 0 according to $\xi' = \xi$ or $\xi' \neq \xi$. Let $X(p) = \{\bar{\xi} \mid \xi \in X^{p+1}\}$. It is easily seen that $C_p(X)$ is a free group with X(p) as a base, i.e., every non-zero element c has a unique canonical representation $c = \sum_{i=1}^{n} t_i \sigma_i$, where $\sigma_1,\ldots,\sigma_n$ are distinct elements of X(p) and $t_1,\ldots,t_n$ are non-zero integers. Define a function from $C_p(X)$ to the subsets of X as follows: For any $\xi \in X^{p+1}$ the image of $\bar{\xi}$ is the set of coordinates of ξ, denoted by $|\bar{\xi}|$. If c is a non-zero element of $C_p(X)$ with canonical representation $c = \sum_{i=1}^{n} t_i \sigma_i$, then the image of c is $|c| = \cup_{i=1}^{n} |\sigma_i|$.

Moreover, the image of the zero element 0 of $C_p(X)$ is $|0| = \emptyset$.

(8.1) <u>If</u> $c = \sum_{j=1}^{m} t_j c_j$ <u>with</u> $c_j \in C_p(X)$ <u>and</u> $t_j \in I$, <u>then</u> $|c| \subset \bigcup_{j=1}^{m} |c_j|$.

Proof. Let $c_j = \sum_{k=1}^{n(j)} s_{jk}\sigma_{jk}$, $j = 1,\ldots,m$, and $c = \sum_{i=1}^{n} s_i\sigma_i$ be canonical representations. Then $\sum_{i=1}^{n} s_i\sigma_i = c = \sum_{j=1}^{m} t_j c_j = \sum_{j=1}^{m}\sum_{k=1}^{n(j)} t_j s_{jk}\sigma_{jk}$. Since $\sigma_1,\ldots,\sigma_n$ are linearly independent, it follows that each σ_i is equal to some σ_{jk}. Hence $|c| = \bigcup_{i=1}^{n} |\sigma_i| \subset \bigcup_{j=1}^{m}\bigcup_{k=1}^{n(j)} |\sigma_{jk}| = \bigcup_{j=1}^{m} |c_j|$. Q.E.D.

There is a homomorphism $\partial: C_{p+1}(X) \longrightarrow C_p(X)$ defined by

$$\partial[x_0,\ldots,x_{p+1}] = \sum_{i=0}^{p+1}(-1)^i[x_0,\ldots,\hat{x}_i,\ldots,x_{p+1}].$$

By definition we have immediately:

(8.2) <u>For</u> <u>each</u> $c \in C_{p+2}(X)$, $\partial\partial c = 0$.

Given any $\sigma \in X(p+1)$ we can easily see that $|\partial\sigma| = |\sigma|$. It follows by (8.1) that

(8.3) <u>For</u> <u>each</u> $c \in C_{p+1}(X)$, $|\partial c| \subset |c|$.

A set is said to be linearly ordered by $<$ if and only if (i) $x \neq x'$ implies $x < x'$ or $x' < x$, (ii) $x < x'$ and $x' < x''$ imply $x < x''$ and (iii) $x < x'$ implies $x \neq x'$. The existence of such an ordering for any given set follows from the well-ordering postulate.

If $(\rho(0),\ldots,\rho(p))$ is a permutation of $(0,\ldots,p)$, a number $o(\rho)$ is an order of this permutation ρ if and only if there exist $o(\rho)$ interchanges carrying $(0,\ldots,p)$ into $(\rho(0),\ldots,\rho(p))$. The number $o(\rho)$ is not unique; but the difference of any two of them is always even.

Let the space X be linearly ordered by $<$. There is a homomorphism $\omega: C_p(X) \longrightarrow C_p(X)$ defined as follows: Let $[x_0,\ldots,x_p] \in X(p)$.

(i) If $x_0,\ldots,x_p$ are not distinct, then

$$\omega[x_0,\ldots,x_p] = 0.$$

(ii) If $x_0,\ldots,x_p$ are distinct, there is a unique permutation $(\rho(0),\ldots,\rho(p))$ of $(0,\ldots,p)$ such that $x_{\rho(0)} < \ldots < x_{\rho(p)}$. Then

$$\omega[x_0,\ldots,x_p] = (-1)^{o(\rho)}[x_{\rho(0)},\ldots,x_{\rho(p)}].$$

The following (8.4) and (8.5) are immediate:

(8.4) $\omega\omega = \omega$.

(8.5) <u>For each</u> $c \in C_p(X)$, $|\omega c| \subset |c|$.

(8.6) <u>If</u> $[x_o,\ldots,x_p] \in X(p)$ <u>and</u> $(\tau(0),\ldots,\tau(p))$ <u>is a permutation of</u> $(0,\ldots,p)$, <u>then</u>

$$\omega[x_{\tau(0)},\ldots,x_{\tau(p)}] = (-1)^{o(\tau)}\omega[x_o,\ldots,x_p].$$

Proof. If $x_o,\ldots,x_p$ are not distinct, then neither are $x_{\tau(0)},\ldots,x_{\tau(p)}$. Therefore $\omega[x_{\tau(0)},\ldots,x_{\tau(p)}] = 0 = (-1)^{o(\tau)}\omega[x_o,\ldots,x_p]$. If $x_o,\ldots,x_p$ are distinct, there is a unique permutation $(\rho(0),\ldots,\rho(p))$ of $(0,\ldots,p)$ such that $x_{\rho(0)} < \ldots < x_{\rho(p)}$. Since $\rho\tau^{-1}$ is a permutation carrying $(\tau(0),\ldots,\tau(p))$ into $(\rho(0),\ldots,\rho(p))$ and $o(\tau) + o(\rho)$ is an order of $\rho\tau^{-1}$, it follows that $\omega[x_{\tau(0)},\ldots,x_{\tau(p)}] = (-1)^{o(\rho)+o(\tau)}[x_{\rho(0)},\ldots,x_{\rho(p)}] = (-1)^{o(\tau)}\omega[x_o,\ldots,x_p]$. Q.E.D.

(8.7) <u>For each</u> $c \in C_{p+1}(X)$, $\partial\omega c = \omega\partial c$.

Proof. It is sufficient to show that $\partial\omega[x_o,\ldots,x_{p+1}]$

$= \omega\partial[x_o,\ldots,x_{p+1}]$ for every $[x_o,\ldots,x_{p+1}] \in X(p+1)$. Fix an element $[x_o,\ldots,x_{p+1}]$ of $X(p+1)$. If $x_o,\ldots,x_{p+1}$ are not distinct, say $x_j = x_k$ with $j < k$, then $\partial\omega[x_o,\ldots,x_{p+1}] = 0$ and $\omega\partial[x_o,\ldots,x_{p+1}] = \omega\sum_{i=0}^{p+1}(-1)^i[x_o,\ldots,\hat{x}_i,\ldots,x_{p+1}] = (-1)^j\,[x_o,\ldots,\hat{x}_j,\ldots,x_k,\ldots,x_{p+1}] + (-1)^k\,[x_o,\ldots,x_j,\ldots,\hat{x}_k,\ldots,x_{p+1}]$. Since there exist $k-j-1$ interchanges carrying $(x_o,\ldots,\hat{x}_j,\ldots,x_k,\ldots,x_{p+1})$ into $(x_o,\ldots,x_j,\ldots,\hat{x}_k,\ldots,x_{p+1})$, it follows by (8.6) that $\omega[x_o,\ldots,x_j,\ldots,\hat{x}_k,\ldots,x_{p+1}] = (-1)^{k-j-1}\,[x_o,\ldots,\hat{x}_j,\ldots,x_k,\ldots,x_{p+1}]$. Hence $\omega\partial[x_o,\ldots,x_{p+1}] = 0 = \partial\omega[x_o,\ldots,x_{p+1}]$. If $x_o,\ldots,x_{p+1}$ are distinct, there is a unique permutation $(\rho(0),\ldots,\rho(p+1))$ of $(0,\ldots,p+1)$ such that $x_{\rho(0)} < \ldots < x_{\rho(p+1)}$. Therefore $\partial\omega[x_o,\ldots,x_{p+1}] = (-1)^{o(\rho)}\partial[x_{\rho(0)},\ldots,x_{\rho(p+1)}] = (-1)^{o(\rho)}\sum_{i=0}^{p+1}(-1)^i[x_{\rho(0)},\ldots,\hat{x}_{\rho(i)},\ldots,x_{\rho(p+1)}]$. Since $(\rho(0),\ldots,\rho(i-1),\rho(i+1),\ldots,\rho(p+1))$ is a permutation of $(0,\ldots,\rho(i)-1,\rho(i)+1,\ldots,p+1)$ $o(\rho) + \rho(i) + i$ is an order of this permutation, it follows that $\partial\omega[x_o,\ldots,x_{p+1}] = \sum_{i=1}^{p+1}(-1)^{\rho(i)}\omega[x_o,\ldots,x_{\rho(i)-1}, x_{\rho(i)+1},\ldots,x_{p+1}] = \omega\partial[x_o,\ldots,x_{p+1}]$. Q.E.D.

For each $x \in X$ there is a homomorphism of $C_p(X)$ into $C_{p+1}(X)$ defined as follows: For each $c \in C_p(X)$ we denote its image by xc. Then for each $[x_o,\ldots,x_p] \in X(p)$,

$$x[x_o,\ldots,x_p] = [x,x_o,\ldots,x_p].$$

(8.8) <u>For every</u> $c \in C_p(X)$, $p > 0$, <u>we have</u> $\partial xc = c - x\partial c$.

Proof. It is sufficient to show the assertion for $c \in X(p)$. Since $p > 0$, then for any $[x_o,\ldots,x_p] \in X(p)$ we have $\partial x[x_o,\ldots,x_p] = \partial[x,x_o,\ldots,x_p] = [x_o,\ldots,x_p] - \sum_{i=0}^{p}(-1)^i[x,x_o,\ldots,\hat{x}_i,\ldots,x_p] = [x_o,\ldots,x_p] - x\sum_{i=0}^{p}(-1)^i[x_o,\ldots,\hat{x}_i,\ldots,x_p] = [x_o,\ldots,x_p] - x\partial[x_o,\ldots,x_p]$. Q.E.D.

(8.9) (Eilenberg) <u>For each non-negative integer</u> p <u>there is a homomorphism</u> $D\colon C_p(X) \longrightarrow C_{p+1}(X)$ <u>such that whenever</u> $c \in C_p(X)$ <u>we have</u>

(i) $|Dc| \subset |c|$.

(ii) $$\partial Dc = \begin{cases} c - \omega c & \text{if } c \in C_o(X); \\ c - \omega c - D\partial c & \text{if } c \in C_p(X),\ p > 0. \end{cases}$$

Proof. The homomorphism D can be constructed inductively as follows. Let $D\colon C_o(X) \longrightarrow C_1(X)$ be the trivial homomorphism. Then for any $c \in C_o(X)$ we have $Dc = 0$ and so $\partial Dc = 0$. Hence both (i) and (ii) are satisfied. Suppose that the homomorphism D has been constructed with the desired behavior for $0,\ldots,p$. For each $\sigma \in X(p+1)$ we choose an element $x_\sigma \in |\sigma|$. Then a homomorphism $D\colon C_{p+1}(X) \longrightarrow C_{p+2}(X)$ is defined by

$$D\sigma = x_\sigma\sigma - x_\sigma\omega\sigma - x_\sigma D\partial\sigma \qquad \text{for every} \quad \sigma \in X(p+1).$$

By (8.5), $|x_\sigma\omega\sigma| = \{x_\sigma\} \cup |\omega\sigma| \subset |\sigma|$ and, by (8.3) and the inductive hypothesis, $|x_\sigma D\partial\sigma| = \{x_\sigma\} \cup |D\partial\sigma| \subset \{x_\sigma\} \cup |\partial\sigma| \subset |\sigma|$. Moreover, $|x_\sigma\sigma| = \{x_\sigma\} \cup |\sigma| = |\sigma|$. We infer by (8.1) that $|D\sigma| \subset |\sigma|$ for every $\sigma \in X(p+1)$. Using (8.1) again, we have $|Dc| \subset |c|$ for every $c \in C_{p+1}(X)$. By (8.2), (8.7), (8.8) and the inductive hypothesis, we have

$$\begin{aligned}
\partial D\sigma &= \partial(x_\sigma\sigma - x_\sigma\omega\sigma - x_\sigma D\partial\sigma) \\
&= \sigma - \omega\sigma - D\partial\sigma - x_\sigma(\partial\sigma - \partial\omega\sigma - \partial D\partial\sigma) \\
&= \sigma - \omega\sigma - D\partial\sigma - x_\sigma(\partial\sigma - \partial\omega\sigma - (\partial\sigma - \omega\partial\sigma)) \\
&= \sigma - \omega\sigma - D\partial\sigma \quad \text{for every} \quad \sigma \in X(p+1).
\end{aligned}$$

It follows that $Dc = c - \omega c - D\partial c$ for every $c \in C_{p+1}(X)$. Hence $D\colon C_{p+1}(X) \longrightarrow C_{p+2}(X)$ behaves as required. Q.E.D.

If A is a subset of X and $\{\lambda_0;\lambda_1\}$ is a cover of A (see p. 25), we denote by $C_p(\{\lambda_0;\lambda_1\})$ the subgroup of $C_p(X)$

generated by the elements σ of $X(p)$ with $|\sigma|$ contained in some $\lambda_o(u)$, $u \in \lambda_1$.

(8.10) (i) $\omega C_p(\{\lambda_o;\lambda_1\}) \subset C_p(\{\lambda_o;\lambda_1\})$.

(ii) $DC_p(\{\lambda_o;\lambda_1\}) \subset C_{p+1}(\{\lambda_o;\lambda_1\})$.

Proof. If $\sigma \in X(p) \cap C_p(\{\lambda_o;\lambda_1\})$, there is some $u \in \lambda_1$ such that $|\sigma| \subset \lambda_o(u)$. By (8.5), $|\omega\sigma| \subset \lambda_o(u)$. Hence $\omega\sigma \in C_p(\{\lambda_o;\lambda_1\})$. Since $C_p(\{\lambda_o;\lambda_1\})$ is generated by the elements in $X(p) \cap C_p(\{\lambda_o;\lambda_1\})$, it follows that $\omega C_p(\{\lambda_o;\lambda_1\}) \subset C_p(\{\lambda_o;\lambda_1\})$. This proves (i). Similarly we have (ii) by using (8.9). Q.E.D.

Let $\overline{C}^p(X)$ be the group of homomorphisms from $C_p(X)$ to the coefficient group G. Then ∂, ω, D induce homomorphisms $\partial^*: \overline{C}^p(X) \longrightarrow \overline{C}^{p+1}(X)$, $\omega^*: \overline{C}^p(X) \longrightarrow \overline{C}^p(X)$, $D^*: \overline{C}^{p+1}(X) \longrightarrow \overline{C}^p(X)$, defined respectively by

$$(\partial^*\varphi)(c) = \varphi(\partial c) \quad \text{for } c \in C_{p+1}(X),\ \varphi \in C^p(X),$$

$$(\omega^*\varphi)(c) = \varphi(\omega c) \quad \text{for } c \in C_p(X),\ \varphi \in C^p(X),$$

$$(D^*\varphi)(c) = \varphi(Dc) \quad \text{for } c \in C_p(X),\ \varphi \in C^{p+1}(X).$$

∂^*, ω^*, D^* are the dual homomorphisms of ∂, ω, D.

For each pair (X,A) let $\bar{C}^p(X,A)$ be the subgroup of $\bar{C}^p(X)$ such that $\varphi \in \bar{C}^p(X)$ is in $\bar{C}^p(X,A)$ if and only if there is a cover $\{\lambda_0;\lambda_1\}$ of A such that $\varphi = 0$ on $C_p(\{\lambda_0;\lambda_1\})$. By (8.4), (8.7), (8.9) and (8.10), we have immediately:

(8.11) (i) $\omega^*\omega^* = \omega^*$ <u>and</u> $\partial^*\omega^* = \omega^*\partial^*$.

(ii) <u>For each</u> $\varphi \in \bar{C}^p(X)$ <u>we have</u>

$$\varphi - \omega^*\varphi = \begin{cases} 0 & \text{if } p = 0, \\ \partial^*D^*\varphi + D^*\partial^*\varphi & \text{if } p > 0. \end{cases}$$

(iii) $\omega^*\bar{C}^p(X,A) \subset \bar{C}^p(X,A)$.

(iv) $D^*\bar{C}^{p+1}(X,A) \subset \bar{C}^p(X,A)$.

Since $C_p(X)$ is a free group with $X(p) = \{\bar{\xi} \mid \xi \in X^{p+1}\}$ as a base, there is a homomorphism $\chi: C^p(X) \longrightarrow \bar{C}^p(X)$ such that given any $\varphi \in C^p(X)$ we have

$$(\chi\varphi)(\bar{\xi}) = \varphi(\xi) \qquad \text{for every } \xi \in X^{p+1}.$$

Since $\xi \rightarrow \bar{\xi}$ is a 1-1 function of X^{p+1} into $X(p)$, there is a homomorphism $\chi': \bar{C}^p(X) \longrightarrow C^p(X)$ such that given any $\psi \in C^p(X)$ we have

$$(\chi'\psi)(\xi) = \psi(\bar{\xi}) \qquad \text{for every } \xi \in X^{p+1}.$$

(8.12) (i) $\varkappa : C^p(X) \longrightarrow \bar{C}^p(X)$ <u>and</u> $\varkappa = \varkappa'^{-1}$.

(ii) $\varkappa \bar{\delta} = \partial^* \varkappa$.

(iii) $\varkappa C^p(X,A) = \bar{C}^p(X,A)$.

Proof. (i) follows from the fact that both $\varkappa\varkappa'$ and $\varkappa'\varkappa$ are the identity isomorphisms. For any $\varphi \in C^p(X)$ and $(x_0,\dots,x_{p+1}) \in X^{p+2}$ we have

$$
\begin{aligned}
(\varkappa\bar{\delta}\varphi)[x_0,\dots,x_{p+1}] &= (\bar{\delta}\varphi)(x_0,\dots,x_{p+1}) \\
&= \textstyle\sum_{i=0}^{p+1}(-1)^i \varphi(x_0,\dots,\hat{x}_i,\dots,x_{p+1}) \\
&= \textstyle\sum_{i=0}^{p+1}(-1)^i (\varkappa\varphi)[x_0,\dots,\hat{x}_i,\dots,x_{p+1}] \\
&= (\varkappa\varphi)(\partial[x_0,\dots,x_{p+1}]) \\
&= (\partial^*\varkappa\varphi)[x_0,\dots,x_{p+1}].
\end{aligned}
$$

This proves (ii). If $\varphi \in C^p(X,A)$, then there is a cover $\{\lambda_0;\lambda_1\}$ of A such that $\varphi = 0$ on $\{\lambda_0;\lambda_1\}^{(p+1)}$. Therefore $\varkappa\varphi = 0$ on $C_p(\{\lambda_0;\lambda_1\})$ and $\varkappa\varphi \in \bar{C}^p(X,A)$. Similarly, if $\psi \in \bar{C}^p(X,A)$, then $\varkappa'\varphi \in C^p(X,A)$. Hence (iii) is proved. Q.E.D.

By (8.12), (i), ω^* and D^* induce homomorphisms Ω: $C^p(X) \longrightarrow C^p(X)$ and $\mathcal{D}$: $C^{p+1}(X) \longrightarrow C^p(X)$ defined respectively by $\Omega = \chi'\omega^{*}\chi$ and $\mathcal{D} = \chi' D^{*}\chi$. According to (8.11) we have

(8.13) (i) $\Omega\Omega = \Omega$ and $\bar{\delta}\Omega = \Omega\bar{\delta}$.

(ii) For each $\varphi \in C^p(X)$ we have

$$\varphi - \Omega\varphi = \begin{cases} 0 & \text{if } p = 0, \\ \bar{\delta}\mathcal{D}\varphi + \mathcal{D}\bar{\delta}\varphi & \text{if } p > 0. \end{cases}$$

(iii) $\Omega C^p(X,A) \subset C^p(X,A)$.

(iv) $\mathcal{D} C^{p+1}(X,A) \subset C^p(X,A)$.

A p-cochain φ of X is said to be alternative if and only if the following conditions hold: Let $(x_o,\ldots,x_p) \in X^{p+1}$.

(i) If $x_o,\ldots,x_p$ are not distinct, then $\varphi(x_o,\ldots,x_p) = 0$.

(ii) If $x_o,\ldots,x_p$ are distinct, then for any permutation $(\tau(0),\ldots,\tau(p))$ of $(o,\ldots,p)$,

$\varphi(x_{\tau(0)},\ldots,x_{\tau(p)}) = (-1)^{o(\tau)}\varphi(x_o,\ldots,x_p)$.

Note that the definition of an alternative cochain is independent of the linear ordering on X used to define ω.

(8.14) $\varphi \in C^p(X)$ is alternative if and only if $\varphi = \Omega\varphi$.

Proof. Let $\varphi \in C^p(X)$ and $(x_0,\ldots,x_p) \in X^{p+1}$. Then $(\Omega\varphi)(x_0,\ldots,x_p) = (\varkappa'\omega^*\varkappa\varphi)(x_0,\ldots,x_p) = (\omega^*\varkappa\varphi)[x_0,\ldots,x_p]$ $= (\varkappa\varphi)(\omega[x_0,\ldots,x_p])$. Suppose that $\varphi = \Omega\varphi$. If $x_0,\ldots,x_p$ are not distinct, then $\varphi(x_0,\ldots,x_p) = (\Omega\varphi)(x_0,\ldots,x_p) =$ $(\varkappa\varphi)(\omega[x_0,\ldots,x_p]) = 0$. If $x_0,\ldots,x_p$ are distinct, then for any permutation $(\tau(0),\ldots,\tau(p))$ of $(0,\ldots,p)$ we have $\varphi(x_{\tau(0)},\ldots,x_{\tau(p)}) = (\Omega\varphi)(x_{\tau(0)},\ldots,x_{\tau(p)}) = (\varkappa\varphi)(\omega[x_{\tau(0)},$ $\ldots,x_{\tau(p)}]) = (-1)^{o(\tau)}(\varkappa\varphi)(\omega[x_0,\ldots,x_p]) = (-1)^{o(\tau)}(\Omega\varphi)(x_0,$ $\ldots,x_p) = (-1)^{o(\tau)}\varphi(x_0,\ldots,x_p)$. Hence φ is alternative. Conversely suppose that φ is alternative. If $x_0,\ldots,x_p$ are not distinct, then $(\Omega\varphi)(x_0,\ldots,x_p) = (\varkappa\varphi)(\omega[x_0,\ldots,x_p]) =$ $0 = \varphi(x_0,\ldots,x_p)$. If $x_0,\ldots,x_p$ are distinct, there is a unique permutation $(\rho(0),\ldots,\rho(p))$ of $(0,\ldots,p)$ such that $x_{\rho(0)} < \ldots < x_{\rho(p)}$. Therefore $(\Omega\varphi)(x_0,\ldots,x_p) =$ $(\varkappa\varphi)(\omega[x_0,\ldots,x_p]) = (-1)^{o(\rho)}(\varkappa\varphi)[x_{\rho(0)},\ldots,x_{\rho(p)}] =$ $(-1)^{o(\rho)}\varphi(x_{\rho(0)},\ldots,x_{\rho(p)}) = \varphi(x_0,\ldots,x_p)$. Hence $\varphi = \Omega\varphi$. Q.E.D.

Let $\tilde{C}^p(X)$ be the subgroup of $C^p(X)$ consisting of all the alternative p-cochains of X. Then $\bar{\delta}\tilde{C}^p(X) \subset \tilde{C}^{p+1}(X)$, since for any $\varphi \in \tilde{C}^p(X)$, $\Omega\bar{\delta}\varphi = \bar{\delta}\Omega\varphi = \bar{\delta}\varphi$. If (X,A) is a pair, we denote

$$\tilde{C}^p(X,A) = C^p(X,A) \cap \tilde{C}^p(X).$$

$$\tilde{C}_o^p(X,A) = C_o^p(X,A) \cap \tilde{C}^p(X).$$

$$\tilde{Z}^p(X,A) = \bar{\delta}^{-1}\tilde{C}_o^{p+1}(X,A) \cap \tilde{C}^p(X,A).$$

$$\tilde{B}^p(X,A) = \begin{cases} 0 & \text{if } p = 0, \\ \bar{\delta}\tilde{C}^{p-1}(X,A) + \tilde{C}_o^p(X,A) & \text{if } p > 0. \end{cases}$$

Clearly $\tilde{B}^p(X,A)$ is a subgroup of $\tilde{Z}^p(X,A)$. Therefore we obtain the p-th alternative Alexander-Kolmogoroff cohomology group of X mod A

$$\tilde{H}^p(X,A) = \tilde{Z}^p(X,A)/\tilde{B}^p(X,A).$$

If $f: (X,A) \longrightarrow (Y,B)$ is a mapping and $f^{\#}: C^p(Y) \longrightarrow C^p(X)$ is defined by (5.4), then $f^{\#}\tilde{Z}^p(Y,B) \subset \tilde{Z}^p(X,A)$ and $f^{\#}\tilde{B}^p(Y,B) \subset \tilde{B}^p(X,A)$ and hence $f^{\#}$ induces a homomorphism $f^{\sim}: \tilde{H}^p(Y,B) \longrightarrow \tilde{H}^p(X,A)$ such that $f^{\sim}\gamma_1^{\sim} = \gamma^{\sim}f^{\#}$ on $\tilde{Z}^p(Y,B)$, where $\gamma^{\sim}: \tilde{Z}^p(X,A) \longrightarrow \tilde{H}^p(X,A)$ and $\gamma_1^{\sim}: \tilde{Z}^p(Y,B) \longrightarrow \tilde{H}^p(Y,B)$ are natural homomorphisms.

Let (X,A) be a pair and let i be the injection of A into X. Then there is a homomorphism $\delta^{\sim}: \tilde{H}^p(A) \longrightarrow \tilde{H}^{p+1}(X,A)$ such that

$$\delta^{\sim}\beta^{\sim}i^{\#} = \gamma^{\sim}\bar{\delta} \quad \text{on} \quad i^{\#-1}\tilde{Z}^p(A) \cap \tilde{C}^p(X),$$

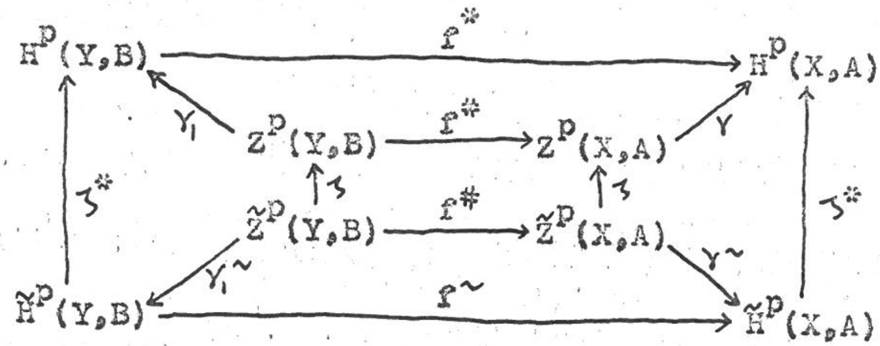
where $\beta^{\sim}: \tilde{Z}^p(A) \longrightarrow \tilde{H}^p(A)$ and $\gamma^{\sim}: \tilde{Z}^{p+1}(X,A) \longrightarrow \tilde{H}^{p+1}(X,A)$ are natural homomorphisms.

(8.15) <u>The alternative Alexander-Kolmogoroff cohomology theory</u> $(\tilde{H}^p, f^{\sim}, \delta^{\sim})$ <u>is equivalent to the Alexander-Kolmogoroff cohomology theory</u> (H^p, f^*, δ) <u>for general spaces.</u>

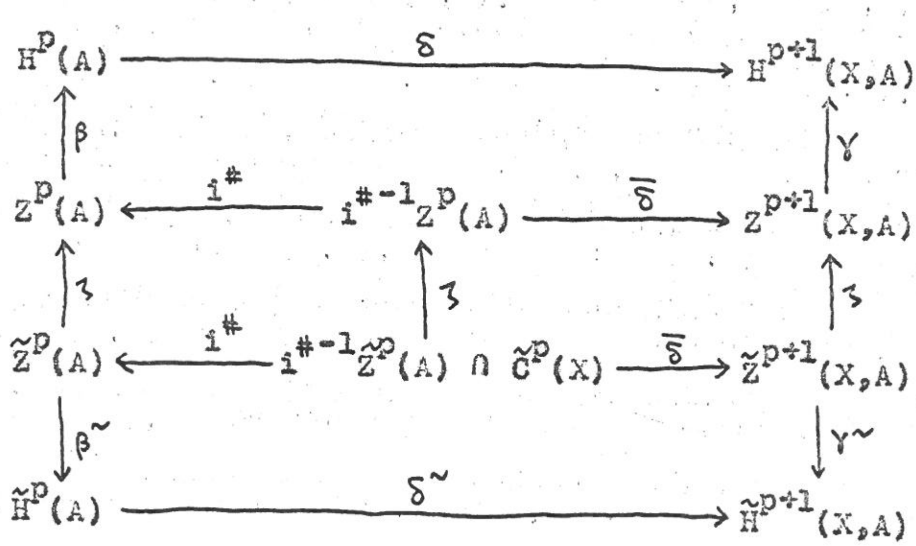
Proof. Given any space X let $\zeta: \tilde{C}^p(X) \longrightarrow C^p(X)$ be the injection. It is clear that $\zeta\bar{\delta} = \bar{\delta}\zeta$ on $\tilde{C}^p(X)$ and for any subset A of X, $\zeta\tilde{C}^p(X,A) \subset C^p(X,A)$. Therefore, if (X,A) is a pair, then $\zeta\tilde{Z}^p(X,A) \subset Z^p(X,A)$ and $\zeta\tilde{B}^p(X,A) \subset B^p(X,A)$. Hence there is a homomorphism $\zeta^*: \tilde{H}^p(X,A) \longrightarrow H^p(X,A)$ such that $\zeta^*\gamma^{\sim} = \gamma\zeta$ on $\tilde{Z}^p(X,A)$, where $\gamma : Z^p(X,A) \longrightarrow H^p(X,A)$ and $\gamma^{\sim}: \tilde{Z}^p(X,A) \longrightarrow \tilde{H}^p(X,A)$ are natural homomorphisms. $\Omega C^p(X) \subset \tilde{C}^p(X)$, since for any $\varphi \in C^p(X)$ we have, by (8.13), (i), $\Omega(\Omega\varphi) = \Omega\varphi$ and hence, by (8.14), $\Omega\varphi \in \tilde{C}^p(X)$. Therefore there is a homomorphism $\eta : C^p(X) \longrightarrow \tilde{C}^p(X)$ defined by $\eta\varphi = \Omega\varphi$ for every $\varphi \in C^p(X)$. By (8.13), (i) and (iii), $\eta\bar{\delta} = \bar{\delta}\eta$ and for any subset A of X, $\eta C^p(X,A) \subset \tilde{C}^p(X,A)$. Therefore η induces a homomorphism $\eta^*: H^p(X,A) \longrightarrow \tilde{H}^p(X,A)$

such that $\eta^*\gamma = \gamma^\sim\eta$.

If $f: (X,A) \longrightarrow (Y,B)$ is a mapping, then in the diagram



$f^*\zeta^*\gamma_1^\sim = f^*\gamma_1\zeta = \gamma f^\#\zeta = \gamma\zeta f^\# = \zeta^*\gamma^\sim f^\# = \zeta^* f^\sim\gamma_1^\sim$. Hence $f^*\zeta^* = \zeta^* f^\sim$. If (X,A) is a pair and $i: A \longrightarrow X$ is the injection, then in the diagram



$\delta\zeta^*\beta^\sim i^\# = \delta\beta\zeta i^\# = \delta\beta i^\#\zeta = \gamma\bar{\delta}\zeta = \gamma\zeta\bar{\delta} = \zeta^*\gamma^\sim\bar{\delta} = \zeta^*\delta^\sim\beta^\sim i^\#$.

Hence $\delta\zeta^* = \zeta^*\delta^\sim$.

Clearly $\eta\zeta: \tilde{C}^p(X) \longrightarrow \mathcal{C}^p(X)$ is the identity isomorphism;

then so is $\eta^* \zeta^* : \tilde{H}^p(X,A) \longrightarrow \tilde{H}^p(X,A)$. Hence ζ^* is an isomorphism into. Since $\zeta\eta = \Omega$, it follows by (8.13), (ii) that for any $\varphi \in C^p(X)$,

$$\varphi - \zeta\eta\varphi = \begin{cases} 0 & \text{if } p = 0, \\ \bar{\delta}\mathcal{D}\varphi + \mathcal{D}\bar{\delta}\varphi & \text{if } p > 0. \end{cases}$$

If $\varphi \in Z^p(X,A)$, $p > 0$, then by (8.13), (iii), $\mathcal{D}\varphi \in C^{p-1}(X,A)$ and $\mathcal{D}\bar{\delta}\varphi \in C_o^p(X,A)$. Therefore $\varphi - \zeta\eta\varphi \in B^p(X,A)$. If $\varphi \in C^o(X,A)$, then $\varphi - \zeta\eta\varphi = 0 \in B^o(X,A)$. Hence $\zeta^*\eta^* : H^p(X,A) \longrightarrow H^p(X,A)$ is the identity isomorphism and ζ^* is onto.

Combining these results, the cohomology theories (H^p, f^*, δ) and $(\tilde{H}^p, f^\sim, \delta^\sim)$ are equivalent for general spaces. Q.E.D.

As a consequence of the proof of (8.15) we have

(8.16) <u>Every Alexander-Kolmogoroff cohomology class contains at least one alternative cocycle.</u>

9. A Proof of the Homotopy Lemma (7.10).

Let (X,A) be a pair such that $(X\times I, A\times I)$ is fully normal, where I is the closed interval from 0 to 1 with the usual topology. Let $h_t: (X,A) \longrightarrow (X\times I, A\times I)$ be defined by $h_t(x) = (x,t)$, $t = 0, 1$. We assert in (7.10) that $h_o^* = h_1^*$. By (8.16), our assertion is equivalent to the following

(9.1) <u>If</u> $\varphi \in \tilde{Z}^p(X\times I, A\times I)$, <u>then</u> $h_o^{\#}\varphi - h_1^{\#}\varphi \in B^p(X,A)$.

By (6.3), (i), there is a covering $\lambda = \{\lambda_o;\lambda_1,\lambda_2\}$ of $(X\times I, A\times I)$ such that $\varphi = 0$ on $(\{\tilde{\lambda}_o;\lambda_2\}^*)^{(p+1)}$ and $\bar{\delta}\varphi = 0$ on $(\{\lambda_o;\lambda_1\}^*)^{(p+2)}$, where $\tilde{\lambda}_o = i^{-1}\lambda_o$ with i being the injection of $A\times I$ into $X\times I$. For each $(x,t) \in X\times I$ there is a neighborhood $U_{(x,t)}$ of x in X and a connected neighborhood $V_{(x,t)}$ of t in I such that $U_{(x,t)}\times V_{(x,t)}$ is contained in some $\lambda_o(u)$, where u is in λ_2 or in $\lambda_1 - \lambda_2$ according as $(x,t) \in A\times I$ or $(x,t) \in (X-A)\times I$. Given any $x \in X$, $\{V_{(x,t)} \mid t \in I\}$ is a collection of open sets whose union covers I. Since each $V_{(x,t)}$ is connected, there exists a finite subcollection $\{V_{(x,t(i))} \mid i = 1,\dots,r(x)\}$ such that

$$0 \in V_{(x,t(1))} - V_{(x,t(2))}, \quad 1 \in V_{(x,t(r(x)))} - V_{(x,t(r(x)-1))};$$

$$V_{(x,t(i))} \cap V_{(x,t(i+1))} \neq \emptyset \quad \text{for } 1 \leq i \leq r(x);$$

$$V_{(x,t(i))} \cap V_{(x,t(j))} = \emptyset \quad \text{for } |i-j| > 1.$$

Let $U_x = \cap_{i=1}^{r(x)} U_{(x,t(i))}$. Let $\rho_1 = X$, $\rho_2 = A$ and define $\rho_0 : \rho_1 \longrightarrow Q(X)$ ($Q(X)$ is the collection of open sets in X) by $\rho_0(x) = U_x$ for every $x \in \rho_1$. Then $\rho = \{\rho_0 ; \rho_1, \rho_2\}$ is a covering of (X,A). Let $\mu_1 = \cup_{x \in X}\{((x,i)) \mid i = 1,\ldots,r(x)\}$, $\mu_2 = \cup_{x \in A}\{((x,i)) \mid i = 1,\ldots,r(x)\}$ and define $\mu_0 : \mu_1 \longrightarrow Q(X \times I)$ by $\mu_0((x,i)) = U_x \times V_{(x,t(i))}$ for all $((x,i)) \in \mu_1$. Then $\mu = \{\mu_0 ; \mu_1, \mu_2\}$ is a covering of $(X \times I, A \times I)$ and $\mu > \lambda$. For each $x \in X$ let $t_x^0 = 0$, $t_x^{r(x)} = 1$ and choose a $t_x^i \in V_{(x,t(i))} \cap V_{(x,t(i+1))}$, $0 < i < r(x)$. Then $t_x^0 < \ldots < t_x^{r(x)}$. For convenience' sake we designate $V_{(x,t(i))}$ and (x, t_x^i) by V_x^i and $[x,i]$ and let $Z = \{[x,i] \mid 1 \leq i \leq r(x) \text{ and } x \in X\}$.

Let $X \times I$ be ordered such that $(x,t) < (x',t')$ if and only if either $t < t'$ or $t = t'$ and $x < x'$. Then $X \times I$ is linearly ordered by $<$. With respect to this ordering there is a homomorphism $\overline{\omega}: C_p(X \times I) \longrightarrow C_p(X \times I)$ analogous to ω. As the dual of $\overline{\omega}$ there is a homomorphism $\overline{\omega}^* : C^p(X \times I)$

$\longrightarrow \bar{C}^p(X\times I)$ defined by $(\bar{\omega}^*\psi)c = \psi(\bar{\omega}c)$ for $\psi \in \bar{C}^p(X\times I)$ and $c \in C_p(X\times I)$. Since there is a natural isomorphism $\bar{\chi}$ of $C^p(X\times I)$ onto $\bar{C}^p(X\times I)$, $\bar{\omega}^*$ induces a homomorphism $\bar{\Omega}$: $C^p(X\times I) \longrightarrow C^p(X\times I)$ defined by $(\bar{\Omega}\psi)(\xi) = (\bar{\omega}^*\bar{\chi}\psi)(\bar{\xi})$ for $\psi \in C^p(X\times I)$ and $\xi \in X^{p+1}$. We can easily see that all the properties of ω, ω^*, Ω obtained above also hold for $\bar{\omega}$, $\bar{\omega}^*$, $\bar{\Omega}$.

Let $\xi = (x_o,\ldots,x_p) \in X^{p+1}$. If x is a coordinate of ξ and $0 < j \leqslant r(x)$, then for each i there is a unique integer $\lambda(i)$ defined by $\lambda(i) = \max\{\lambda \mid [x,j] \not< [x_i,\lambda]\}$. We denote

$$\theta_{[x,j]}[x_o,\ldots,x_p] = [[x_o,\lambda(0)],\ldots,[x_p,\lambda(p)]].$$

If $x = \max\{x_o,\ldots,x_p\}$, then we denote

$$\theta_{[x,0]}[x_o,\ldots,x_p] = [[x_o,0],\ldots,[x_p,0]].$$

For each $[x,j] \in Z$ there is a homomorphism $d_{[x,j]}$: $C_p(X) \longrightarrow C_{p+1}(X\times I)$ defined as follows : Let $\xi = (x_o,\ldots,x_p) \in X^{p+1}$.

(i) If x is not a coordinate of ξ, then

$$d_{[x,j]}[x_o,\ldots,x_p] = 0;$$

(ii) If x is a coordinate of ξ, then

$$d_{[x,j]}[x_o,\ldots,x_p] = \bar{\omega}[x,j-1]\,\theta_{[x,j]}[x_o,\ldots,x_p].$$

In case (ii) if $x_o,\ldots,x_p$ are distinct, $x = x_n$ $(0 \leqslant n \leqslant p)$ and $[y,k]$ is the maximum of $[x_o,\Delta(0)],\ldots,[x_{n-1},\Delta(n-1)]$, $[x_n,\Delta(n)-1],[x_{n+1},\Delta(n+1)],\ldots,[x_p,\Delta(p)]$, then

$$\theta_{[y,k]}[x_o,\ldots,x_p] = [[x_o,\Delta(0)],\ldots,[x_{n-1},\Delta(n-1)],$$
$$[x_n,\Delta(n)-1],[x_{n+1},\Delta(n+1)],\ldots,[x_p,\Delta(p)]].$$

It follows that for $p > 0$,

$$d_{[x,j]}\partial[x_o,\ldots,x_p] = \sum_{i=0}^{p}(-1)^i d_{[x,j]}[x_o,\ldots,\hat{x}_i,\ldots,x_p]$$
$$= \sum_{i=0,i\neq n}^{p}(-1)^i\bar{\omega}[x,j-1]\,\theta_{[x,j]}[x_o,\ldots,\hat{x}_i,\ldots,x_p].$$

$$\partial d_{[x,j]}[x_o,\ldots,x_p] = \partial\bar{\omega}[x,j-1]\,\theta_{[x,j]}[x_o,\ldots,x_p]$$
$$= \bar{\omega}\partial[x,j-1]\,\theta_{[x,j]}[x_o,\ldots,x_p]$$
$$= \bar{\omega}\,\theta_{[x,j]}[x_o,\ldots,x_p] - \bar{\omega}[x,j-1]\partial\theta_{[x,j]}[x_o,\ldots,x_p]$$
$$= \bar{\omega}\,\theta_{[x,j]}[x_o,\ldots,x_p] - \bar{\omega}\sum_{i=0}^{p}(-1)^i[[x,j-1],$$
$$[x_o,\Delta(0)],\ldots,\widehat{[x_i,\Delta(i)]},\ldots,[x_p,\Delta(p)]]$$
$$= \bar{\omega}\,\theta_{[x,j]}[x_o,\ldots,x_p] - \bar{\omega}\,\theta_{[y,k]}[x_o,\ldots,x_p]$$
$$- \sum_{i=0,i=n}^{p}(-1)^i\bar{\omega}[x,j-1]\,\theta_{[x,j]}[x_o,\ldots,\hat{x}_i,\ldots,x_p].$$

Therefore

(9.2) $\quad (d_{[x,j]}\partial + \partial d_{[x,j]})[x_o,\ldots,x_p]$

$$= \bar{\omega}\theta_{[x,j]}[x_0,\ldots,x_p] - \bar{\omega}\theta_{[y,k]}[x_0,\ldots,x_p].$$

If $p = 0$, then $\partial d_{[x,j]}[x] = \partial\bar{\omega}[x,j-1]\theta_{[x,j]}[x] =$ $\partial\bar{\omega}[[x,j-1],[x,j]] = \partial[[x,j-1],[x,j]] = [x,j] - [x,j-1]$, or

(9.2') $$\partial d_{[x,j]}[x] = [x,j] - [x,j-1].$$

Given any $\xi \in X^{p+1}$, $d_{[x,j]}(\bar{\xi}) \neq 0$ only if x is a coordinate of ξ. Therefore a homomorphism $D\colon C_p(X) \longrightarrow C_{p+1}(X\times I)$ is defined by

$$D(\bar{\xi}) = \sum\{d_{[x,j]}\bar{\xi} \mid d_{[x,j]}\bar{\xi} \neq 0\}.$$

Let $h_{i\#}\colon C_p(X) \longrightarrow C_p(X\times I)$ be the homomorphism defined by $h_{i\#}[x_0,\ldots,x_p] = [(x_0,i),\ldots,(x_p,i)]$ for every $(x_0,\ldots,x_p) \in X^{p+1}$, $i = 0,1$.

(9.3) $$\bar{\omega}h_{1\#} - \bar{\omega}h_{0\#} = \begin{cases} D\omega\partial + \partial D\omega & \underline{\text{on}}\ C_p(X),\ p > 0, \\ \partial D\omega & \underline{\text{on}}\ C_0(X). \end{cases}$$

Proof. It is sufficient to show that for every $\xi \in X^{p+1}$,

$$\bar{\omega}h_{1\#}\bar{\xi} - \bar{\omega}h_0\bar{\xi} = \begin{cases} D\omega\partial\bar{\xi} + \partial D\omega\bar{\xi} & \text{if } p > 0, \\ \partial D\omega\bar{\xi} & \text{if } p = 0. \end{cases}$$

For any $x \in X$ we have

$$\partial D\omega[x] = \partial D[x] = \partial \sum_{i=1}^{r(x)} d_{[x,i]}[x]$$

$$= \sum_{i=1}^{r(x)} ([x,i] - [x,i-1])$$

$$= [x,r(x)] - [x,0]$$

$$= \bar{\omega} h_{1\#}[x] - \bar{\omega} h_{0\#}[x].$$

The second part is proved.

Now fix an $\xi = (x_0,\ldots,x_p) \in X^{p+1}$, $p > 0$. If $x_0,\ldots,x_p$ are not distinct, then

$$\bar{\omega} h_{1\#}\bar{\xi} - \bar{\omega} h_{0\#}\bar{\xi} = 0 = D\omega\partial\bar{\xi} + \partial D\omega\bar{\xi}.$$

Suppose that $x_0,\ldots,x_p$ are distinct and let $\{[x_i,\ell] \mid 0 \leq \ell \leq r(x_i) \text{ and } 0 \leq i \leq p\}$ be arranged in a descending sequence

$$z(1),\ldots,z(R),z(R+1),\ldots,z(R+p+1),$$

where $R = \sum_{i=0}^{p} r(x_i)$. Then $z(1) = (\max_i x_i, 1)$ and $z(R+1) = (\max_i x_i, 0)$. Therefore

$$\Theta_{z(1)}[x_0,\ldots,x_p] = [(x_0,1),\ldots,(x_p,1)] = h_{1\#}[x_0,\ldots,x_p],$$

$$\Theta_{z(R+1)}[x_0,\ldots,x_p] = [(x_0,0),\ldots,(x_p,0)] = h_{0\#}[x_0,\ldots,x_p].$$

Moreover,

$$D[x_0,\ldots,x_p] = \sum_{j=1}^{R} d_{z(j)}[x_0,\ldots,x_p].$$

If $j \leq R$ with $\Theta_{z(j)}[x_0,\ldots,x_p] = [[x_0,\ell(0)],\ldots,[x_p,\ell(p)]]$

and $z(j) = [x_n, \mathcal{A}(n)]$, then $z(j+1) \not< [x_i, \mathcal{A}(i)]$ for all i different from n and $z(j+1) \not< [x_n, \mathcal{A}(n)-1]$. Hence

$$\theta_{z(j+1)}[x_o,\dots,x_p] = [[x_o,\mathcal{A}(0)],\dots,[x_{n-1},\mathcal{A}(n-1)],$$
$$[x_n,\mathcal{A}(n)-1],[x_{n+1},\mathcal{A}(n+1)],\dots,[x_p,\mathcal{A}(p)]].$$

and, by (9.2),

$$(d_{z(j)}\partial + \partial d_{z(j)})[x_o,\dots,x_p] = \bar{\omega}\theta_{z(j)}[x_o,\dots,x_p]$$
$$- \bar{\omega}\theta_{z(j+1)}[x_o,\dots,x_p].$$

Consequently,

$$(D\partial + \partial D)[x_o,\dots,x_p] = \textstyle\sum_{j=1}^{R}(d_{z(j)}\partial + \partial d_{z(j)})[x_o,\dots,x_p]$$
$$= \textstyle\sum_{j=1}^{R}(\bar{\omega}\theta_{z(j)}[x_o,\dots,x_p] - \bar{\omega}\theta_{z(j+1)}[x_o,\dots,x_p])$$
$$= \bar{\omega}\theta_{z(1)}[x_o,\dots,x_p] - \bar{\omega}\theta_{z(R+1)}[x_o,\dots,x_p]$$
$$= (\bar{\omega}h_{1\#} - \bar{\omega}h_{o\#})[x_o,\dots,x_p].$$

Since $\omega\omega = \omega$ and $\bar{\omega}h_{1\#} = h_{1\#}\omega$, it follows that

$$(D\omega\partial + \partial D\omega)[x_o,\dots,x_p] = (D\partial + \partial D)\omega[x_o,\dots,x_p]$$
$$= (\bar{\omega}h_{1\#} - \bar{\omega}h_{o\#})\omega[x_o,\dots,x_p]$$
$$= (\bar{\omega}h_{1\#} - \bar{\omega}h_{o\#})[x_o,\dots,x_p],$$

completing the proof of (9.3). Q.E.D.

As the dual of D there is a homomorphism $D^*: C^{p+1}(X\ I) \longrightarrow C^p(X)$ defined by

$$(D^*\psi)c = \psi(Dc)$$

for $\psi \in \bar{C}^{p+1}(X\times I)$ and $c \in C_p(X)$. Since there is a natural isomorphism $\bar{\chi}$ of $C^{p+1}(X\times I)$ onto $\bar{C}^{p+1}(X\times I)$ and a natural isomorphism χ of $C^p(X)$ onto $\bar{C}^p(X)$, D^* induces a homomorphism $\mathcal{D} : C^{p+1}(X\times I) \longrightarrow C^p(X)$ such that $D^*\bar{\chi} = \chi\mathcal{D}$, i.e.,

$$(\mathcal{D}\psi)(\xi) = (D^*\bar{\chi}\psi)(\bar{\xi})$$

for $\psi \in C^{p+1}(X\times I)$ and $\xi \in X^{p+1}$. By (9.3), we can easily show that

(9.4) __For each__ $\psi \in C^p(X\ I)$ __we have__

$$h_1^{\#}\bar{\Omega}\psi - h_0^{\#}\bar{\Omega}\psi = \begin{cases} \bar{\delta}\Omega\mathcal{D}\psi + \Omega\mathcal{D}\bar{\delta}\psi & \underline{\text{if}}\ p > 0, \\ \Omega\mathcal{D}\bar{\delta}\psi & \underline{\text{if}}\ p = 0. \end{cases}$$

Proof of (9.1). Since $\varphi \in \tilde{Z}^p(X\times I, A\times I)$, then $\bar{\Omega}\varphi = \varphi$. It follows by (9.4) that

$$h_1^{\#}\varphi - h_0^{\#}\varphi = \begin{cases} \bar{\delta}\Omega\mathcal{D}\varphi + \Omega\mathcal{D}\bar{\delta}\varphi & \text{if } p > 0, \\ \Omega\mathcal{D}\bar{\delta}\varphi & \text{if } p = 0. \end{cases}$$

Therefore we have only to show that $\Omega\mathcal{D}\varphi \in C^{p-1}(X,A)$ for $p > 0$ and $\Omega\mathcal{D}\bar{\delta}\varphi \in C_0^p(X,A)$ for $p \geqslant 0$. Here only a proof for $\Omega\mathcal{D}\varphi \in C^{p-1}(X,A)$, $p > 0$, will be given below, since a parallel proof for the other can be done without difficulty.

Fix an $\xi = (x_0,\ldots,x_{p-1}) \in \{\tilde{\rho}_0;\rho_2\}^{(p)}$, say $\xi \in (A \cap U_y)^p$ with $y \in A$. If $x_0,\ldots,x_{p-1}$ are not distinct, then $\Omega\partial\varphi(\bar{\xi}) = (\varkappa\partial\varphi)(\omega\bar{\xi}) = 0$. Suppose that $x_0,\ldots,x_{p-1}$ are distinct. Let x be a coordinate of ξ and $1 \leq j \leq r(x)$. Let $t_x^j \in V_y^m$ and $d_{[x,j]}[x_0,\ldots,x_{p-1}] = [[x,j-1],[x_0,\lambda(0)],\ldots,[x_{p-1},\lambda(p-1)]]$. Then for each i either $\lambda(i) = r(i)$ and $[x,j] \nless [x_i,\lambda(i)]$ or $[x,j] < [x_i,\lambda(i)+1]$ and $[x,j] \nless [x_i,\lambda(i)]$. In the first case we have $t_{x_i}^{\lambda(i)} = t_x^j = 1$ and hence $(x_i,1) \in \tilde{\mu}_0((x_i,\lambda(i))) \cap \tilde{\mu}_0((y,m))$. Therefore

$$[x_i,\lambda(i)] = (x_i,1) \in \tilde{\mu}_0^*((y,m)).$$

In the second case we have $[x_i,\lambda(i)], [x_i,\lambda(i)+1] \in \tilde{\mu}_0((x_i,\lambda(i)+1))$ and hence $t_x^j \in V_{x_i}^{\lambda(i)+1}$ and $(x_i,t_x^j) \in \tilde{\mu}_0((x_i,\lambda(i)+1)) \cap \tilde{\mu}_0((y,m))$. Therefore

$$[x_i,\lambda(i)] \in \tilde{\mu}_0((x_i,\lambda(i)+1)) \subset \tilde{\mu}_0^*((y,m)).$$

Since $[x,j] \in \tilde{\mu}_0((x,j)) \cap \tilde{\mu}_0((y,m))$, we have also

$$[x,j-1] \in \tilde{\mu}_0((x,j)) \subset \tilde{\mu}_0^*((y,m)).$$

Hence

$$d_{[x,j]}[x_0,\ldots,x_{p-1}] \in C_p(\{\tilde{\mu}_0;\tilde{\mu}_2\}^*).$$

By the definition of ω, $\omega[x_0,\ldots,x_{p-1}]$ is either 0 or $(-1)^{o(\rho)}[x_{\rho(0)},\ldots,x_{\rho(p-1)}]$ with $(\rho(0),\ldots,\rho(p-1))$ being the permutation of $(0,\ldots,p-1)$ such that $x_{\rho(0)} < \ldots < x_{\rho(p-1)}$. We infer that

$$d_{[x,j]}\omega[x_0,\ldots,x_{p-1}] \in C_p(\{\tilde{\mu}_0;\mu_2\}^*).$$

Consequently,

$$D\omega[x_0,\ldots,x_{p-1}] = \Sigma\{d_{[x,j]}\omega[x_0,\ldots,x_{p-1}] \mid d_{[x,j]}\overline{\omega}[x_0,\ldots,x_{p-1}] \neq 0\} \in C_p(\{\tilde{\mu}_0;\mu_2\}^*).$$

Since $\varphi = 0$ on $(\{\tilde{\lambda}_0;\lambda_2\}^*)^{(p+1)}$ and $\mu > \lambda$, it follows that $\varkappa\varphi = 0$ on $C_p(\{\tilde{\mu}_0;\mu_2\}^*)$. Hence $(\Omega\mathcal{D}\varphi)(x_0,\ldots,x_{p-1}) = (\overset{*}{\omega}D^*\varkappa\varphi)[x_0,\ldots,x_{p-1}] = (\varkappa\varphi)(D\omega[x_0,\ldots,x_{p-1}]) = 0$, proving that $\Omega\mathcal{D}\varphi = 0$ on $\{\tilde{\rho}_0;\rho_2\}^{(p)}$ and $\Omega\mathcal{D}\varphi \in C^{p-1}(X,A)$.

Q.E.D.

BIOGRAPHY

Chung-Tao Yang was born May 4, 1923, in Pingyang, Chekiang, China. After graduating from Wenchow Middle School in 1942 he attended National Chekiang University and received the degree of Bachelor of Science in Mathematics in 1946. He was associated with National Chekiang University and National Academia Sinica as an Assistant of Mathematics and was an Instructor of Mathematics at National Taiwan University. In September 1950 he came to the United States and entered Tulane University as a graduate student.

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