MATH-GA2120 Linear Algebra II

Bilinear and Sesquilinear Forms
Quadratic Forms

Deane Yang

Courant Institute of Mathematical Sciences
New York University

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Bilinear Form on Real Vector Space

- A **bilinear form** on a real vector space V is a bilinear function

$$B : V \times V \to \mathbb{R}$$

- I.e., for any $a^1, a^2 \in \mathbb{R}$ and $v_1, v_2, v \in V$,

$$B(a^1 v_1 + a^2 v_2, v) = a^1 B(v_1, v) + a^2 B(v_2, v)$$
$$B(v, a^1 v_1 + a^2 v_2) = a^1 B(v, v_1) + a^2 B(v, v_2)$$

- An inner product is an example of a bilinear form

Bilinear Form on Inner Product Space

- Given a linear map $L : V \to V$, the function
$$\begin{aligned} B : V \times V &\to \mathbb{R} \\ (v_1, v_2) &\mapsto (L(v_1), v_2) \end{aligned}$$
is a bilinear form
- Conversely, if $B : V \times V \to \mathbb{R}$ is a bilinear form, then there is a map
$$\begin{aligned} \delta_B : V &\to V^* \\ w &\mapsto \ell_w, \end{aligned}$$
where for any $v \in V$,
$$\langle \ell_w, v \rangle = B(v, w)$$
- If V has an inner product and
$$L = (\delta^{-1} \circ \delta_B)^* : V \to V,$$
then
$$B(v, w) = \langle \delta_B(w), v \rangle = (v, \delta^{-1}(\delta_B(w))) = (L(v), w))$$

Bilinear Form as Matrix

- If $(e_1, \ldots, e_n)$ is a basis of V and
$$v = e_j v^j \text{ and } w = e_k w^k,$$
then
$$\begin{aligned} B(v, w) &= B(e_j v^j, e_k w^k) \\ &= v^j w^k B(e_j, e_k) \\ &= v^j w^k M_{jk} \end{aligned}$$
- Therefore, given a basis $(e_1, \ldots, e_n)$ of V, B is uniquely determined by the n-by-n matrix M, where
$$M_{jk} = B(e_j, e_k)$$
- Conversely, given any n-by-n matrix M, we can define a bilinear form B, where
$$B(e_j v^j, e_k w^k) = M_{jk} v^j w^k$$
- Two bilinear forms are equal if and only if their matrices (with respect to a basis) are equal

Symmetric Bilinear Forms

- A bilinear form B on a real vector space V is **symmetric** if for any $v_1, v_2 \in V$,
$$B(v_2, v_1) = B(v_1, v_2)$$
- Given a basis $(e_1, \ldots, e_n)$ of V, a bilinear form B is symmetric if and only if
$$B(e_j v^j, e_k w^k) = M_{jk} v^j w^k \text{ and } M_{kj} = M_{jk}$$
- An inner product on V is an example of a bilinear form

Quadratic Form on Real Vector Space

- A function $Q : V \to \mathbb{R}$ is a **quadratic form** if there exists a symmetric bilinear form $B : V \times V \to \mathbb{R}$ such that for each $v \in V$,
$$Q(v) = B(v, v)$$
- Equivalently, if $(e_1, \ldots, e_n)$ is a basis of V, then there exist coefficients $b_{ij} = b_{ji}$, $1 \leq i, j \leq n$, such that for any $v = e_k v^k$,
$$Q(v, v) = b_{ij} v^i v^j$$
- Examples:
  -
$$Q(e_1x^1 + e_2x^2 + e_3x^3) = (x^1)^2 + (x^2)^2 - (x^3)^2$$
$$Q(e_1x^1 + e_2x^2 + e_3x^3) = x^1x^2$$
- The right side is always a homogeneous polynomial of degree 2
  - Homogeneous means every term has same degree

Inner Product on Complex Vector Space

- Inner product on complex vector space V looks different from one on real vector space
- For any $v, v_1, v_2 \in V$ and $c \in \mathbb{C}$,

$$\begin{aligned}
(v_1 + v_2, v) &= (v_1, v) + (v_2, v) \\
(v, v_1 + v_2) &= (v, v_1) + (v, v_2) \\
(cv_1, v_2) &= c(v_1, v_2) \\
(v_1, cv_2) &= \bar{c}(v_1, v_2)
\end{aligned}$$

Sesquilinear Form on Complex Vector Space

- A sesquilinear form is a function

$$B : V \times V \to \mathbb{C}$$

with the following properties, similar to above:

$$\begin{aligned} B(v_1 + v_2, v) &= B(v_1, v) + B(v_2, v) \\ B(v, v_1 + v_2) &= B(v, v_1) + B(v, v_2) \\ B(cv_1, v_2) &= cB(v_1, v_2) \\ B(v_1, cv_2) &= \bar{c}B(v_1, v_2) \end{aligned}$$

Space of Linear Maps and Space of Sesquilinear Forms

- Let V be an inner product space
- Let $\mathcal{L}(V)$ be the space of all linear maps $L : V \to V$
- If $L_1, L_2 \in \mathcal{L}(V)$ and $c^1, c^2 \in \mathbb{F}$, then

$$c^1 L_1 + c^2 L_2 \in \mathcal{L}(V)$$

- Let $\mathcal{B}(V)$ be the space of all sesquilinear forms $B : V \times V \to \mathbb{F}$
- If $B_1, B_2 \in \mathcal{L}(V)$ and $c^1, c^2 \in \mathbb{F}$, then

$$c^1 B_1 + c^2 B_2 \in \mathcal{B}(V)$$

Isomorphism between Spaces of Linear Maps and of Sesquilinear Forms

- There is a linear map $\mathcal{L}(V) \to \mathcal{B}(V)$, where each $L \in \mathcal{L}(V)$ maps to $B \in \mathcal{V}$ such that for any $v, w \in V$,

$$B(v, w) = (L(v), w)$$

- If L lies in the kernel of this map, then for any $v, w \in V$, $B = 0$ and therefore

$$0 = B(v, w) = (L(v), w)$$

- This implies that $L(v) = 0$ for any $v \in V$, which implies $L = 0$
- Given $B \in \mathcal{B}(V)$ and $w \in V$,

Sesquilinear Form as Matrix

- If $(e_1, \dots, e_n)$ is a basis of V and $v = e_j v^j,\ w = e_k w^k$, then

$$\begin{aligned} B(v, w) &= B(e_j v^j, e_k w^k) \\ &= v^j \bar{w}^k B(e_j, e_k) \\ &= v^j \bar{w}^k M_{jk} \end{aligned}$$

- Therefore, given a basis $(e_1, \dots, e_n)$ of V, B is uniquely determined by the n-by-n matrix M, where

$$M_{jk} = B(e_j, e_k)$$

- Conversely, given any n-by-n matrix M, we can define a bilinear form B, where

$$B(e_j v^j, e_k w^k) = M_{jk} v^j w^k$$

- Two bilinear forms are equal if and only if their matrices (with respect to a basis) are equal
- Sometimes, we write

$$M_{j\bar{k}} = B(e_j, e_k)$$

to keep track of the fact that

Different Notation Conventions

- We are using the following convention:

$$B(cv, w) = cB(v, w)$$
$$B(v, cw) = \bar{c}B(v, w)$$

- Some use the following convention:

$$B(cv, w) = \bar{c}B(v, w)$$
$$B(v, cw) = cB(v, w)$$

- When reading a paper or book, look carefully to see which convention is used

Hermitian Forms

- A sequilinear form B on a complex vector space V is **hermitian** if for any $v_1, v_2 \in V$,

$$B(v_2, v_1) = \overline{B(v_1, v_2)}$$

- Given a basis $(e_1, \ldots, e_n)$ of V, B is hermitian if and only if its matrix $M_{jk} = B(e_j, e_k)$ satisfies

$$M_{kj} = \bar{M}_{jk}$$

- An inner product on V is an example of a hermitian form

Quadratic Form on Complex Vector Space

- A function $Q : V \to \mathbb{R}$ is a **quadratic form** if there exists a hermitian form $B : V \times V \to \mathbb{C}$ such that for each $v \in V$,

$$Q(v) = B(v, v)$$

- Equivalently, if $(e_1, \ldots, e_n)$ is a basis of V, then there is a hermitian matrix M such that for any $v = e_k v^k$,

$$Q(v, v) = M_{ij} v^i \overline{v}^j$$

Example

- If $Q(e_1 v^1 + e_2 v^2 + e_3 v^3) = |v^1|^2 + |v^2|^2 - |v^3|^2$, then $Q(v) = B(v, v)$, where

$$
\begin{aligned}
B(e_1, e_1) &= 1 \\
B(e_2, e_2) &= 1 \\
B(e_3, e_3) &= -1 \\
B(e_2, e_3) &= B(e_3, e_1) = B(e_1, e_2) = 0
\end{aligned}
$$

Example

- If $Q(e_1 v^1 + e_2 v^2 + e_3 v^3) = v^1 \bar{v}^2$, then $Q(v) = B(v, v)$, where

$$
\begin{aligned}
B(e_1, e_1) &= 0 \\
B(e_2, e_2) &= 0 \\
B(e_1, e_2) &= B(e_2, e_1) = \frac{1}{2} \\
B(e_2, e_3) &= B(e_3, e_1) = 0
\end{aligned}
$$

Example

- If $Q(e_1 v^1 + e_2 v^2 + e_3 v^3) = i v^1 \bar{v}^2$, then $Q(v) = B(v, v)$, where

$$
\begin{aligned}
B(e_1, e_1) &= 0 \\
B(e_2, e_2) &= 0 \\
B(e_1, e_2) &= \frac{i}{2} \\
B(e_2, e_1) &= -\frac{i}{2} \\
B(e_2, e_3) &= B(e_3, e_1) = 0
\end{aligned}
$$

Hermitian Form as $(1,1)$-Polynomial

- Observe that

$$Q(e_k x^k) = B(e_j x^j, e_k x^k) = x^j \bar{x}^k B(e_j, e_k) = M_{jk} x^j \bar{x}^k,$$

  - which is called a polynomial of degree $(1,1)$

Change of Basis Formula for Quadratic Form

- On a complex vector space V, let Q be a quadratic form. $E = (e_1, \ldots, e_n)$ be a basis of V, and M be the hermitian matrix such that

$$Q(e_k v^k) = v^j \bar{v}^k M_{jk}$$

- If $F = (f_1, \ldots, f_n)$ is another basis such that

$$f_k = e_j A^j_k,$$

then

$$\begin{aligned} Q(f_p w^p) &= Q(e_j A^j_p w^p) \\ &= B(e_j A^j_p w^p, e_k A^k_q w^q) \\ &= w^p A^j_p B(e_j, e_k) \bar{A}^k_q \bar{w}^q \\ &= w^p \bar{w}^q N_{pq}, \end{aligned}$$

where

$$N_{pq} = A^j_p M_{jk} \bar{A}^k_q, \text{ i.e., } N = AMA^*$$

Diagonalization of a Quadratic Form

- Recall that since M is a hermitian matrix, its eigenvalues are real and there exists a unitary matrix U such that
$$M = UDU^*,$$
where D is a diagonal matrix with the eigenvalues of M along its diagonal
- In particular, if
$$e_p = f_k U_p^k,$$
then
$$\begin{aligned} Q(f_j, f_k) &= Q(e_p U_j^p, e_q U_k^q) \\ &= U_j^p Q(e_p, e_q) U_k^q \\ &= U_j^p M_{pq} U_k^q \\ &= (U^* M U)_{jk} \\ &= D_{jk} \end{aligned}$$
- Observe that no inner product on V is used here