MATH-GA2120 Linear Algebra II

Self-Adjoint Maps and Matrices
Positive Definite Self-Adjoint Maps
Normal Form of Linear Map
Polar Decomposition

Deane Yang

Courant Institute of Mathematical Sciences
New York University

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Self-Adjoint Maps and Symmetric Matrices

- Given a Hermitian vector space V, a linear map $L : V \to V$ is **self-adjoint** if
$$L^* = L$$
- A complex matrix M is **Hermitian** if
$$M^* = M$$

Eigenvalues of a Self-Adjoint Map are Real

- Let $L : V \to V$ be a hermitian linear map with basis $(e_1, \cdots, e_n)$
- If v is an eigenvector of L with eigenvalue λ, then

$$\begin{aligned}
\lambda\|v\|^2 &= \langle L(v), v\rangle \\
&= \langle v, L(v)\rangle \\
&= \overline{\langle L(v), v\rangle} \\
&= \overline{\lambda\|v\|^2} \\
&= \overline{\lambda}\|v\|^2
\end{aligned}$$

Eigenspaces of a Self-Adjoint Map are Orthogonal

- Suppose λ, μ are two different eigenvalues of a self-adjoint operator $L : V \to V$ with eigenvectors v, w respectively
- It follows that

$$\begin{aligned} 0 &= \langle L(v), w \rangle - \langle v, L(w) \rangle \\ &= \langle \lambda v, w \rangle - \langle v, \mu w \rangle \\ &= (\lambda - \mu) \langle v, w \rangle \text{ since } \mu \in \mathbb{R} \end{aligned}$$

- Since $\lambda - \mu \neq 0$, it follows that $\langle v, w \rangle = 0$

Self-Adjoint Map Has Unitary Basis of Eigenvectors

- **Theorem.** Given a self-adjoint map $L : V \to V$, there exists a unitary basis of eigenvectors
- **Corollary.** Given a Hermitian matrix M, there exists a unitary matrix $U \in U(n)$ and real diagonal matrix D such that

$$M = U^* DU,$$

Proof of Theorem

- Given a linear map $L : V \to V$, by the Schur decomposition, there exists a unitary basis $(u_1, \ldots, u_n)$ such that

$$L(e_k) = e_k M^k_k + \cdots e_n M^n_k, \text{ for each } 1 \leq k \leq n$$

- Equivalently, for any $1 \leq k \leq n$ and $1 \leq j < k$,

$$(L(e_k), e_j) = 0$$

- If L is self-adjoint, then for any $1 \leq k \leq n$ and $1 \leq j < k$,

$$0 = (L(e_k), e_j) = (e_k, L^*(e_j)) = (e_k, L(e_j)) = \overline{(L(e_j), e_k)},$$

  which implies $(L(e_j), e_k) = 0$
- Therefore, for any $1 \leq k \leq n$ and $1 \leq j < k$,

$$(L(e_j), e_k) = 0$$

- It follows that for each $1 \leq k \leq n$,

$$L(e_k) = M^k_k e_k$$

Every Self-Adjoint Matrix is Diagonalizable by a Unitary Matrix

- Given a square matrix M, by the Schur decomposition, there exists a unitary matrix U such that

$$M = UTU^*,$$

where T is upper triangular

- If M is self-adjoint, then

$$UTU^* = M = M^* = (U^*)^* T^* U^* = UT^* U^*$$

- Therefore,

$$T = U^* UTU^* U = U^*(UTU^*)U = U^*(UT^* U^*)U = T^*,$$

which implies T is self-adjoint

- Since T is upper triangular,

$$T^j_k = 0, \text{ if } j < k \leq n$$

- Since $T^* = T$,

$$T^k_j = (T^*)^k_j = \bar{T}^j_k = 0, \text{ if } j < k \leq n$$

Positive Definite Self-Adjoint Maps

- A self-ajoint map $L : V \to V$ is **positive definite** if for any $v \neq 0$,
$$\langle L(v), v \rangle > 0$$
- If L is a positive definite self-adjoint map, we write $L > 0$
- $L > 0$ if and only if the eigenvalues of L are all positive
- We write $L \geq 0$ if the eigenvalues of L are all nonnegative

Powers and Roots of a Positive Definite Self-Adjoint Map

- Let L be a self-adjoint map such that $L \geq 0$ and $(u_1, \ldots, u_n)$ be a unitary basis of eigenvectors
- There is a unique self-adjoint map $\sqrt{L} \geq 0$ such that
$$\sqrt{L} \circ \sqrt{L} = L$$
  - Let
$$\sqrt{L}(u_k) = \sqrt{\lambda_k} u_k$$
- If k is a nonnegative integer and $L \geq 0$, then there is a unique self-adjoint map $L^{1/k}$ such that
$$(L^{1/k}) = L$$
  - Let
$$L^{1/k}(u_j) = \lambda_j^{1/k} u_j$$
- If $L > 0$ and $k \in \mathbb{Z}$, then there is a unique self-adjoint map L^{-k} such that
$$L^k L^{-k} = I$$

Singular Values of a Linear Map

- Let $L : X \to Y$ be *any* linear map (not necessarily self-adjoint)
- The map $L^*L : X \to X$ is self-adjoint, because for any $x_1, x_2 \in X$,

$$\langle L^*(L(x_1)), x_2 \rangle_X = \langle L(x_1), L(x_2) \rangle_Y = \langle x_1, L^*(L(x_2)) \rangle_X$$

- $L^*L \geq 0$, because for any $x \in X$,

$$\langle L^*L(x), x \rangle = \langle L(x), L(x) \rangle \geq 0$$

- We can denote $|L| = \sqrt{L^*L}$
- The eigenvalues of $|L|$ are called the **singular values** of L
- Singular values are always real and nonnegative
- Since $\ker L^*L = \ker L$, if $k = \dim \ker L$, then exactly k singular values are zero

Normal Form of a Linear Map (Part 1)

- Let X and Y be complex vector spaces such that $\dim X = m$ and $\dim Y = n$
- Let $L : X \to Y$ be a linear map with rank r
- If $\dim \ker L = k$, then, by Rank Theorem, $r + k = m$
- Let $(u_{r+1}, \ldots, u_m)$ be a unitary basis of $\ker L$
- This can be extended to a unitary basis of eigenvectors of $|L| = \sqrt{L^* L}$
- Therefore,
$$|L|(u_j) = s_j u_j, 1 \leq j \leq m,$$
where $s_1, \ldots, s_m$ are the singular values of L
- Observe that
$$\begin{aligned} s_1, \ldots, s_r &> 0 \\ s_{r+1} = \cdots = s_m &= 0 \end{aligned}$$

Normal Form of a Linear Map (Part 2)

- Let $\tilde{v}_j = L(u_j)$, $1 \le j \le r$
- The set $\{\tilde{v}_1, \ldots, \tilde{v}_r\}$ is linearly independent because if

$$a^1 \tilde{v}_1 + \cdots + a^k \tilde{v}_r = 0,$$

then

$$L(a^1 u_1 + \cdots + a^k u_r) = a^1 \tilde{v}_1 + \cdots + a^k \tilde{v}_r = 0$$

which implies that $a^1 u_1 + \cdots + a^k u_r \in \ker L$ and therefore

$$a^1 u_1 + \cdots + a^k u_r = b^{r+1} u_{r+1} + \cdots + b^m u_m,$$

which implies that $a^1 = \cdots = a^k = b^{r+1} = \cdots = b^m = 0$

- Moreover, if $1 \le i, j \le k$, then $s_j \ne 0$ and therefore

$$\langle \tilde{v}_i, \tilde{v}_j \rangle = \langle L(u_i), L(u_j) \rangle = \langle u_i, (L^* L)(u_j) \rangle = s_j^2 \langle u_i, u_j \rangle = s_j^2 \delta_{ij}$$

Normal Form for a Linear Map (Part 3)

- If
$$v_j = s_j^{-1}\tilde{v}_j = s_j^{-1}L(u_j),\ 1 \leq j \leq k,$$
then $(v_1, \ldots, v_r)$ is a unitary basis of $L(X) \subset Y$ and therefore a unitary set in Y
- This can be extend, using Gram-Schmidt, to a unitary basis $(v_1, \ldots, v_n)$ of Y
- Therefore, there is a unitary basis $(u_1, \cdots, u_m)$ of X and a unitary basis $(v_1, \ldots, v_n)$ of Y such that

$$L(u_j) = \begin{cases} s_j v_j & \text{if } 1 \leq j \leq \dim\ker L \\ 0 & \text{if } j > \dim\ker L \end{cases}$$

Normal Form for a Linear Map (Part 4)

- $$\begin{aligned} &\left[\begin{array}{ccc|ccc} L(u_1) & \cdots & L(u_r) & L(u_{r+1}) & \cdots & L(u_m) \end{array}\right] \\ &= \left[\begin{array}{ccc|ccc} v_1 & \cdots & v_r & v_{r+1} & \cdots v_n \end{array}\right] \left[\begin{array}{c|c} D & 0_{r\times(m-r)} \\ \hline 0_{(n-r)\times r} & 0_{(n-r)\times(m-r)} \end{array}\right], \end{aligned}$$

  where D is the r-by-r diagonal matrix such that

  $$D^i_j = \begin{cases} s_j & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

- $$\begin{aligned} &L(a^1 u_1 + \cdots + a^m u_m) \\ &\qquad = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \left[\begin{array}{c|c} D & 0_{r\times(m-r)} \\ \hline 0_{(n-r)\times r} & 0_{(n-r)\times(m-r)} \end{array}\right] \begin{bmatrix} a^1 \\ \vdots \\ a^m \end{bmatrix} \end{aligned}$$

Normal Form for a Rectangular Complex Matrix

- If M is an n-by-m complex matrix with rank r and whose positive singular values are $s_1, \dots, s_r$, then there are unitary matrices $P \in U(m)$ and $Q \in U(n)$ such that

$$M = QSP,$$

where

$$S = \left[\begin{array}{c|c} D & 0_{r\times(m-r)} \\ \hline 0_{(n-r)\times r} & 0_{(n-r)\times(m-r)} \end{array}\right]$$

and D is the r-by-r diagonal matrix such that

$$D^i_j = \begin{cases} s_j & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Normal Form for a Rectangular Real Matrix

- If M is an n-by-m real matrix with rank r and whose positive singular values are $s_1, \ldots, s_r$, then there are orthogonal matrices $P \in O(m)$ and $Q \in O(n)$ such that

$$M = QSP,$$

where

$$S = \left[\begin{array}{c|c} D & 0_{r\times(m-r)} \\ \hline 0_{(n-r)\times r} & 0_{(n-r)\times(m-r)} \end{array}\right]$$

and D is the r-by-r diagonal matrix such that

$$D^i_j = \begin{cases} s_j & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$