MATH-GA2120 Linear Algebra II

Unitary Basis
Adjoint of Linear Map
Unitary Maps and Matrices
Schur Representation

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Hermitian Inner Product on Complex Vector Space

- If V is a complex vector space then a **Hermitian inner product** on V is a function of two vectors v_1, v_2, written

$$\langle v_1, v_2 \rangle \in \mathbb{C}$$

that satisfies the following properties

$$\begin{aligned}
\langle a^1 v_1 + a^2 v_2, w \rangle &= a^1 \langle v_1, w \rangle + a^2 \langle v_2, w \rangle \\
\langle v, b^1 w_1 + b^2 w_2 \rangle &= \bar{b}^1 \langle v, w_1 \rangle + \bar{b}^2 \langle v, w_2 \rangle \\
\langle v, w \rangle &= \overline{\langle w, v \rangle} \\
\langle v, v \rangle &> 0 \text{ if } v \neq 0
\end{aligned}$$

- A complex vector space with a Hermitian inner product is called a *Hermitian vector space*

Hermitian Inner Product With Respect To Basis

- Let V be a complex vector space and let $(b_1, \ldots, b_n)$ be a basis of V
- Any inner product on V is uniquely determined by the matrix A, where
$$A_{ij} = \langle b_i, b_j \rangle$$
- The matrix A satisfies the following properties
  - Hermitian:
$$A_{ij} = \langle b_i, b_j \rangle = \overline{\langle b_j, b_i \rangle} = \bar{A}_{ji}$$
(In particular, since $A_{ii} = \bar{A}_{ii}$, it follows that $A_{ii} \in \mathbb{R}$)
  - Positive definite: For any nonzero $v = a^k b_k = Ba \in V$,
$$0 < \langle v, v \rangle = \langle a^j b_j, a^k b_k \rangle = a^j \bar{a}^k \langle b_j, b_k \rangle = a^T A \bar{a}$$
- Conversely, given the basis $(b_1, \ldots, b_n)$ of V, any positive definite Hermitian matrix A defines an inner product where
$$\langle b_i, b_j \rangle = A_{ij}$$

Standard Hermitian Inner Product on $\mathbb{C}^n$

- Let $(e_1, \ldots, e_n)$ be the standard basis of $\mathbb{C}^n$
- Define the standard hermitian inner product of $v = v^i e_i, w = w^i e_i$ to be

$$\langle v, w \rangle = v \cdot \bar{w} = v^1 \bar{w}^1 + \cdots + v^n \bar{w}^n$$

Orthogonality and Orthogonal Projection

- Two vectors $v, w \in V$ are **orthogonal** if

$$\langle v, w \rangle = 0.$$

- If v is a unit vector and w is any vector, then

$$\begin{aligned} \langle w - \langle w, v \rangle v, v \rangle &= \langle w, v \rangle - \langle \langle w, v \rangle v, v \rangle \\ &= \langle w, v \rangle - \langle w, v \rangle \|v\|^2 \\ &= 0 \end{aligned}$$

- But order matters

$$\begin{aligned} \langle w - \langle v, w \rangle v, v \rangle &= \langle w, v \rangle - \langle \langle v, w \rangle v, v \rangle \\ &= \langle w, v \rangle - \langle v, w \rangle \|v\|^2 \\ &= \langle w, v \rangle - \overline{\langle v, w \rangle} \end{aligned}$$

Unitary Set

- A set $(e_1, \ldots, e_k)$ is called **unitary** if

$$\langle e_i, e_j \rangle = \delta_{ij}, 1 \leq i, j \leq k$$

- A unitary set is linearly independent
  - If $a^1 e_1 + \cdots + a^k e_k = 0$, then for each $1 \leq j \leq k$,

$$a^j = \langle a^1 e_1 + \cdots + a^k e_k, e_j \rangle = 0$$

- If $\dim V = n$, then a unitary set with n elements is a **unitary basis**

Gram-Schmidt

- **Lemma.** *Any (possibly empty) unitary set can be extended to a unitary basis*
- Suppose $S = \{e_1, \ldots, e_k\}$ is a unitary set, where $k < \dim V$
- The span of S is not all of V and therefore there is a nonzero vector $v \in V$ such that $v \notin S$
- Let $\hat{v} = v - \langle v, e_1 \rangle e_1 - \cdots - \langle v, e_k \rangle e_k$
- $\hat{v} \neq 0$, because $v \notin$ the span of S
- $\hat{v}$ is orthogonal to S, because for each $1 \leq j \leq k$,

  $$\langle \hat{v}, e_j \rangle = \langle v - \langle v, e_1 \rangle e_1 - \cdots - \langle v, e_k \rangle e_k, e_j \rangle = \langle v, e_j \rangle - \langle v, e_j \rangle = 0$$

- If

  $$e_{k+1} = \frac{\hat{v}}{\|\hat{v}\|},$$

  then $\|e_{k+1}\| = 1$ and $\langle e_{k+1}, e_j \rangle = 0$ for each $1 \leq j \leq k$
- Therefore, $\{e_1, \ldots, e_{k+1}\}$ is a unitary set

Adjoint of a Linear Map

- Let X and Y be Hermitian vector spaces (i.e., complex vector spaces with Hermitian inner products)
- Let $L : X \to Y$ be a linear map
- The **adjoint** of L is the operator $L^* : Y \to X$ such that for any $x \in X$ and $y \in Y$,

$$\langle L(x), y\rangle = \langle x, L^*(y)\rangle$$

  and therefore

$$\langle y, L(x)\rangle = \overline{\langle L(x), y\rangle} = \overline{\langle x, L^*(y)\rangle} = \overline{\langle L^*(x), y\rangle}$$

- Observe that if $L^{**} = (L^*)^* : X \to Y$, then for every $x \in X$ and $y \in Y$,

$$\langle y, L^{**}(x)\rangle = \langle L^*(y), x\rangle = \langle y, L(x)\rangle$$

  and therefore $L^{**} = L$

Adjoint Map With Respect to Basis

- Let $(e_1, \ldots, e_m)$ be a unitary basis of X and $(f_1, \ldots, f_n)$ be a unitary basis of Y
- Let M and M^* be the matrices such that for every $1 \leq k \leq m$,

$$L(e_k) = M_k^1 f_1 + \cdots + M_k^n f_n$$

and for every $1 \leq a \leq n$,

$$L^*(f_a) = (M^*)_a^1 e_1 + \cdots + (M^*)_a^m e_m$$

- It follows that

$$(M^*)_a^k = \langle L^*(f_a), e_k \rangle = \overline{\langle f_a, L(e_k) \rangle} = \overline{M}_k^a$$

- In other words, $M^* = \overline{M}^T$
- Given a complex matrix $M \in \mathcal{M}_{n \times m}$, we define the **adjoint matrix** of M to be

$$M^* = \overline{M}^T$$

Examples of Adjoint Matrices

- $$\begin{bmatrix} 1 & -i & 1+i \\ 1 & i & 1-i \end{bmatrix}^* = \begin{bmatrix} 1 & 1 \\ i & -i \\ 1-i & 1+i \end{bmatrix}$$

- $$\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}^* = \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$$

- Self-adjoint matrix

$$\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}^* = \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$$

Unitary Maps

- If V is a Hermitian vector space, a linear map $L : V \to V$ is **unitary**, if for any $v, w \in V$, if any of the following equivalent statements hold:

$$\langle L(v), L(w)\rangle = \langle v, w\rangle$$
$$\langle L^* L(v), w\rangle = \langle v, w\rangle$$
$$L^* \circ L = I$$
$$L \text{ is invertible and } L^{-1} = L^*$$

- It also follows that $L \circ L^* = I$

Unitary Matrices

- Let $L : V \to V$ be a unitary map
- If $(u_1, \ldots, u_n)$ is a unitary basis of V and $L(u_k) = M^j_k u_j$, then

$$\begin{aligned}\delta_{jk} &= \langle u_j, u_k \rangle \\ &= \langle L(u_j), L(u_k) \rangle \\ &= \langle u_j, (L^* \circ L)(u_k) \rangle \\ &= \langle u_j, (M^* M)^i_k u_i \rangle \\ &= (M^* M)^j_k\end{aligned}$$

- $$M^* M = I$$

- A matrix M is **unitary** if $M^* M = M M^* = I$

Properties of unitary maps and matrices

- If L_1, L_2 are unitary maps, then so is $L_1 \circ L_2$
  - If M_1, M_2 are unitary matrices, then so is $M_1 M_2$
- If L is unitary, then L is invertible and $L^{-1} = L^*$ is unitary
  - If M is unitary, then M is invertible and $M^{-1} = M^*$ is unitary
- The identity map is unitary
  - The identity matrix is unitary

Unitary Group

- Define the unitary group $U(V)$ of a Hermitian vector space V to be the set of all unitary transformations
- Denote
$$U(n) = U(\mathbb{C}^n)$$
using the standard Hermitian inner product on $\mathbb{C}^n$
- Both satisfy the properties of an abstract group G
  - Any ordered pair $(g_1, g_2) \in G \times G$ uniquely determine a third, denoted $g_1 g_2 \in G$
  - (Associativity) $(g_1 g_2) g_3 = g_1 (g_2 g_3)$
  - (Identity element) There exists an element $e \in G$ such that $ge = eg = g$ for any $g \in G$
  - (Inverse of an element) For each $g \in G$, there exists an element $g^{-1} \in G$ such that $gg^{-1} = g^{-1} g = e$
- $U(n)$ is an example of a matrix group
- Both $U(V)$ and $U(n)$ are examples of Lie groups

Schur Representation of a Real Linear Map

- Let V be a finite dimensional real inner product space
- **Theorem:** Given any linear map $L : V \to V$ with only real eigenvalues, there exists an **orthonormal** basis $(u_1, \dots, u_n)$ of V such that for each $1 \leq k \leq n$, $L(u_k)$ is a linear combination of $u_1, \dots, u_k$,

$$L(u_k) = M_k^k u_k + \cdots + M_k^n u_n$$

- **Corollary:** Given any real matrix M with only real eigenvalues, there is an orthogonal matrix O such that the matrix $O^t MO$ is triangular

Schur Representation of a Complex Linear Map

- Let V be a finite dimensional Hermitian vector space
- **Theorem:** Given any linear map $L : V \to V$, there exists a **unitary** basis $(u_1, \ldots, u_n)$ of V such that for each $1 \leq k \leq n$, $L(u_k)$ is a linear combination of $u_1, \ldots, u_k$,

$$L(u_k) = M_k^k u_k + \cdots + M_k^n u_n$$

- **Corollary:** Given any complex matrix M, there is a unitary matrix O such that the matrix $O^t M O$ is triangular

Proof (Part 1)

- Proof by induction
- Theorem holds when $\dim V = 1$
- Suppose theorem holds when $\dim V = n - 1$
- Consider a linear map $L : V \to V$, where $\dim V = n$ with eigenvalues $\lambda_1, \ldots, \lambda_n$
- Let u_n be a unit eigenvector for the eigenvalue λ_n, i.e.,

$$\|u_n\| = 1 \text{ and } L(u_n) = \lambda_n u_n$$

- Let

$$u_n^\perp = \{v \in V \ : \ \langle v, u_n \rangle = 0\}$$

- Recall that the orthogonal projection map onto $u_n^\perp$ is given by

$$\begin{aligned} \pi^\perp : V &\to u_n^\perp \\ v &\to v - \langle v, u_n \rangle u_n \end{aligned}$$

Proof (Part 2)

- If $(v_1, \ldots, v_{n-1})$ is a basis of $u_n^\perp$, then $(v_1, \ldots, v_{n-1}, u_n)$ is a basis of V
- Let M be the matrix such that

$$\begin{aligned} L(v_k) &= M_k^1 v_1 + \cdots + M_k^{n-1} v_{n-1} + M_k^n u_n \\ L(u_n) &= M_n^1 v_1 + \cdots + M_n^{n-1} v_{n-1} + M_n^n u_n \end{aligned}$$

- Since $L(u_n) = \lambda_n u_n$,

$$M_n^1 = \cdots = M_n^{n-1} = 0 \text{ and } M_n^n = \lambda_n$$

- Let $L^\perp : u_n^\perp \to u_n^\perp$ be the linear map given by

$$L^\perp(v_k) = M_k^1 v_1 + \cdots + M_k^{n-1} v_{n-1},\ 1 \leq k \leq n-1$$

- Since $\dim u_n^\perp = n - 1$, there is a basis $(u_1, \ldots, u_{n-1})$ such that

$$L^\perp(u_k) = M_k^k u_k + \cdots + M_k^{n-1} u_{n-1},\ 1 \leq k \leq n-1$$

Proof (Part 3)

- Since

$$L^{\perp}(u_k) = M_k^k u_k + \cdots + M_k u^{n-1} u_{n-1},\ 1 \leq k \leq n-1,$$

it follows that

$$L(u_k) = M_k^k u_k + \cdots + M_k u^{n-1} u_{n-1} + M_k^n u_n,\ 1 \leq k \leq n-1$$

- Also,

$$L(u_n) = \lambda_n u_n$$

- Therefore,

$$L(u_k) = M_k^k u_k + \cdots + M_k u^{n-1} u_{n-1} + M_k^n u_n,\ 1 \leq k \leq n,$$

where $M_n^n = \lambda_n$