# MATH-GA2120 Linear Algebra II

## Determinant of a Composition
## Eigenvalues, Eigenvectors
## Characteristic Polynomial
## Triangular Matrices

Deane Yang

Courant Institute of Mathematical Sciences
New York University

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# Determinant of a Composition

- If $L_1, L_2$ are linear maps from $V$ to $V$, then

$$\begin{aligned}
\det(L_2 \circ L_1)D(v_1, \dots, v_n) &= (L_2 \circ L_1)^*(D(v_1, \dots, v_n)) \\
&= D((L_2 \circ L_1)(v_1)), \dots, (L_2 \circ L_1)(v_n))) \\
&= D(L_2(L_1(v_1)), \dots, L_2(L_1(v_n))) \\
&= \det(L_2)D(L_1(v_1), \dots, L_1(v_n)) \\
&= \det(L_2)\det(L_1)
\end{aligned}$$

- Since all three definitions of $\det(L)$ are equivalent, this is another proof that

$$\det(M_2 M_1) = \det(M_2)\det(M_1)$$

# Eigenvalues and Eigenvectors of a Linear Transformation

- Consider a linear map $L : V \to V$
- A scalar $\lambda \in \mathbb{F}$ is called an **eigenvalue** of $L$ if there is a nonzero vector $v \in V$ such that any of the following equivalent statements hold

$$L(v) = \lambda v \iff (L - \lambda I)v = 0 \iff v \in \ker(L - \lambda I)$$

  The vector $v$ is called an **eigenvector**
- $\lambda$ is an eigenvalue of $L \iff$ the linear maps $L - \lambda I$ is singular
- Any nonzero $v \in \ker(L - \lambda I)$ is an eigenvector for the eigenvalue $\lambda$
- Any nonzero $v \in \ker L$ is an eigenvector for the eigenvalue 0
- The eigenspace for an eigenvalue $\lambda$ of $L$ is the subspace

$$E_\lambda(L) = \ker(L - \lambda I) = \{v \in V \ : \ L(v) = \lambda v\}$$

  A nonzero vector $v$ is an eigenvector for the eigenvalue $\lambda$ if and only if $v \in E_\lambda(L)$

# Eigenvalues and Eigenvectors of a Square Matrix

- A scalar $\lambda$ is an **eigenvalue** of a matrix $M \in M_{n \times n}$ if there is a nonzero vector $v \in \mathbb{F}^n$ such that

$$Mv = \lambda v$$

  The vector $v$ is called an **eigenvector** for the eigenvalue $\lambda$
- If $v$ is an eigenvector for $\lambda$, then so is any nonzero scalar multiple of it
- $\lambda$ is an eigenvalue of $M$ if and only if the matrix $M - \lambda I$ is singular
- Therefore,

$$\lambda \text{ is an eigenvalue of } M \iff \det(M - \lambda I) = 0$$

- $\ker(M - \lambda I) \backslash \{0\}$ is the set of all eigenvectors for the eigenvalue $\lambda$

# Determinants, Eigenvalues, and Eigenvectors of Similar Matrices

- Two matrices $M$ and $N$ are called **similar** if there is an invertible matrix $H$ such that

$$M = HNH^{-1}$$

  or, equivalently, there is an invertible matrix $G$ such that

$$N = GMG^{-1}$$

- If $M$ and $N$ are similar, then $\det M = \det N$
- $M$ and $N$ have the same eigenvalues, because

$$Mv = \lambda v \iff HNH^{-1}v = \lambda v \iff N(H^{-1}v) = \lambda(H^{-1}v)$$

# Characteristic Polynomial of a Matrix

- Let $\delta^j_k$ be the element in the $j$-th row and $k$-column of the identity matrix, i.e.,

$$\delta^j_k = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

- Observe that the function $p_M : \mathbb{F} \to \mathbb{F}$ given by

$$\begin{aligned} p_M(\lambda) &= \det(M - \lambda I) \\ &= \sum_{\sigma \in S_n} \epsilon(\sigma)(M - \lambda I)^{\sigma(1)}_1 \cdots (M - \lambda I)^{\sigma(n)}_n \\ &= \sum_{\sigma \in S_n} \epsilon(\sigma)(M^{\sigma(1)}_1 - \lambda\delta^{\sigma(1)}_1) \cdots (M^{\sigma(n)}_n - \lambda\delta^{\sigma(n)}_n) \end{aligned}$$

  is a polynomial in $\lambda$ of degree $n$
- The roots of $p_M$ are eigenvalues of $M$

# Similar Matrices Have the Same Characteristic Polynomial

- If $M = HNH^{-1}$, then

$$M - \lambda I = HNH^{-1} - \lambda H(I)H^{-1} = H(N - \lambda I)H^{-1}$$

- Therefore,

$$\begin{aligned} p_M(\lambda) &= \det(M - \lambda I) \\ &= (\det H)(\det(N - \lambda I))(\det H^{-1}) \\ &= \det(N - \lambda I) \\ &= p_N(\lambda) \end{aligned}$$

- This is another proof that the eigenvalues of $M$ are the same as the eigenvalues of $N$

# Eigenvalues of Linear Transformation and its Matrix

- Consider a linear transformation $L : V \to V$ on an $n$-dimensional vector space $V$
- Given a basis $E = (e_1, \ldots, e_n)$ of $V$, there is a matrix $M \in M_{n \times n}$ such that for any $v = Ea$,

$$L(v) = L(Ea) = E(Ma)$$

- Observe that for any $v \in V$,

$$\begin{aligned} L(v) = \lambda v &\iff L(Ea) = \lambda Ea \\ &\iff E(Ma) = E(\lambda a) \qquad \iff Ma = \lambda a \end{aligned}$$

- Therefore, $\lambda$ is an eigenvalue of $L$ iff it is an eigenvalue of $M$

# Change of Basis formula for a Linear Transformation

- If $F$ is another basis of $V$, then there is an invertible square matrix $H$ such that $F = EH$
- If $v = Ea = Fb$, then

$$v = Fb = EHb,$$

  which implies that $a = Hb$ and $b = H^{-1}a$
- If $L : V \to V$ is a linear transformation, then there are matrices $M$ and $N$ such that

$$L(Ea) = E(Ma) \text{ and } L(Fb) = F(Nb) = EHNH^{-1}a,$$

  which implies that $M = HNH^{-1}$ and $N = H^{-1}MH$
- In other words, $M$ and $N$ are similar and therefore have the same characteristic polynomial

# Characteristic Polynomial of a Linear Transformation

- It follows that

$$p_M(\lambda) = \det(M - \lambda I) = \det(N - \lambda I) = p_N(\lambda)$$

- We can therefore define the characteristic polynomial $p_L$ of a linear transformation $L : V \to V$ to be

$$p_L = p_M,$$

  where $p_M$ is the characteristic polynomial of the matrix $M$ associated to $L$ and a basis of $V$

- The polynomial is the same, no matter what basis of $V$ is used

# Examples

- Let
$$Z = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$
- $Zv = 0v$ for any $v \in \mathbb{R}^2$ and therefore 0 is the only eigenvalue
- Any nonzero vector $v \in \mathbb{R}^2$ is an eigenvector
- The characteristic polynomial is
$$p_Z(x) = \det(Z - \lambda I) = \det\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = -\lambda^2$$

# Examples

- If $D = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, then $D \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} av^1 \\ bv^2 \end{bmatrix}$
- If $\lambda = a = b$, then the only eigenvalue is $\lambda$
  - Every $v \in \mathbb{R}^2$ is an eigenvector
- If $a \neq b$, then the only eigenvalues are $a$ and $b$
  - The eigenvectors for the eigenvalue $a$ are

$$\begin{bmatrix} x \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; x \in \mathbb{F} \backslash \{0\}$$

  - The eigenvectors for the eigenvalue $b$ are

$$\begin{bmatrix} 0 \\ x \end{bmatrix} = x \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; x \in \mathbb{F} \backslash \{0\}$$

- The characteristic polynomial is

$$p_D(x) = \det(D - \lambda I) = \lambda \begin{bmatrix} a - \lambda & 0 \\ 0 & b - \lambda \end{bmatrix} = (a - \lambda)(b - \lambda)$$

# Examples

- If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $A \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} v^2 \\ v^1 \end{bmatrix}$
- The only eigenvalues are $1, -1$
- The eigenvectors for the eigenvalue 1 are

$$\begin{bmatrix} x \\ x \end{bmatrix}, \ x \in \mathbb{F} \backslash \{0\}$$

- The eigenvectors for the eigenvalue $-1$ are

$$\begin{bmatrix} x \\ -x \end{bmatrix}, \ x \in \mathbb{F} \backslash \{0\}$$

- The characteristic polynomial is

$$p_A(x) = \det \left( \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} \right) = \det \left( \begin{bmatrix} -\lambda & 1 \\ 1 & \lambda \end{bmatrix} \right) = 1 - \lambda^2$$

# Examples

- If $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, then $B \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} -v^2 \\ v^1 \end{bmatrix}$
- There are no real eigenvalues
- The complex eigenvalues are $i, -i$
- The eigenvectors for the eigenvalue $i$ are

$$\begin{bmatrix} ix \\ -x \end{bmatrix} = x \begin{bmatrix} i \\ 1 \end{bmatrix}, \; x \in \mathbb{F} \backslash \{0\}$$

- The eigenvectors for the eigenvalue $-i$ are

$$\begin{bmatrix} x \\ ix \end{bmatrix} = x \begin{bmatrix} 1 \\ i \end{bmatrix}, \; x \in \mathbb{F} \backslash \{0\}$$

- The characteristic polynomial is

$$\begin{aligned} p_B(x) &= \det(B - \lambda I) \\ &= \det \left( \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} \right) \\ &= 1 + \lambda^2 \end{aligned}$$

# Complex Versus Real Eigenvalues

- If an $n - by - n$ matrix contains only real entries, it can have anywhere from 0 to $n$ eigenvalues
- A polynomial with complex coefficients

$$p(x) = a_0 + a_1 x + \cdots a_n x^n,$$

where $a_n \neq 0$ with complex coefficients can always be factored into $n$ linear factors

$$p(x) = a_n(r_1 - x) \cdots (r_n - x)$$

- A complex matrix $A$ always has anywhere from 1 to $n$ eigenvalues, where an eigenvalue might appear more than once in the factorization of $p_A$
- The **multiplicity** of an eigenvalue $\lambda$ is the number of linear factors equal to $(\lambda - x)$ in $p_A$

# Examples

- Let $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$
- The eigenvalues of $D$ are $-2, 3$
- The characteristic polynomial of $D$ is

$$p_D(\lambda) = (x-3)(x+2)(x-3) = (x-3)^2(x+2)$$

- The eigenvalue 3 has multiplicity 2, and the eigenvalue 2 has multiplicity 1
- The eigenvectors for the eigenvalue $-2$ are

$$\begin{bmatrix} 0 \\ x \\ 0 \end{bmatrix} = x \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ x \in \mathbb{F} \backslash \{0\}$$

- The eigenvectors for the eigenvalue 3 are

$$\begin{bmatrix} x^1 \\ 0 \\ x^2 \end{bmatrix} = x^1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x^2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ x \in \mathbb{F} \backslash \{0\}$$

# Examples

- Let $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
- The characteristic polynomial of $M$ is

$$p_M(\lambda) = \det(M - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}\right) = (1-\lambda)^2$$

- The only eigenvalue is 1 with multiplicity 2
- Since

$$M \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} v^1 \\ v^1 + v^2 \end{bmatrix},$$

the eigenvectors of the eigenvalue 1 are

$$\begin{bmatrix} 0 \\ x \end{bmatrix} = x \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

# Diagonal Matrices

- An $n$-by-$n$ matrix $M$ is **diagonal** if
$$M_k^j = 0 \text{ if } j \neq k$$
- In particular, the $k$-th column of $M$ is
$$C_k = Me_k = M_k^k e_k \text{ (no sum over } k\text{)},$$
where $(e_1, \ldots, e_n)$ is the standard basis of $\mathbb{R}^n$
- The determinant of $M$ is, by multilinearity,
$$\begin{aligned} D(C_1, \ldots, C_n) &= D(M_1^1 e_1, M_2^2 e_2, \ldots, M_n^n e_n) \\ &= D(e_1, \ldots, e_n) \\ &= (M_1^1 \cdots M_n^n) \end{aligned}$$
- Since $M - \lambda I$ is also diagonal, it follows that the characteristic polynomial of $M$ is
$$p_M(\lambda) = \det(M - \lambda I) = (M_1^1 - \lambda) \cdots (M_n^n - \lambda)$$
- The diagonal elements of $M$ are its eigenvalues

# Triangular Matrices

- An $n$-by-$n$ matrix $M$ is **upper triangular** if it is of the form

$$M = \begin{bmatrix} M_1^1 & M_2^1 & \cdots & M_{n-1}^1 & M_n^1 \\ 0 & M_2^2 & \cdots & M_{n-1}^2 & M_n^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & M_{n-1}^{n-1} & M_n^{n-1} \\ 0 & 0 & \cdots & 0 & M_n^n \end{bmatrix}$$

- I.e., $M_k^j = 0$ if $j > k$
- An $n$-by-$n$ matrix $M$ is **lower triangular** if it is of the form

$$M = \begin{bmatrix} M_1^1 & 0 & \cdots & 0 & 0 \\ M_1^2 & M_2^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ M_1^{n-1} & M_2^{n-1} & \cdots & M_{n-1}^{n-1} & 0 \\ M_1^n & M_2^n & \cdots & M_{n-1}^n & M_n^n \end{bmatrix}$$

- I.e., $M_k^j = 0$ if $j < k$

# Columns of an Upper Triangular Matrix

- Let $M$ be an upper triangular matrix and consider the matrix $T = M - \lambda I$
- $T$ is itself an upper triangular matrix
- Choose a value of $\lambda \in \mathbb{F}$ such that every element on the diagonal of $T$ is nonzero
- Let $(e_1, \ldots, e_n)$ be the standard basis of $\mathbb{R}^n$
- Let $(C_1, \ldots, C_n)$ be the columns of $T$
- By assumption, $C_1^1, C_2^2, \cdots, C_n^n$ are all nonzero

# Columns of Upper Triangular Matrix (Part 2)

- Each column can therefore be written as

$$C_k = C_k^k \hat{C}_k,$$

where

$$\hat{C}_k = \begin{bmatrix} \hat{C}_k^1 \\ \vdots \\ \hat{C}_k^{k-1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ and } \hat{C}_k^j = \frac{C_k^j}{C_k^k}, \text{ for each } 1 \leq j, k \leq n$$

# Determinant of Upper Triangular Matrix (Part 1)

- Let $(C_1, \ldots, C_n)$ be the columns of $T$ and recall that the determinant of $T$ is

$$\det(T) = D(C_1, \ldots, C_n)$$

  where $D \in \Lambda^n V^*$ satisfies $D(e_1, \ldots, e_n) = 1$
- By the multilinearity of $D$,

$$\begin{aligned} D(C_1, \ldots, C_n) &= D(C_1^1 \hat{C}_1, C_2^2 \hat{C}_2, \ldots, C_n^n \hat{C}_n) \\ &= (C_1^1 C_2^2 \cdots C_n^n) D(\hat{C}_1, \ldots, \hat{C}_n) \end{aligned}$$

# Determinant of Upper Triangular Matrix (Part 2)

- Since $T$ is lower triangular, its columns are of the form

$$\begin{aligned} C_1 &= C_1^1 e_1 \\ C_2 &= C_2^1 e_1 + C_2^2 e_2 \\ C_3 &= C_3^1 e_1 + C_3^2 e_2 + C_3^3 e_3 \\ \vdots \ &\ \ \vdots \\ C_n &= C_n^1 e_1 + C_n^2 e_2 + C_n^3 e_3 + \cdots + C_n^n e_n \end{aligned}$$

- Similarly,

$$\begin{aligned} \hat{C}_1 &= e_1 \\ \hat{C}_2 &= \hat{C}_2^1 e_1 + e_2 \\ \hat{C}_3 &= \hat{C}_3^1 e_1 + \hat{C}_3^2 e_2 + e_3 \\ \vdots \ &\ \ \vdots \\ \hat{C}_n &= \hat{C}_n^1 e_1 + \hat{C}_n^2 e_2 + \hat{C}_n^3 e_3 + \cdots + \hat{C}_n^{n-1} e_{n-1} + e_n \end{aligned}$$

# Determinant of Upper Triangular Matrix (Part 3)

- Therefore,

$$
\begin{aligned}
& D(\hat{C}_1, \ldots, \hat{C}_n) \\
& = D(e_1, \hat{C}_2, \ldots, \hat{C}_n) \\
& = D(e_1, \hat{C}_2^1 e_1 + e_2, \hat{C}_3^1 e_1 + \hat{C}_3^2 e_2 + e_3, \ldots, \hat{C}^1 e_1 + \cdots + e_n) \\
& = D(e_1, e_2, \hat{C}_3^2 e_2 + e_3, \ldots, \hat{C}_n^2 e_2 + \cdots + e_n) \\
& = D(e_1, e_2, e_3, \ldots, \hat{C}_n^3 e_3 + \cdots + \cdots + e_n) \\
& \vdots \qquad \vdots \\
& = D(e_1, e_2, \ldots, e_n) \\
& = 1
\end{aligned}
$$

# Characteristic Polynomial and Determinant of $M$

- It follows that if $\lambda$ is not equal to any of $C_1^1, \cdots, C_n^n$,

$$\begin{aligned} p_M(\lambda) &= \det(T) \\ &= D(C_1, \ldots, C_n) \\ &= C_1^1 C_2^2 \cdots C_n^n D(\hat{C}_1, \ldots, \hat{C}_n) \\ &= C_1^1 C_2^2 \cdots C_n^n \\ &= (M_1^1 - \lambda I) \cdots (M_n^n - \lambda I) \end{aligned}$$

- If follows that the polynomial

$$r(\lambda) = p_M(\lambda) - (M_1^1 - \lambda I) \cdots (M_n^n - \lambda I)$$

has infinitely many roots

- This implies that $r$ is the zero polynomial
- Therefore, the characteristic polynomial of an upper triangular matrix $M$ is

$$p_M(\lambda) = (M_1^1 - \lambda I) \cdots (M_n^n - \lambda I)$$

- In particular, $\det(M) = p_M(0) = M_1^1 \cdots M_n^n$