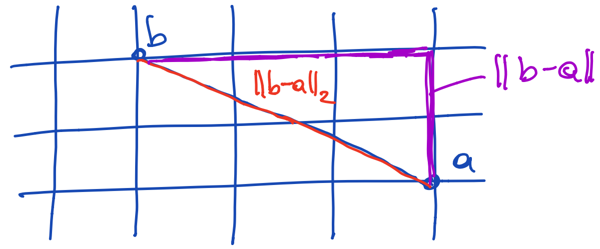
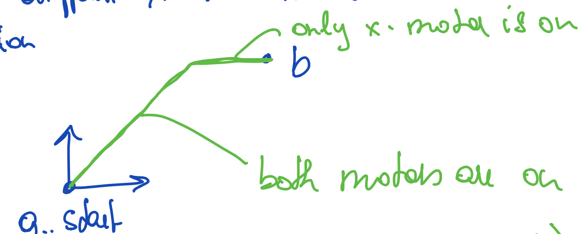
Summary: Norms in $\mathbb{R}^n$: $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$

Example for 1-norm:



Example for ∞-norm: machine head that moves from a to b using different motors in each coordinate direction



$$\|b-a\|_\infty = \max\left(|b_1-a_1|, |b_2-a_2|\right)$$

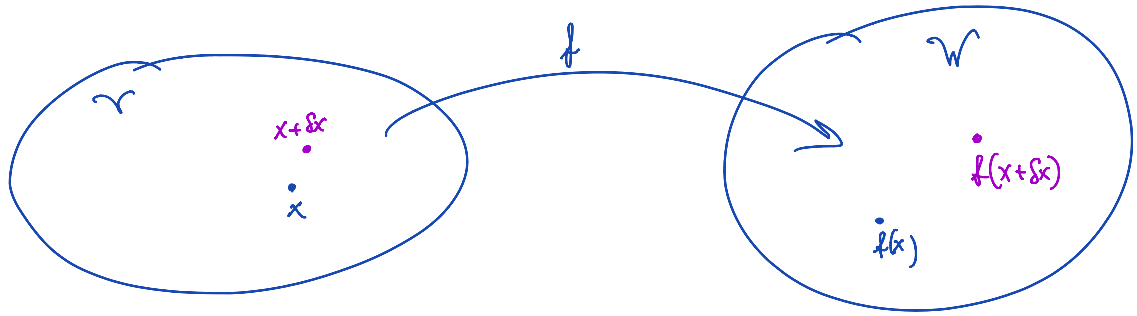
Vector norms imply | induce matrix norms:

$$A \in \mathbb{R}^{n\times n} \qquad \|A\| = \max_{\substack{v\in\mathbb{R}^n \\ v\neq 0}} \frac{\|Av\|}{\|v\|} = \max_{\|v\|=1} \|Av\|$$

matrix norm

vector norms

Condition numbers: measure of sensitivity of the output to changes in the input of a map



Absolute local condition number

$$\mathrm{Cond}_x(f) = \sup_{\substack{\delta x \neq 0 \\ x+\delta x \in V \\ \delta x \to 0}} \frac{\|f(x+\delta x) - f(x)\|_W}{\|\delta x\|_V}$$

Relative local condition number

$$\mathrm{cond}_x(f) = \sup_{\substack{\delta x \neq 0 \\ x+\delta x \in V \\ \delta x \to 0}} \frac{\|f(x+\delta x) - f(x)\|_W / \|f(x)\|_W}{\|\delta x\|_V / \|x\|_V}$$

$\mathrm{cond}_x(f)$ or $\mathrm{Cond}_x(f) \gg 1$ "very large" ill-conditioned

~ 1 "moderate" well-conditioned

Condition number of

$$A^{-1}: \begin{cases} \mathbb{R}^n \to \mathbb{R}^n \\ b \longmapsto A^{-1}b \end{cases}$$

(for solving $Ax=b$)

$A \in \mathbb{R}^{n\times n}$, invertible

$$\mathrm{Cond}_b(A^{-1}) = \sup_{\substack{\delta b \neq 0 \\ \delta b \to 0}} \frac{\|A^{-1}(b+\delta b) - A^{-1}b\| / \|A^{-1}b\|}{\|\delta b\| / \|b\|}$$

$$= \sup_{\delta b \neq 0} \frac{\|A^{-1}\delta b\| / \|A^{-1}b\|}{\|\delta b\| / \|b\|} = \|A^{-1}\| \frac{\|b\|}{\|A^{-1}b\|}$$

$b = A(A^{-1}b) \Longrightarrow \|b\| = \|A(A^{-1}b)\| \leq \|A\| \, \|A^{-1}b\|$

$$\Longrightarrow \mathrm{cond}_b(A^{-1}) \leq \|A^{-1}\| \frac{\|A\| \, \|A^{-1}b\|}{\|A^{-1}b\|} = \|A\| \, \|A^{-1}\|$$

Def: $\kappa(A) = \|A\| \, \|A^{-1}\|$ is called condition number of the matrix A

Thm: Condition number depends on the norm!

$$\kappa_1(A) = \|A\|_1 \|A^{-1}\|_1$$
$$\kappa_2(A) = \|A\|_2 \|A^{-1}\|_2$$
$$\kappa_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty$$

Example:

$$A = \begin{bmatrix} 1 & & & 0 \\ 1 & \ddots & & \\ \vdots & & \ddots & \\ 1 & 0 & \cdots & 0 \;\; 1 \end{bmatrix} \in \mathbb{R}^{n\times n} \qquad A^{-1} = \begin{bmatrix} 1 & & & 0 \\ -1 & \ddots & & \\ \vdots & & \ddots & \\ -1 & 0 & \cdots & 0 \;\; 1 \end{bmatrix} \in \mathbb{R}^{n\times n}$$

$$\kappa_1(A) = \|A\|_1 \|A^{-1}\| = n \cdot n = n^2$$
$$\kappa_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty = 2\cdot 2 = 4$$
$$\kappa_2(A) = \max_i \left(\lambda_i(AA^T)\right)^{\frac{1}{2}} \cdot \max_i \left(\lambda_i(A^{-1}A^{-T})\right)^{\frac{1}{2}}$$

Thm: $A \in \mathbb{R}^{n\times n}$ regular, $b \in \mathbb{R}^n$, $Ax = b$, $A(x+\delta x) = b + \delta b$

$$\boxed{\Longrightarrow \frac{\|\delta x\|}{\|x\|} \leq \kappa(A) \frac{\|\delta b\|}{\|b\|}}$$

[Note: $\kappa(A) \geq 1$ since: $1 = \|I\| = \|AA^{-1}\| \leq \|A\| \|A^{-1}\| = \kappa(A)$]

Proof: $Ax = b$, $A(x+\delta x) = b + \delta b \Longrightarrow$

$$\Longrightarrow A\,\delta x = b + \delta b - Ax$$
$$\Longrightarrow \delta x = A^{-1}(b+\delta b) - x = A^{-1}\delta b$$

$\|b\| \leq \|A\| \|x\|$,

$$\|\delta x\| \leq \|A^{-1}\| \|\delta b\|$$

$$\xrightarrow{\text{multiply}} \|b\| \|\delta x\| \leq \kappa(A) \|x\| \|\delta b\|$$

$$\Longrightarrow \frac{\|\delta x\|}{\|x\|} \leq \kappa(A) \frac{\|\delta b\|}{\|b\|}$$

$\square$.

§ 2.9 Linear least squares

Consider an overdetermined linear system, e.g.

$$\begin{pmatrix} 3 & 1 \\ 1 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \qquad A \in \mathbb{R}^{m \times n},\ b \in \mathbb{R}^m,\ m \geq n$$

does <u>not</u> have (in general) a solution, but we can try to find $x \in \mathbb{R}^n$ such that $Ax \approx b$, or:

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|_2$$

obviously equivalent to

$$\min_{x \in \mathbb{R}^n} \|Ax-b\|_2^2 = \min_{x \in \mathbb{R}^n} (Ax-b)^T(Ax-b)$$