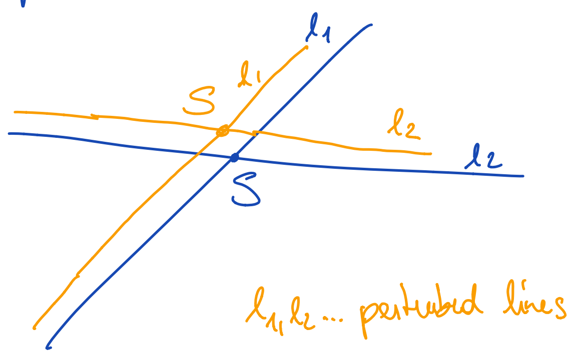
§2.7 Norms and condition numbers

We ask the question what consequence a small perturbation in A or b can have on the solution of the linear system

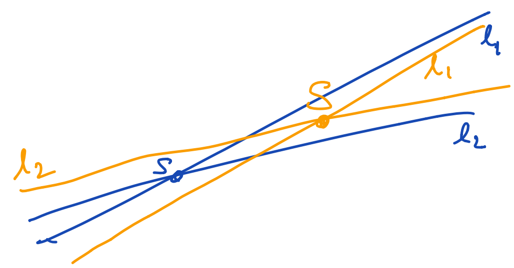
$$Ax = b$$

Can small changes have a huge influence on x? If so, when?

Solving a 2×2 system means to compute the intersection point between two lines.



small influence on S
" well-conditioned "



large influence on S
" ill-conditioned "

We consider a perturbed system

$$A(x + \Delta x) = b + \Delta b,$$

where Δx is a consequence of Δb perturbation

Want estimates like:

$$\|\Delta x\| \leq \kappa \|\Delta b\| \quad , \quad \kappa > 0 \qquad \text{(absolute)}$$

$$\frac{\|\Delta x\|}{\|x\|} \leq \tilde{\kappa} \frac{\|\Delta b\|}{\|b\|} \quad , \quad \tilde{\kappa} > 0 \qquad \text{(relativ)}$$

Q: How to compute $\kappa, \tilde{\kappa}$? What norms choose?

Def: Norms : $\|\cdot\| : V \longrightarrow \mathbb{R}$, V linear space over $\mathbb{R}$

- $\|v\| = 0 \iff v = 0$ f.a. $v \in V$
- $\|\lambda v\| = |\lambda| \, \|v\|$ f.a. $\lambda \in \mathbb{R}$, $v \in V$
- $\|v + w\| \leq \|v\| + \|w\|$ f.a. $v, w \in V$

Examples:

- $\|v\|_2 = \left(\sum_{i=1}^{n} v_i^2 \right)^{\frac{1}{2}}$, $v \in \mathbb{R}^n$, 2-norm, Euclidean norm
- $\|v\|_1 = \sum_{i=1}^{n} |v_i|$ 1-norm
- $\|v\|_\infty = \max_{i=1}^{n} |v_i|$ ∞-norm

Examples for norm in $\mathbb{R}^{n \times n}$:

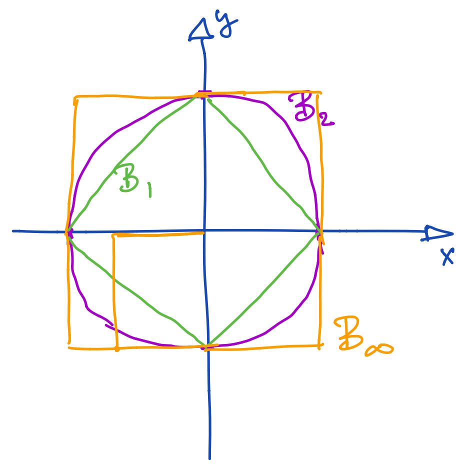
$$\|A\| = \|A\|_F = \left(\sum_{j,i=1}^{n} a_{ij}^2 \right)^{\frac{1}{2}}$$ Frobenius norm

Unit circles/spheres in $\mathbb{R}^2$

$B_1 = \{ v \in \mathbb{R}^n \mid \|v\|_1 = 1 \}$

$B_2 =$ —"— $\|v\|_2 = 1 \}$

$B_\infty =$ —"— $\|v\|_\infty = 1 \}$



Matrix norms induced by vector norms

"subordinate" norms : For a norm $\|\cdot\|$ on $\mathbb{R}^n$, the induced matrix norm is

$$\boxed{\|A\| = \max_{\substack{v \in \mathbb{R}^n \\ v \neq 0}} \frac{\|Av\|}{\|v\|} \qquad A \in \mathbb{R}^{n \times n}}$$

matrix norm

vector norms

i.e. matrix norm depends on norm in $\mathbb{R}^n$, e.g. $\|\cdot\|_2$, $\|\cdot\|_1$, $\|\cdot\|_\infty$

equivalent definition:

$$\boxed{\|A\|_* = \max_{\substack{\|v\|=1 \\ v\in\mathbb{R}^n}} \|Av\|}$$

Proof that $\|A\| = \|A\|_*$:

$$\|A\| = \max_{\substack{v\in\mathbb{R}^n \\ v\neq 0}} \frac{\|Av\|}{\|v\|} \geq \max_{\|v\|=1} \|Av\| = \|A\|_*$$

$$\Longrightarrow \|A\| \geq \|A\|_*$$

$v \neq 0$ $$\frac{\|Av\|}{\|v\|} = \left\| A \frac{v}{\|v\|} \right\| \leq \max_{\|w\|=1} \|Aw\| = \|A\|_*$$

$$\|A\| = \max_{v\neq 0} \frac{\|Av\|}{\|v\|} \leq \|A\|_* \Longrightarrow \|A\| \leq \|A\|_*$$

$$\Longrightarrow \checkmark$$

Note: The induced matrix norms satisfy:

matrix norm

$$\|Av\| \leq \|A\| \, \|v\|$$

vector-norms

Thm: Induced matrix norm for $\|\cdot\|_\infty$ vector norm is

$$\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$$ "maximum row sum"

Ex: $$A = \begin{pmatrix} 1 & -2 \\ -3 & 1 \end{pmatrix} \Longrightarrow \|A\|_\infty = 4$$

Thm: The matrix norm induced by the $\|\cdot\|_1$-norm is

$$\|A\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|$$ "largest column sum"

$$\|A\|_1 = 4$$

Thm: Eigenvalues of A^TA are $\lambda_1, \dots, \lambda_n$. Then

$$\|A\|_2 = \max_i \sqrt{\lambda_i}$$

If A is symmetric, then $A^TA = A^2$ and the eigenvalues of A^TA are squares of the eigenvalues of A

$$\Longrightarrow \|A\|_2 = \max_i |\lambda_i(A)|$$

Ex: $A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$ $\quad \|A\|_2 = 2$

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next class

$\kappa = \|A\| \, \|A^{-1}\|$ condition number of A