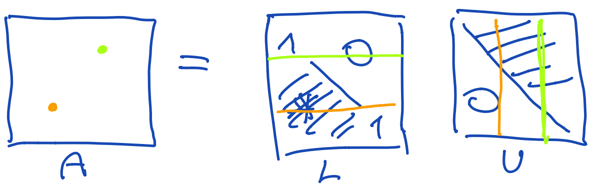
Summary: Solve $Ax=b$, $A\in\mathbb{R}^{n\times n}$, $x,b\in\mathbb{R}^{n}$

$$L_N \cdots L_1 A = U$$

U... upper triangular

L_i... unit lower triangular

$$\Longrightarrow A = \underbrace{L_1^{-1}\cdot \ldots \cdot L_N^{-1}}_{L}\, U = L\,U$$



How can $A=LU$ be used to solve $Ax=b$?

$$Ax=b=L\underbrace{Ux}_{y}=b$$

$$Ax=b \Longleftrightarrow \begin{array}{l} Ly=b \\ Ux=y \end{array}$$

easy to solve using forward & backward substitution!

direct computation:

$$a_{ij} = \sum_{k=1}^{n} l_{ik}u_{kj} \qquad 1\le i,j\le n$$

$$a_{ij} = \sum_{k=1}^{j} l_{ik}u_{kj} \qquad 1\le j< i\le n$$

$$a_{ij} = \sum_{k=1}^{i} l_{ik}u_{kj} \qquad 1\le i\le j\le n$$

$$\Longrightarrow l_{ij} = \frac{1}{u_{jj}}\left(a_{ij} - \sum_{k=1}^{j-1} l_{ik}u_{kj}\right) \qquad \begin{array}{l} i=2,3,\ldots,n \\ j=1,2,\ldots,i-1 \end{array}$$

$$\xRightarrow{l_{ii}=1} u_{ij} = a_{ij} - \sum_{k=1}^{i-1} l_{ik}u_{kj} \qquad \begin{array}{l} i=1,\ldots,n \\ j=i,\ldots,n \end{array}$$

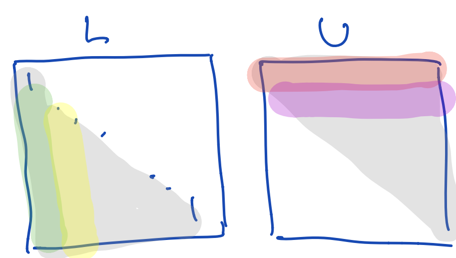
order of computation:

$u_{1j} = a_{1j}$ $\quad j=1,2,\ldots,n$

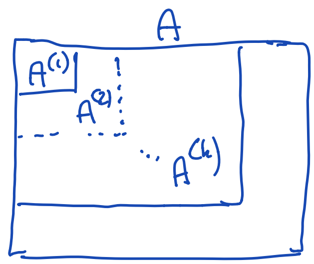
$l_{11}=1$, $l_{i1} = \frac{a_{i1}}{u_{11}}$ $\quad i=2,\ldots,n$

$u_{2j} = a_{2j} - l_{21}u_{12}$ $\quad j=i\ldots n$

$l_{2j} = \ldots$



Def: $A \in \mathbb{R}^{n\times n}$ leading principle submatrices A^k coincide with A on the "upper left" $k\times k$ entries.



Thm: $A \in \mathbb{R}^{n\times n}$, every leading principle submatrix $A^{(k)}$ is non-singular, $k = 1, \dots, n-1$. Then $A = LU$ exists with a unit lower triangular matrix L, and upper triangular matrix U.

Proof: Induction over matrix size n

$n=2$ $\qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a \neq 0$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix} \begin{pmatrix} u & v \\ 0 & \eta \end{pmatrix}$$

$u = a$, $v = b$, $mu = c$, $mv + \eta = d$

$m = c/u$ ok as $u = a \neq 0$

$\eta = d - mb \longrightarrow$ LU decomposition exists.

Assume $A \in \mathbb{R}^{(k+1)\times(k+1)}$ for which all principle leading submatrices of order k and smaller are invertible, and thus they have an LU-decomposition

$$A = \left[\begin{array}{c|c} A^{(k)} & b \\ \hline c^T & d \end{array}\right]$$

$A^{(k)}$ is non-singular,

$A^{(k)} = L^{(k)} U^{(k)}$

$$A = \left[\begin{array}{c|c} L^{(k)} & \begin{matrix}0\\ \vdots \\ 0\end{matrix} \\ \hline m^T & 1 \end{array}\right] \left[\begin{array}{c|c} U^{(k)} & v \\ \hline 0 \cdots 0 & \eta \end{array}\right] = \left[\begin{array}{c|c} A^{(k)} & b \\ \hline c^T & d \end{array}\right]$$

block multiplication

$$\Longrightarrow A^{(k)} = L^{(k)} U^{(k)}, \quad L^{(k)} v = b, \quad m^T U^{(k)} = c^T,$$
$$m^T v + \eta = d$$

$$L^{(k)} v = b \Longrightarrow v = L^{(k)^{-1}} b \checkmark$$

$$U^{(k)^T} m = c \Longrightarrow$$

$$A^{(k)} = L^{(k)} U^{(k)}$$
$$0 \neq \det(A^{(k)}) = \underbrace{\det(L^{(k)})}_{=1} \det(U^{(k)})$$
$$\Longrightarrow \det(U^{(k)}) \neq 0$$

$$\Longrightarrow m = U^{(k)^{-T}} c \checkmark$$

$$\Longrightarrow \eta = d - m^T v.$$

$\Longrightarrow$ LU factorization exists for A. $\square$

§2.4 Pivoting

What do we do if $u_{ii} = 0$ (or very small)

Example:

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix}$$

no LU decomposition exists, but exchanging 1st & 2nd row

$$\longrightarrow \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix}$$

now LU decomposition exists!

Pivoting exchanges rows to make sure we don't find a zero (or something very small in diagonal)

Def: Permutation matrices, $P \in \mathbb{R}^{n\times n}$, only contain zeros and ones, and each column & row contains exactly one non-zero.

Examples:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \dots$$

Properties:
- products of permutation matrices are again permutation matrices
- det is ± 1
- they are products of "interchange matrices"
- inverse is again a permutation matrix

Thm: $A \in \mathbb{R}^{n\times n}$, There exists a permutation matrix $P \in \mathbb{R}^{n\times n}$, L unit lower triangular, U upper triangular matrix such that

$$PA = LU.$$

Proof (next time).

§ 2.5 How can this be used to solve linear systems

$$\begin{aligned} & Ax = b \\ \Longleftrightarrow\ & PAx = Pb \\ \Longleftrightarrow\ & LUx = Pb \\ \Longleftrightarrow\ & Ly = Pb, \quad Ux = y \end{aligned}$$

Thus, to solve $Ax=b$:

1.) Compute LU-factorization $PA=LU$ cost $\sim n^3$

2.) Solve $Ly=Pb$ (forward substitution) $\sim n^2$

3.) Solve $Ux=y$ (backward substitution) $\sim n^2$