§2.1 Solving linear systems

We are interested in solving the linear system

$$Ax = b, \quad A \in \mathbb{R}^{n\times n}, \quad x, b \in \mathbb{R}^n$$

i.e.:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

Recall: $A^{-1} \in \mathbb{R}^{n\times n}$, $AA^{-1} = A^{-1}A = I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \in \mathbb{R}^{n\times n}$

inverse exists if $\det(A) \neq 0$,
we call these matrices "non-singular", "regular", "invertible".

Cramer's rule: $x_i = \dfrac{\det(A_i^b)}{\det(A)}$ $\quad A_i^b$ ... matrix A with i-th column replaced by b

requires $n+1$ determinant computations to find $x \in \mathbb{R}^n$.
each determinant computation is expensive / computationally intensive, e.g. for $n=4$

$$\det \begin{bmatrix} a_{11} & \cdots & a_{14} \\ \vdots & & \vdots \\ a_{41} & \cdots & a_{44} \end{bmatrix} = a_{11}(-1)^{1+1} \det \begin{bmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & & \vdots \\ a_{42} & \cdots & a_{44} \end{bmatrix} +$$

$$+ a_{21}(-1)^{2+1} \det \begin{bmatrix} a_{12} & a_{13} & a_{14} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{bmatrix} + \cdots$$

it's possible to show that computing a determinant of an $n\times n$-matrix requires $\sim e\, n!$ operations (additions, summations), "flop"

floating point operations

§2.2 Gaussian elimination

Example
$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & 5 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 16 \\ -3 \end{bmatrix}$$

Generate triangular form by adding multiples of rows to other rows [solution remains the same!]

add (−2) first row $\longrightarrow$ to 2nd row

add first row to 3rd
$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 6 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 3 \end{bmatrix}$$

identical to multiplication of system with

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I + \mu_{21} E^{21}, \quad \text{then by } L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = I + \mu_{31} E^{31}$$

$$E^{21} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Continue...

$(-3) \times$ 2nd row $\longrightarrow$ added to 3rd row

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ -9 \end{bmatrix} \qquad (*)$$

$$L_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

is an upper triangular matrix

(*) can be solved easily by "backward substitution"

$$\rightarrow \quad x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Def: $L \in \mathbb{R}^{n \times n}$ is <u>lower triangular</u> if $l_{ij} = 0$ for all $1 \le i < j \le n$:

$$\begin{bmatrix} l_{11} & & 0 \\ \vdots & \ddots & \\ l_{n1} & \cdots & l_{nn} \end{bmatrix}$$

and <u>unit lower triangular</u> if $l_{11} = l_{22} = \ldots l_{nn} = 1$

Analog definitions hold for <u>upper triangular</u>.

Thm (Properties of lower triangular systems — identical results hold for upper triangular systems)

(i) products of lower triangular matrices are lower triangular

(ii) — " — unit — " — unit — " —

(iii) non-singular if $l_{11} \neq 0$, $l_{22} \neq 0 \ldots$

(iv) lower triangular matrices that are invertible have lower triangular inverses

(v) — " — for <u>unit</u> lower triangular matrices

<u>Proof of iv:</u> Induction over matrix size,

<u>$n=2$</u>

$$L = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \quad L^{-1} = \begin{pmatrix} d & e \\ f & g \end{pmatrix}$$

$$LL^{-1} = I \implies ae = 0, \implies e = 0$$

since $a \neq 0$

$n \mapsto n+1$

$$L = \left[\begin{array}{c|c} L_1 & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline -\, r^T \,- & \alpha \end{array}\right] \in \mathbb{R}^{(n+1)\times(n+1)} \quad L^{-1} = \left[\begin{array}{c|c} L_1' & \begin{matrix} | \\ c \\ | \end{matrix} \\ \hline r^{T\prime} & \mu \end{array}\right]$$

Since $LL^{-1} = I \in \mathbb{R}^{(n+1)\times(n+1)} \implies$

$$L_1 L_1' = I \in \mathbb{R}^{n \times n}, \quad L_1 c = 0, \quad r^T L_1' = 0, \quad r^T c + \alpha\mu = 1$$

$\implies L_1' = L_1^{-1}$ is lower triangular. by induction assumption for n.

$L_1 c = 0 \implies c = 0 \implies L^{-1}$ is lower triangular $\square$.

Elimination process:

$$L_N \cdots L_2 L_1 A = U$$

upper triangular system

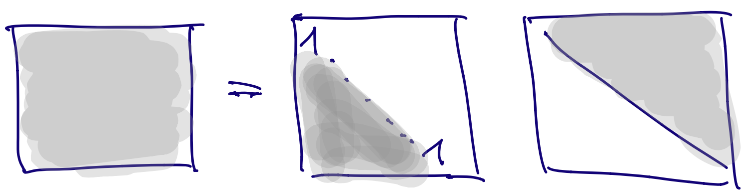
$$N = \frac{n(n-1)}{2}$$

$$L_j = I + \mu_{rs} E^{rs}$$

$$E^{rs} = \begin{array}{c|c} & s \\ \hline r & \begin{bmatrix} 0 & \cdots & 0 \\ & 1 & \\ & & 0 \end{bmatrix} \end{array}$$

$$\Longrightarrow A = \underbrace{L_1^{-1} L_2^{-1} \cdots L_N^{-1}}_{L \text{ unit lower triangular}} \cdot U$$

$$A = L \; U$$



LU-decomposition of A.