Summary of Newton's method to solve $f(x)=0$, $f:[a,b]\to\mathbb{R}$

Choose $x_0 \in [a,b]$ (initialization)

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - f'(x_k)^{-1} f(x_k) \quad k=0,1,2,\dots$$

Theorem: (Convergence of Newton's method)

f cont, f'' continuous on $I_\delta = [\xi-\delta, \xi+\delta]$, $\delta>0$

$f(\xi)=0$, $f''(\xi)\neq 0$, $\exists A>0$:

$$\left|\frac{f''(x)}{f'(y)}\right| \le A \quad \text{for all } x,y\in I_\delta$$

If $|\xi - x_0| \le h := \min(\delta, 1/A)$, then the Newton sequence

$x_k \longrightarrow \xi$ quadratically.

$$\frac{|x_{k+1}-\xi|}{|x_k-\xi|^2} \longrightarrow \mu < \infty$$

Remark: implies that $f'(\xi)\neq 0$.

Newton's method requires $f'(x_k)$, in every iteration.

Example: Newton to solve $f(x) = x^3 = 0$

Obvious solution $x=0$

§1.5 Secant method

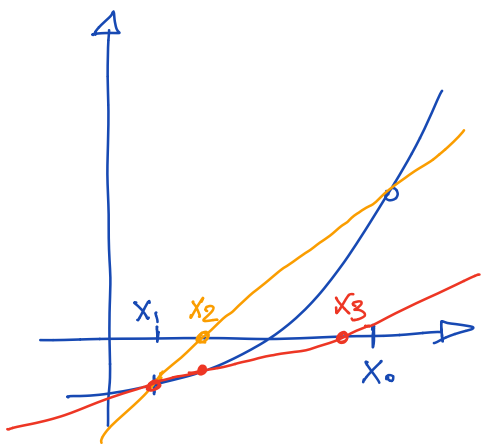
does not require f', as it approximates derivatives by difference quotients:

$$f'(x_k) \approx \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} \quad \text{if } x_k \neq x_{k-1}$$

Secant method:

x_0, x_1 initialization

$$x_{k+1} = x_k - \underbrace{\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}}_{\approx f'(x_k)^{-1}} f(x_k), \quad k = 1, 2, 3 \dots$$



Theorem: f continuously differentiable on $[\xi - h, \xi + h]$, $h > 0$
$f(\xi) = 0$, $f'(\xi) \neq 0$
Then the secant method converges at least linearly to ξ if x_0, x_1 are sufficiently close to ξ.

Remark: One can show that the secant method converges faster:

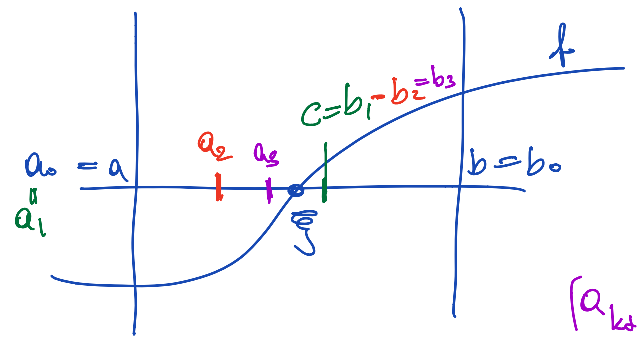
$$\lim_{k \to \infty} \frac{|x_{k+1} - \xi|}{|x_k - \xi|^q} = \mu < \infty$$

$$\text{for } q = \frac{1}{2}\left(1 + \sqrt{5}\right) \sim 1.6$$

The method is slower than Newton's method, but "cheaper" since I do not require $f'(x_k)$.

§ 1.6. Bisection method

$f:[a,b]\to\mathbb{R}$, $\xi\in[a,b]$, $f(\xi)=0$, f continuous.



(a_0,b_0)

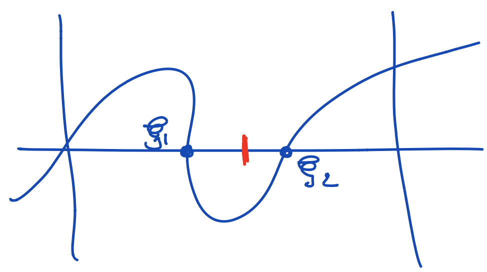
check if there is a sign change between a_0, c or c, b_0

$$(a_{k+1},b_{k+1}) = \begin{cases}(a_k,c_k) & \text{if } f(c_k)f(b_k)>0\\ (c_k,b_k) & \text{else}\end{cases}$$

$$c_k = \frac{a_k+b_k}{2}$$

- simple, robust
- slower than Newton's method, at least asymptotically.
- Issues if there are several roots of f.

depending on how we choose the new (a_k,b_k), we converge to either ξ_1 or ξ_2



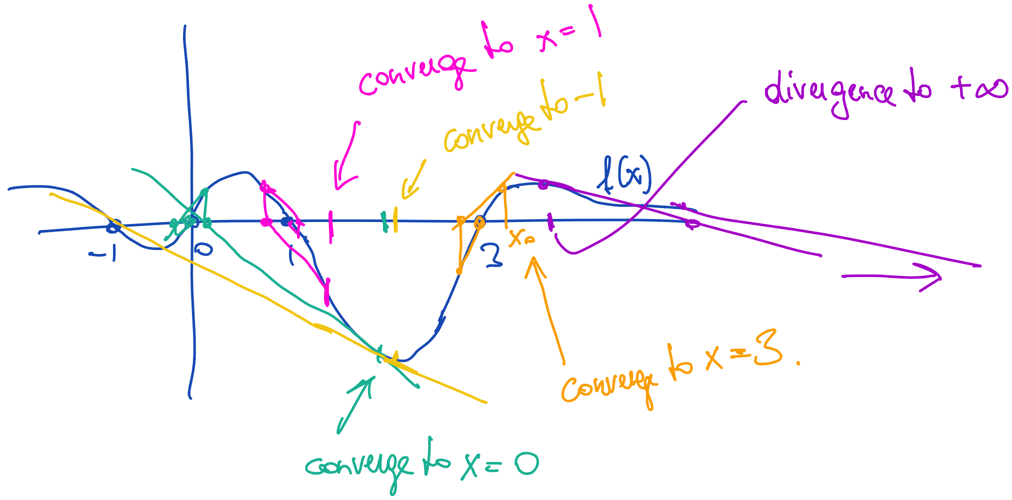
§ 1.7 Global behavior of Newton's method

Convergence if starting point x_0 is close to ξ; general behavior can be very complicated

Example: $f(x) = x(x^2-1)(x-3)\exp\left(-\frac{1}{2}(x-1)^2\right)$

roots:

$x=0$
$x=\pm 1$
$x=3$



Newton's method for systems of equations

$$f: \mathbb{R}^2 \to \mathbb{R}^2 \qquad f(x,y) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$$

Look for $(\xi_1, \xi_2) \in \mathbb{R}^2$: $f(\xi_1, \xi_2) = 0$

Newton's method: Choose $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathbb{R}^2$ initialization

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} x_k \\ y_k \end{pmatrix} - \underbrace{Df(x_k, y_k)^{-1}}_{\in \mathbb{R}^{2\times 2}} \underbrace{f(x_k, y_k)}_{\in \mathbb{R}^2} \qquad k = 0, 1, 2, \ldots$$

$$Df(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} \in \mathbb{R}^{2\times 2}$$

Example: $f(x,y) = \begin{pmatrix} x^2 + y^2 - 2 \\ 2x^2 - y^2 - 1 \end{pmatrix} = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$

solution is $(\xi_1, \xi_2) = (1,1)$, also: $(-1,1)$, $(1,-1)$, $(-1,-1)$

$$Df(x,y) = \begin{pmatrix} 2x & 2y \\ 4x & -2y \end{pmatrix}$$

Newton: Start $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} x_k \\ y_k \end{pmatrix} - \begin{pmatrix} 2x_k & 2y_k \\ 4x_k & -2y_k \end{pmatrix}^{-1} \begin{pmatrix} x_k^2 + y_k^2 - 2 \\ 2x_k^2 - y_k^2 - 1 \end{pmatrix}$$

Remarks:
- requires the Jacobian matrix ("expensive")
- requires matrix inverse (or to solve a linear system)
- similar convergence results hold, i.e. "quadratic convergence close to solution"