Runge-Kutta methods

$$y_{n+1} = y_n + h\,(a k_1 + b k_2)$$

$$k_1 = f(x_n, y_n)$$

$$k_2 = f(x_n + \alpha h,\ y_n + \beta h k_1)$$

stages

$$\beta = \alpha, \quad a = 1 - \frac{1}{2\alpha}, \quad b = \frac{1}{2\alpha}, \quad \alpha \neq 0$$

$\rightsquigarrow$ second-order accurate, i.e. $|T_n| \leq C h^2$

$\underline{\alpha = \frac{1}{2}}$ modified Euler:

$$y_{n+1} = y_n + h\, f\left(x_n + \tfrac{1}{2}h,\ y_n + \tfrac{1}{2} h\, f(x_n, y_n)\right)$$

$\underline{\alpha = 1:}$ improved Euler:

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f\big(\overbrace{x_n + h}^{x_{n+1}},\ \overbrace{y_n + h f(x_n, y_n)}^{\approx y_{n+1}}\big)\right]$$

similar to trapezoidal, but this is $f(x_{n+1}, y_{n+1})$ in trapezoidal rule

High-order Runge-Kutta (classic 4^{th}-order RK method):

$$y_{n+1} = y_n + \frac{1}{6}h\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$k_1 = f(x_n, y_n)$$
$$k_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right)$$
$$k_3 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right)$$
$$k_4 = f\left(x_n + h, y_n + hk_3\right)$$

4 stages, each stage only depends on previous stage

Note that each step requires evaluation of f 4 times! It's 4^{th}-order, i.e. the error is bounded by Ch^4.

§12.6 Linear multistep methods

Attempts to reduce the number of function evaluations of f by using a history of points, e.g.

x_{n-1}, x_n, x_{n+1}

e.g.: $$y_{n+1} = y_{n-1} + \frac{1}{3}h\left[f(x_{n-1}, y_{n-1}) + 4f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]$$

implicit

involves 3 steps: y_{n+1}, y_n, y_{n-1}

This formula comes from
$$y(x_{n+1}) = y(x_{n-1}) + \underbrace{\int_{x_{n-1}}^{x_{n+1}} f(x, y(x))\,dx}_{\text{approximate with Simpson's rule}}$$

rule: $\approx \dfrac{f(x_{n-1}, y_{n-1}) + 4f(x_n, y_n) + f(x_{n+1}, y_{n+1})}{6}$

Explicit Euler rule is a 1-step rule, and so is the implicit Euler rule:

backwards Euler: $$y_{n+1} = y_n + hf(x_{n+1}, y_{n+1})$$

Adams - Bashford: explicit 4 - step method

$$y_{n+4} = y_{n+3} + \frac{h}{24}\left(\underbrace{55 f_{n+3}}_{f(x_{n+3},\, y_{n+3})} - 59 f_{n+2} + 37 f_{n+1} - 9 f_n\right)$$

explicit (no y_{n+4} on the right)

requires start up using lower - step schemes, e.g:

$y_0 \longrightarrow$ known, initial value
$y_1 \longrightarrow$ compute with 1- step method (Euler)
$y_2 \longrightarrow$ —— " —— 2 - step method
$y_3 \longrightarrow$ —— " —— —— " ——
$y_4 \longrightarrow$ Adams Bashford
$\vdots$

Adams - Moulton:

$$y_{n+3} = y_{n+2} + \frac{h}{24}\left(9 f_{n+3} + 19 f_{n+2} - 5 f_{n+1} + f_n\right)$$

implicit