Implicit one-step method

$$\underline{y_{n+1}} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, \underline{y_{n+1}})\right]$$

implicit

$$T_n = \frac{y(x_{n+1}) - y(x_n)}{h} - \underbrace{\frac{1}{2}\left[f(x_n, y(x_n)) + f(x_{n+1}, y(x_{n+1}))\right]}_{\phi(x, y; h)}$$

truncation error

$$|T_n| \le \frac{1}{12} h^2 M_3 \qquad M_3 = \max_{x \in [x_0, x_N]} |y'''(x)|$$

higher-order than Euler, where it was h

Thm:

$$|e_n| \le \frac{T}{L_\phi}\left(e^{L_\phi (x_n - x_0)} - 1\right) \qquad n = 0, 1, 2, \ldots, N$$

we have T, which is the maximum truncation error

L_ϕ can be bounded by Lipschitz constant of f.

$$\Longrightarrow \quad |e_n| \le \frac{M_3}{12 L_\phi}\left(e^{L_\phi (x_n - x_0)} - 1\right) h^2$$

higher-order!

Example

$$y' = y^2 - g(x), \quad y(0) = 2$$

$$g(x) = \frac{x^4 - 6x^3 + 12x^2 - 14x + 9}{(1+x)^2}$$

$$y_{n+1} = y_n + \frac{h}{2}\left[\overbrace{y_n^2 - g(x_n)}^{f(x_n, y_n)} + \overbrace{y_{n+1}^2 - g(x_{n+1})}^{f(x_{n+1}, y_{n+1})}\right]$$

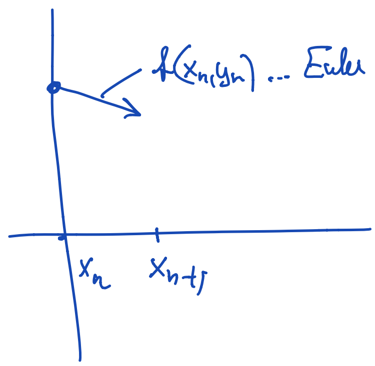
quadratic equation in y_{n+1} — can be solved analytically or using Newton's method.

$$\longrightarrow y_{n+1}^2 - \frac{2}{h} y_{n+1} + \left(\frac{2}{h} y_n + y_n^2 - g(x_n) - g(x_{n+1})\right) = 0$$

$$\xrightarrow{\text{solve}} y_{n+1}^{(1,2)} = \frac{1}{h} \pm \sqrt{\frac{1}{h^2} - \left(\frac{2}{h} y_n + y_n^2 - g(x_n) - g(x_{n+1})\right)}$$

Runge-Kutta methods

Is it possible to have an explicit rule that has higher-order accuracy compared to Euler's method? Yes, through intermediate evaluation of f.



$$y_{n+1} = y_n + h(a k_1 + b k_2)$$

"stages" evaluate f twice

$$\to k_1 = f(x_n, y_n)$$

$$\to k_2 = f(x_n + \alpha h, y_n + \beta h k_1)$$

$$a, b, \alpha, \beta \in \mathbb{R}$$

We want $a + b = 1$.

$$\phi(x, y; h) = a f(x_n, y_n) + b f(x_n + \alpha h, y_n + \beta h f(x_n, y_n))$$

We want to estimate the truncation error

$$T_n = \frac{y(x_n + h) - y(x_n)}{h} - \phi(x_n, y_n; h)$$

We use Taylor expansion for $y(x_n+h)$, $\phi(x_n, y_n; h)$:

$$y(x_n+h) = y(x_n) + h\, y'(x_n) + \frac{h^2}{2} y''(x_n) + \frac{h^3}{3!} y'''(x_n) + \mathcal{O}(h^4)$$

$$y'(x_n) = f(x_n, y(x_n)) = f \qquad \text{ODE}$$

$$y''(x_n) = f_x + f_y y' = f_x + f_y f$$

$$y'''(x_n) = f_{xx} + f_{xy} f + (f_{xy} + f_{yy} f) f + f_y (f_x + f_y f)$$

$$\phi(x_n, y_n; h) = a f + b\Big(f + \alpha h f_x + \beta h f f_y + \tfrac{1}{2}(\alpha h)^2 f_{xx} + \alpha\beta h^2 f f_{xy} + \tfrac{1}{2}(\beta h)^2 f^2 f_{yy} + O(h^3)\Big)$$

$$\Longrightarrow \quad T_n = \frac{y(x_n+h) - y(x_n)}{h} - \phi(x_n, y_n; h)$$

$$= \cancel{f} + \tfrac{1}{2} h (f_x + f f_y)$$
$$+ \tfrac{1}{6} h^2 \left[f_{xx} + 2 f_{xy} f + f_{yy} f^2 + f_y (f_x + f_y f) \right]$$
$$- \Big\{ \cancel{a f} + b \Big[\cancel{f} + \alpha h f_x + \beta h f f_y + \tfrac{1}{2} (\alpha h)^2 f_{xx} + \alpha \beta h^2 f f_{xy} + \tfrac{1}{2} (\beta h)^2 f^2 f_{yy} \Big] \Big\} + O(h^3)$$

$$f(1-a-b) = 0$$

$\tfrac{1}{2} h (f_x + f f_y) - b\alpha h f_x - b\beta h f f_y$ is zero if

$$b\alpha = \tfrac{1}{2}, \quad b\beta = \tfrac{1}{2}$$

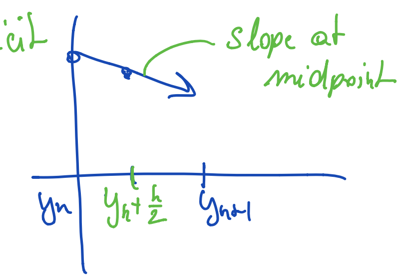
$$\Longrightarrow \boxed{\beta = \alpha, \quad a = 1 - \frac{1}{2\alpha}, \quad b = \frac{1}{2\alpha}, \quad \alpha \neq 0}$$

For any $\alpha \neq 0$, we found a second-order RK method

$\underline{\alpha = \tfrac{1}{2}}$: (modified Euler method):

$$y_{n+1} = y_n + h f\left(x_n + \tfrac{1}{2} h, \; y_n + \tfrac{1}{2} h f(x_n, y_n)\right)$$

2 evaluations of f, explicit, second-order accurate



$\underline{\alpha = 1}$ (improved Euler)

$$y_{n+1} = y_n + \frac{1}{2} h \left[f(x_n, y_n) + f\big(x_n + h, \underbrace{y_n + h f(x_n, y_n)}_{\approx y_{n+1} \text{ from explicit Euler}}\big) \right]$$

2 evaluations, 2nd-order, explicit