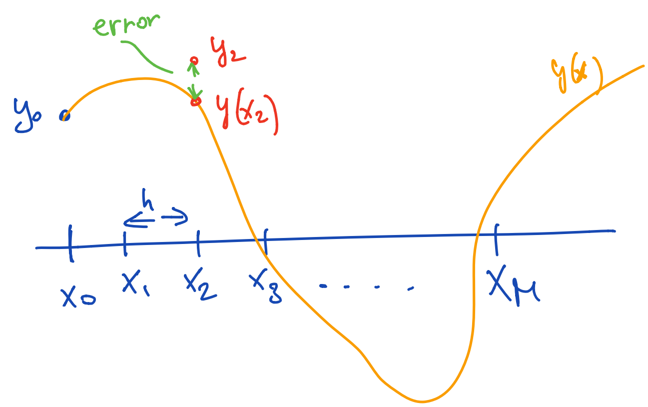
Initial value problems (IVP)

$$\begin{cases} y'(x) = f(x, y(x)) \\ y(x_0) = y_0 \end{cases} \qquad f: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$$

$y: [x_0, X_M] \longrightarrow \mathbb{R}$, $\quad y = y(x)$ solution of the IVP



Approximation:

points $x_0, x_1, x_2, \dots x_M$,

$x_n = x_0 + n h$, $\quad h = \dfrac{x_M - x_0}{N}$

$y_k \approx y(x_k)$

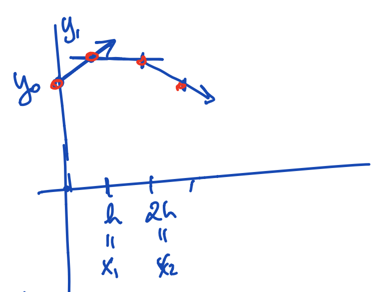
One-step methods:

y_{n+1} is computed just from y_n

(different from k-step methods, where y_{n+1} is computed from $y_n, y_{n-1}, \dots, y_{n-k+1}$)

Simplest 1-step method is **Euler's method**

$$y_{n+1} = y_n + h f(x_n, y_n)$$



Where does this come from:

Taylor expansion of $y(x_{n+1}) = y(x_n + h)$

$$\underset{y_{n+1}}{\underset{\wr}{y(x_n+h)}} = \underset{y_n}{\underset{\wr}{y(x_n)}} + h \cdot \underbrace{f(x_n, y(x_n))}_{f(x_n, y_n)} + \cancel{O(h^2)}$$

One step method, general form:

$$y_{n+1} = y_n + h\,\phi(x_n, y_n; h) \qquad n = 0, 1, 2, \ldots$$

$e_n = y(x_n) - y_n$ <u>error</u>

$$T_n = \frac{y(x_{n+1}) - y(x_n)}{h} - \phi(x_n, y(x_n); h)$$ <u>truncation error</u>

<u>Theorem:</u> Consider a one-step method, where

$$|\phi(x, u; h) - \phi(x, v; h)| \le L_\phi |u - v|$$

Assuming $|y_n - y_0| \le C$ for $n = 1, 2, \ldots$

$$|e_n| \le \frac{T}{L_\phi}\left(e^{L_\phi (x_n - x_0)} - 1\right) \qquad n = 0, 1, 2, \ldots$$

larger for larger n! (exponential growth of error bound)

$T = \max\limits_{0 \le n \le N-1} |T_n|$ maximum truncation error.

truncation error for Euler's method:

$$T_n = \frac{y(x_{n+1}) - y(x_n)}{h} - f(x_n, y(x_n))$$

$y(x) \in C^2$ $\qquad y(x_{n+1}) = y(x_n + h) = y(x_n) + h y'(x_n) + \frac{h^2}{2!} y''(\xi_n)$

$x_n \le \xi_n \le x_{n+1}$

$$\Longrightarrow T_n = \frac{1}{2} h\, y''(\xi_n)$$

$$M_2 = \max_{x \in [x_0, x_N]} |y''(x)| \Longrightarrow |T_n| \le T = \frac{1}{2} h M_2$$

$$\xrightarrow{\text{Theorem}} |e_n| \le \frac{1}{2} M_2 \left[\frac{e^{L(x_n - x_0)} - 1}{L}\right] h$$

allows to make error small.

Example: $y' = \tan^{-1}(y)$, $y(0) = y_0$

Compute L and M_2 explicitly

$f(x,y) = \tan^{-1} y \implies |f(x,u) - f(x,v)| =$

$$= \left|\frac{\partial f}{\partial y}(x,\eta)(u-v)\right| = \left|\frac{\partial f}{\partial y}(x,\eta)\right| |u-v| \qquad \eta \in (u,v)$$

$$\left|\frac{\partial f}{\partial y}(x,y)\right| = \left|\frac{1}{1+y^2}\right| \le 1 \implies \underline{L = 1}$$

M_2 depends on $|y''|$ which we don't know - but we can still estimate it by differentiating $y' = f(x,y) = \tan^{-1}(y)$

$$y'' = \frac{d}{dx}\left(\tan^{-1} y\right) = \frac{1}{1+y^2}\frac{dy}{dx} = \frac{1}{1+y^2}\tan^{-1} y$$

"$y' = \tan^{-1} y$"

$$\implies |y''| \le M_2 = \frac{\pi}{2}$$

$$\implies |e_n| \le \frac{1}{4}\pi\left(e^{x_n} - 1\right)h$$

I can use this to choose h such that $|e_n| \le tol \ (=: 10^{-5})$.

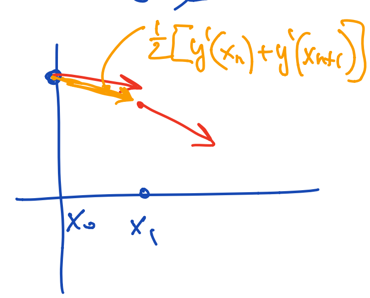
§ 12.4 An implicit one-step method

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]$$

Can be found from

$$y(x_{n+1}) - y(x_n) = \int_{x_n}^{x_{n+1}} y'(x)\, dx$$

approximate integral using trapezoidal rule, i.e. $\int \dots = \frac{h}{2}\left(y'(x_n) + y'(x_{n+1})\right)$



Main difference: The right hand side depends on y_{n+1}!

So we need to solve a nonlinear equation

$$\underbrace{y_{n+1}}_{\text{unknown}} - \frac{h}{2} f(x_{n+1}, \underbrace{y_{n+1}}_{\text{unknown}}) = y_n + \frac{h}{2} f(x_n, y_n)$$