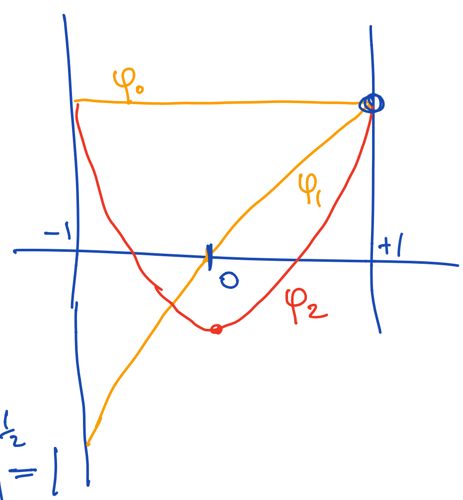
Orthogonal polynomials on $[0,1]$, $w(x) \equiv 1$

$$\varphi_0(x) = 1$$
$$\varphi_1(x) = x - \frac{1}{2}$$
$$\varphi_2(x) = x^2 - x + \frac{1}{6}$$
$$\varphi_3(x) = x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}$$
$$\vdots$$

To find an orthogonal family of polynomials w.r.to $w(x) \equiv 1$ on $[a,b]$, we use a linear transformation $x \longmapsto (b-a)x + a$, then the resulting system is orthogonal on $[a,b]$. For $[a,b] = [-1,1]$ these polynomials are called <u>Legendre polynomials</u>:

$$\varphi_0(x) = 1$$
$$\varphi_1(x) = x$$
$$\varphi_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$$
$$\varphi_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$$



These are scaled such that $\varphi_i(1) = 1$, one could scale them such that $\|\varphi_i\|_2 = \left(\int_{-1}^{1} \varphi_i(x)^2 \, dx \right)^{\frac{1}{2}} = 1$

(we then call them <u>orthonormal</u>)

<u>Theorem:</u> Given $f : [a,b] \to \mathbb{R}$, there exists a unique polynomial $p_n \in \mathbb{P}_n$ such that

$$\|f - p_n\|_2 = \min_{q \in \mathbb{P}_n} \|f - q\|_2$$

<u>Proof:</u> $\varphi_0, \dots, \varphi_n$ family of orthogonal polynomials, and we normalize them

$$\psi_j = \frac{\varphi_j}{\|\varphi_j\|_2} \qquad j = 0, \dots, n.$$

As a result: $\langle \psi_j, \psi_\ell \rangle = \begin{cases} 0 & \text{if } j \neq \ell \\ 1 & \text{if } j = \ell \end{cases}$

Every $q_n \in \mathcal{P}_n$ is of the form

$$q_n(x) = \beta_0 \psi_0(x) + \dots + \beta_n \psi_n(x), \quad \beta_i \in \mathbb{R}$$

We want to choose $\beta_0, \dots, \beta_n$ such that q_n minimizes $\|f - q_n\|_2$ over all $q_n \in \mathcal{P}_n$.

Consider $E : \mathbb{R}^{n+1} \longrightarrow E(\beta_0, \dots, \beta_n) = \|f - q_n\|_2^2$

$$E(\beta_0, \dots, \beta_n) = \|f - q_n\|_2^2 = \langle f - q_n, f - q_n \rangle$$

$$= \langle f, f \rangle - 2 \langle f, q_n \rangle + \langle q_n, q_n \rangle$$

$$= \|f\|_2^2 - 2 \sum_{j=0}^{n} \beta_j \langle f, \psi_j \rangle + \underbrace{\sum_{j=0}^{n} \sum_{k=0}^{n} \beta_j \beta_k \langle \psi_j, \psi_k \rangle}_{\sum_{j=0}^{n} \beta_j^2}$$

$$= \sum_{j=0}^{n} \left\{ \beta_j - \langle f, \psi_j \rangle \right\}^2 + \underbrace{\|f\|_2^2 - \sum_{j=0}^{n} |\langle f, \psi_j \rangle|^2}_{\text{indep. of } \beta_i}$$

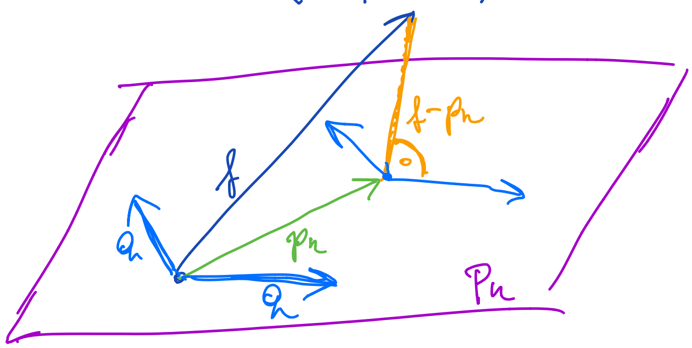
Minimum will be attained for

$$\beta_j^* = \langle f, \psi_j \rangle \qquad j = 0, \dots, n$$

$\Longrightarrow p_n(x) = \beta_0^* \psi_0(x) + \dots + \beta_n^* \psi_n(x)$ is the unique minimizer. $\square$

Theorem $p_n \in \mathcal{P}_n$ is the best fit polynomial for $f : [a,b] \to \mathbb{R}$ in 2-norm if and only if $f - p_n$ is orthogonal to every $p \in \mathcal{P}_n$, i.e.

$$\langle f - p_n, q_n \rangle = 0 \quad \text{for all } q_n \in \mathcal{P}_n$$



Practical computation for given f:

$\varphi_0, \dots, \varphi_n$ orthogonal, $\psi_j = \dfrac{\varphi_j}{\|\varphi_j\|_2}$

$$\beta_j^* = \langle f, \psi_j \rangle$$

$$p_n(x) = \beta_0^* \psi_0(x) + \dots + \beta_n^* \psi_n(x)$$

$$= \beta_0^* \|\varphi_0\|_2^{-1} \varphi_0(x) + \dots + \beta_n^* \|\varphi_n\|_2^{-1} \varphi_n(x)$$

$$= \gamma_0^* \varphi_0(x) + \dots + \gamma_n^* \varphi_n(x)$$

$$\gamma_j^* = \frac{\beta_j^*}{\|\varphi_j\|_2} = \frac{\langle f, \psi_j \rangle}{\|\varphi_j\|_2} = \frac{\langle f, \varphi_j \rangle}{\langle \varphi_j, \varphi_j \rangle}$$

Example: Best fit polynomial in $\mathcal{P}_2$ of $f : x \mapsto e^x$ over $(0,1)$, with $w(x) \equiv 1$

$$\varphi_0(x) = 1, \quad \varphi_1(x) = x - \frac{1}{2}, \quad \varphi_2(x) = x^2 - x + \frac{1}{6}$$

$$\gamma_0^* = \frac{\langle f, \varphi_0 \rangle}{\langle \varphi_0, \varphi_0 \rangle} = \frac{\int_0^1 e^x \cdot 1 \, dx}{\int_0^1 1 \cdot 1 \, dx} = \frac{e^1 - e^0}{1} = e - 1$$

$$\gamma_1^* = \frac{\langle f, \varphi_1 \rangle}{\langle \varphi_1, \varphi_1 \rangle} = \frac{\int_0^1 e^x (x - \frac{1}{2}) \, dx}{\int_0^1 (x - \frac{1}{2})^2 \, dx} = 18 - 6e$$

$$\gamma_2^* = 210e - 570$$

$$\Rightarrow p_2(x) = \underline{(e-1) + (18-6e)\left(x - \tfrac{1}{2}\right) + (210e - 570)\left(x^2 - x + \tfrac{1}{6}\right)}$$

Numerical integration / quadrature §10

$f : [a,b] \to \mathbb{R}$ continuous & diff'able

$$\int_a^b w(x) f(x) \, dx \qquad w(x) > 0 \text{ for } x \in (a,b)$$

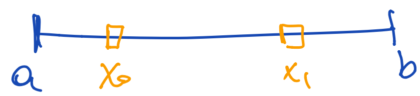
Newton-Cotes allowed us to compute integrals exactly for polynomials up to degree n. In Newton-Cotes, we fixed $x_0, \dots, x_n$ as being uniform. In Gauss quadrature, we allow those points to change and hope to find more accurate quadrature rules

$$\int_a^b w(x) f(x) \, dx \approx \sum_{i=0}^{n} w_i f(x_i)$$

We'll find that the nodes $x_0, \dots, x_n$ can be chosen in an optimal way that allows us to integrate polynomials of degree $2n+1$ exactly using the quadrature rule (compared to n for Newton-Cotes)

These optimal $x_0, \dots, x_n$ will be the roots of orthogonal polynomials. Quadrature nodes for Gauss:

$\underline{n=1}$



"Gauss points"

$\underline{n=2}$