Eg 1.2 Consider the eqn. $f(x) = 0$ on $[1,2]$ with $f(x) = e^x - 2x \mp 1$.

This eqn has soln. $\xi$, which is a fixed pt. of $g(x) = \ln(2x+1)$.

Note that $g$ is defined and cts and diff-able on $[1,2]$ ~~and diff-able on (1,2)~~.

By the mean value thm, for any $x, y \in [1,2]$ $\exists\ \eta \in [1,2]$ s.t.

$$g(x) - g(y) = g'(\eta)(x-y).$$

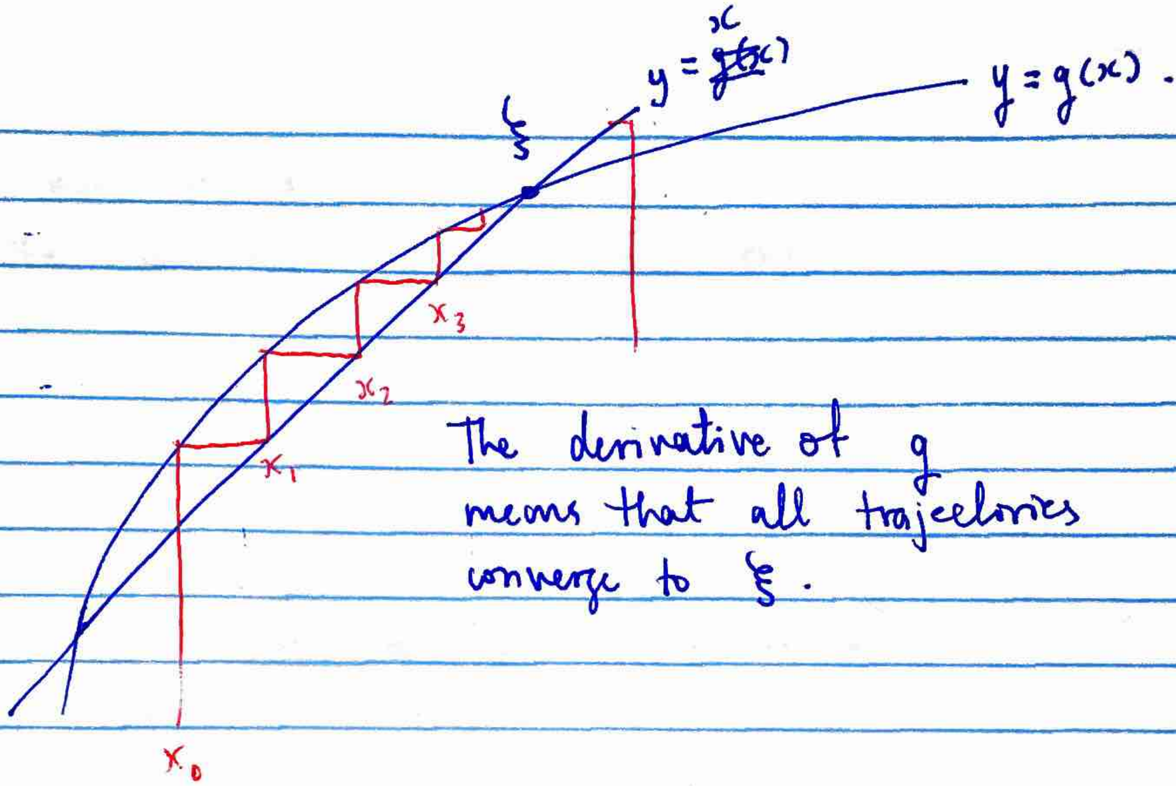
And
$$g'(x) = \frac{2}{2x+1}$$
$$g''(x) = -\frac{4}{(2x+1)^2} < 0 \quad \text{on } [1,2]$$

$\therefore$ $g'$ is monotone decreasing on $[1,2]$. and hence

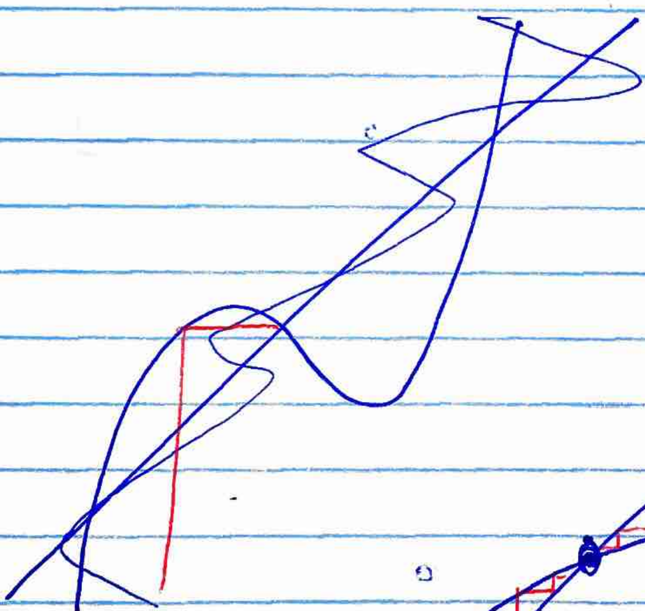
$$g'(1) \geq g'(\eta) \geq g'(2).$$
$$\frac{2}{3} \geq g'(\eta) \geq \frac{2}{5}.$$

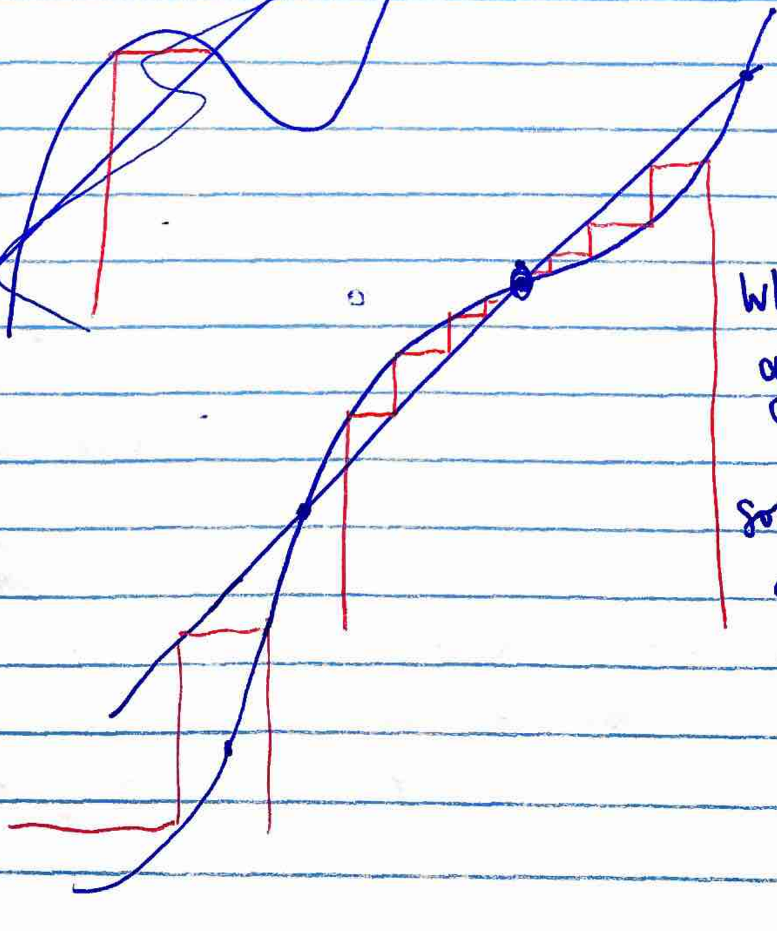
$$\therefore \quad |g'(\eta)| < \frac{2}{3}.$$

$$|g(x) - g(y)| \leq |g'(\eta)|\,|x-y| < \frac{2}{3}|x-y| \quad \therefore\ L = \frac{2}{3}.$$




The derivative of g means that all trajectories converge to $\xi$.





When $|g'(x)|$ goes above and below 1, some fixed pts attract the $x_n$ and some don't.

By ~~thm~~ contraction mapping thm, the sequence $x_k$ defined by

$$x_{k+1} = \ln(2x_k + 1)$$

converges to $\xi$ for any $x_0 \in [1,2]$.

We also have a local version of the contraction mapping thm.

<u>Thm 1.5</u> : Suppose $g$ is a real, bdd fn, cts on $[a,b]$, and $g(x) \in [a,b]$ $\forall$ $x \in [a,b]$.

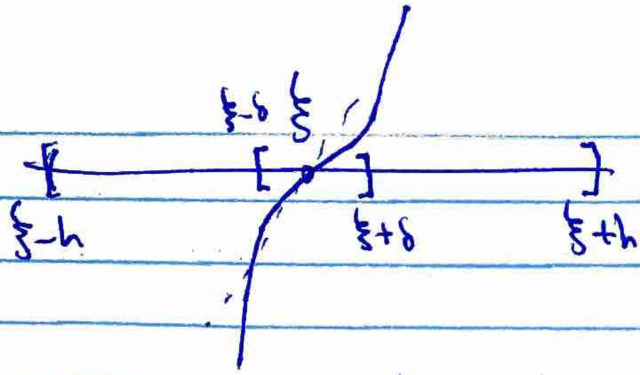
Let $\xi = g(\xi)$ be ~~the~~ a fixed pt. (guaranteed to exist).

Suppose $g$ is diff-able in some nbhd of $\xi$ and that $|g'(\xi)| < 1$.

Then $(x_k)$ defined by $x_{k+1} = g(x_k)$ converges to $\xi$ as $k \to \infty$, provided that $x_0$ is sufficiently close to $\xi$.

<u>Proof</u>: By hypothesis, $\exists$ $h > 0$ s.t. $g'$ is cts in the interval $[\xi - h, \xi + h]$. Since $|g'(\xi)| < 1$, we can find an smaller interval

$$I_\delta = [\xi - \delta, \xi + \delta], \text{ where } \delta \leq h.$$




s.t. $|g'(x)| \le L < 1$ for all $x \in I_\delta$.

Now suppose $x_k$ lies in $I_\delta$. Then

$$x_{k+1} - \xi = g(x_k) - g(\xi)$$
$$= g'(\eta_k)(x_k - \xi) \quad \text{(MVT)}.$$

where $\eta_k$ lies in $(x_k, \xi)$, hence $\eta_k \in I_\delta$.

$$\Longrightarrow |x_{k+1} - \xi| \le |g'(\eta_k)|\,|x_k - \xi|$$
$$< L\,|x_k - \xi|.$$

~~A simple induction shows~~

This shows that $x_{k+1} \in I_\delta$ also.
By a simple induction, if $x_0 \in I_\delta$ then we will have $x_k \in I_\delta$ $\forall$ $k \ge 0$, and hence

$$|x_{k+1} - \xi| < L\,|x_k - \xi| < \dots < L^{k+1}\,|x_0 - \xi|.$$

This implies $x_k \to \xi$ as $k \to \infty$ $\square$.

Thm 1.6: Suppose that $\xi = g(\xi)$, $g$ has cts derivative in a nbhd of $\xi$, and let $|g'(\xi)| > 1$.

Then the sequence $(x_k)$ defined by $x_{k+1} = g(x_k)$ does not converge to $\xi$ for any $x_0 \neq \xi$.

Proof: Suppose $x_0 \neq \xi$., we can find an interval $I_\delta = [\xi - \delta, \xi + \delta]$ in which

$|g'(x)| \geq L > 1$.

If $x_k$ lies in $I_\delta$ then

$$|x_{k+1} - \xi| = |g(x_k) - g(\xi)|$$
$$= |g'(\eta_k)(x_k - \xi)| \geq L|x_k - \xi|.$$

for $\eta_k$ in $(x_k, \xi)$. If $x_{k+1}$ is still in $\xi$ then we have

$$|x_{k+2} - \xi| \geq L^2 |x_k - \xi|.$$

$\vdots$

Similarly, $|x_{k+m} - \xi| \geq L^m |x_k - \xi|$.

Since $L > 1$, then $|x_{k+m} - \xi|$ is getting larger and we must eventually have $x_{k+m} \notin I_\delta$.

Hence, there is no value $k_0 = k_0(\delta)$ s.t.
$|x_k - \xi| \leq \delta \quad \forall\, k \geq k_0(\delta)$.

$\therefore\ x_k \not\to \xi$.

$\square$

~~We now disc~~
This leads to a classification of fixed pts.

Defn 1.3: $\xi$ is a stable fixed pt if
$x_k \to \xi$ for $x_0$ sufficiently close

and $\xi$ is an unstable fixed pt. if
$x_k \not\to \xi$ for any $x_0 \neq \xi$, sufficiently close.

From the theorems we know that

$$\boxed{\begin{array}{l} |g'(\xi)| < 1 \implies \text{stable fixed pt.} \\ |g'(\xi)| > 1 \implies \text{unstable fixed pt.} \end{array}}$$

We can measure speed of convergence via

$$\lim_{k\to\infty} \frac{|x_{k+1}-\xi|}{|x_k-\xi|} = \lim_{k\to\infty} \frac{|g(x_k)-g(\xi)|}{|x_k-\xi|}$$

$$= \lim_{k\to\infty} |g'(\eta_k)| = |g'(\xi)|.$$

(Recall that $\eta_k$ between $x_k$ and $\xi$)

If $\lim_{k\to\infty} \frac{|x_{k+1}-\xi|}{|x_k-\xi|} = \mu$, $\mu \in (0,1)$

we say that $x_k \to \xi$ linearly.

If $\mu = 1$, sublinearly

and if $\mu = 0$, superlinearly.

Eg. 1.5. $g(x) = \frac{1}{2}(x^2+c)$. , $c \in \mathbb{R}$ fixed constant

Fixed pts must satisfy

$$x^2 - 2x + c = 0.$$

$$\Rightarrow x = \frac{2 \pm \sqrt{4-4c}}{2} = 1 \pm \sqrt{1-c}.$$

(Must have $c \leq 1$).

Call $\xi_1 = 1 - \sqrt{1-c}$

$\xi_2 = 1 + \sqrt{1-c}$.

$\xi_1 < 1 < \xi_2$.

$g'(x) = x$ $\Rightarrow$ $\cancel{|g'(\xi_1)| < 1 \text{ and } |g'(}$

$\therefore |g'(\xi_2)| > 1. \Rightarrow \xi_2$ unstable.

But $|g'(\xi_1)| = |1 - \sqrt{1-c}|$

$< 1$ if $1 - \sqrt{1-c} > -1$.

$\sqrt{1-c} < 2$.

$1 - c < 4$

$\boxed{c > -3}$.

$> 1$ if $1 - \sqrt{1-c} < -1 \Rightarrow \boxed{c < -3}$.

$\therefore$ $\xi_1$ stable iff $\Leftrightarrow$ $c \in (-3, 1)$
$\xi_1$ unstable if $c < -3$.

Note that, if $c \in (-3,1)$ and $x_0$ close to $\xi_1$

$$\lim_{k \to \infty} \frac{|x_{k+1} - \xi_1|}{|x_k - \xi_1|} = |f'(\xi_1)|. = |\xi_1|.$$

If $c$ is close to 1 then convergence is linear, since $|\xi_1| = |1 - \sqrt{1-c}| \approx 1$.
(also, the larger $\mu$, the slower the convergence).

If $c = 0$ then $\xi_1 = 0$ and the convergence is <u>superlinear</u>.