Inner products between functions, $f, g : [a,b] \to \mathbb{R}$

$$\langle f, g \rangle := \int_a^b w(x) f(x) g(x)\, dx \qquad w(x) > 0,\ x \in (a,b)$$

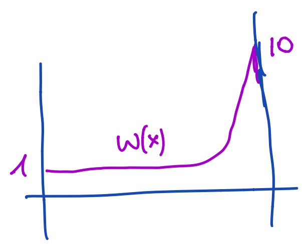
induced norm:

$$\|f\| = \sqrt{\langle f, f \rangle} \quad \text{is a norm}$$

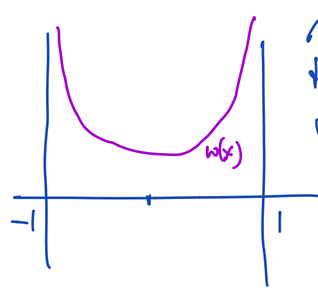
$$= \sqrt{\int w(x) f(x)^2\, dx}, \qquad \text{"2-norm" of functions, depends on the weight } w(x)$$

Examples for weights:

-) puts more "weight" on the points on the right



-) $w(x) \equiv 1 \longrightarrow$ usual 2-norm / 2-inner product
-) $a = -1,\ b = 1,\quad w = \dfrac{1}{\sqrt{1-x^2}}$



makes points closer to ± 1 more important

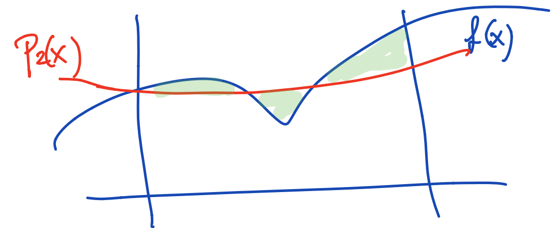
We say that f, g are <u>orthogonal</u> if $\langle f, g \rangle = 0$. Note that this depends on $w(x)$!

Best approximation in 2-norm

Given $f:[a,b] \longrightarrow \mathbb{R}$. Find $p_n \in P_n$ such that

$$\|f-p_n\|_2 = \inf_{q_n \in P_n} \|f-q_n\|_2$$

p_n is "closest" to f in the 2-norm, $\|f\|_2 = \sqrt{\int_a^b w(x) f^2(x)\,dx}$



Example: $f:[0,1] \longrightarrow \mathbb{R}$, $p_n(x) = c_0 + c_1 x + \ldots + c_n x^n$

$w(x) \equiv 1$ $\quad c_i \in \mathbb{R}$

$$\min_{p_n \in P_n} \|f-p_n\|_2 = \left(\int_0^1 \left(f(x)-p_n(x)\right)^2 dx \right)^{\frac{1}{2}}$$

square is monotone, so we can square both sides without changing the minimum.

$$\min_{c_0, c_1, \ldots, c_n} E(c_0, \ldots, c_n) = \int_0^1 \left(f(x)-p_n(x)\right)^2 dx$$

$$= \int_0^1 f(x)^2 dx - 2 \sum_{j=0}^{n} c_j \int_0^1 f(x) x^j dx +$$

$$+ \sum_{j=0}^{n} \sum_{k=0}^{n} c_j c_k \int_0^1 x^k x^j dx$$

At the minimum, all partial derivatives of E with respect to c_j are zero, which leads to a linear system for c_j:

$$\sum_{k=0}^{n} M_{jk} c_k = b_j \qquad j = 0, \dots, n$$

n+1 equations in
n+1 unknowns

$$b_j = \int_0^1 f(x) x^j \, dx = \langle f, x^j \rangle$$

$$M_{jk} = \langle x^k, x^j \rangle = \int_0^1 x^{k+j} \, dx = \frac{1}{k+j+1}$$

The matrix $(M_{jk})_{j,k=0,\dots,n}$ is the Hilbert matrix, which is poorly conditioned — so it is prone to errors and we want to avoid using it.

Orthogonal polynomials

$\{1, x, x^2, \dots, x^n\}$ is a basis for P_n. If we had a different basis $\{\varphi_0, \varphi_1, \dots, \varphi_n\}$ that is orthogonal, the system to be solved for optimal 2-norm interpolation is diagonal.

<u>Def:</u> Given a weight function $w(x)$, $w>0$, φ_j polynomials are orthogonal on (a,b) with respect to $w(x)$ if φ_j has degree j, and

$$\langle \varphi_j, \varphi_k \rangle = \int_a^b w(x) \varphi_j(x) \varphi_k(x) \, dx = \begin{cases} 0 & \text{if } j \neq k \\ \neq 0 & \text{else} \end{cases}$$

<u>Example:</u> $\{\varphi_0, \varphi_1, \varphi_2\}$ on $[0,1]$ with respect to $w(x) \equiv 1$

$\varphi_0(x) \equiv 1$

$\varphi_1(x) = x - c_0 \varphi_0(x)$ to compute $c_0 \in \mathbb{R}$, we want

$$0 = \langle \varphi_0, \varphi_1 \rangle = \langle 1, x - c_0 1 \rangle$$

$$= \int_0^1 x \cdot 1 \, dx - c_0 \int_0^1 1 \cdot 1 \, dx = \frac{1}{2} - c_0$$

$\Rightarrow c_0 = \frac{1}{2}$, and thus

$\varphi_1(x) = x - \frac{1}{2}$

$\varphi_2(x) = x^2 - d_1 \varphi_1(x) - d_0 \varphi_0(x)$

$$0 = \langle \varphi_2, \varphi_0 \rangle = \int_0^1 \left(x^2 - d_1 \varphi_1(x) - d_0 \varphi_0(x)\right) \cdot \varphi_0(x)\, dx$$

$$= \int_0^1 x^2\, dx - d_0 \int_0^1 1\, dx = \frac{1}{3} - d_0$$

$$\Rightarrow d_0 = \frac{1}{3}$$

$$0 = \langle \varphi_2, \varphi_1 \rangle = \int_0^1 \left(x^2 - d_1 \varphi_1(x) - d_0 \varphi_0(x)\right) \varphi_1(x)\, dx$$

$$= \int_0^1 x^2 \left(x - \frac{1}{2}\right) dx - d_1 \int_0^1 \left(x - \frac{1}{2}\right)^2 dx$$

$$= \frac{1}{4} - \frac{1}{6} - d_1 \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{4}\right)$$

$$= \frac{6-4}{24} - d_1 \left(\frac{1}{12}\right)$$

$$\Rightarrow \underline{d_1 = 1}$$

$$\varphi_2(x) = x^2 - d_1 \varphi_1(x) - d_0 \varphi_0(x) = x^2 - \left(x - \frac{1}{2}\right) - \frac{1}{3}$$

$$= x^2 - x + \frac{1}{6}$$