$$\int_a^b f(x)\,dx \approx \int_a^b p_n(x)\,dx = \sum_{k=0}^{n} \underbrace{w_k}_{\text{quadrature weights}} \underbrace{f(x_k)}_{\text{quadrature point/nodes}}$$

$n=1 \longrightarrow$ trapezoidal rule

$n=2 \longrightarrow$ Simpson's rule

<u>Simpson's rule:</u>

$$|E_2(f)| = \left| \int_a^b f(x)\,dx - \int_a^b p_2(x)\,dx \right|$$

$$\leq \frac{(b-a)^4}{196} M_3 \qquad M_3 = \max_{x\in[a,b]} |f^{(3)}(x)|$$

<u>Theorem:</u> (Improved estimate for Simpson), $f:[a,b]\to\mathbb{R}$, $f^{(4)}$ exists & continuous

$$\Longrightarrow \underbrace{\int_a^b f(x)\,dx - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]}_{E_2(f)} = -\frac{(b-a)^5}{2880} f^{(4)}(\xi)$$

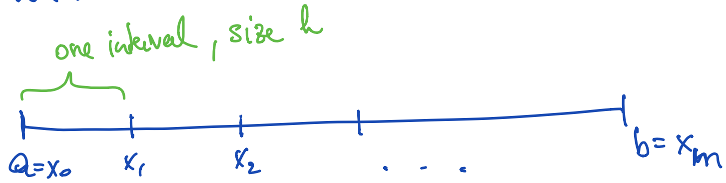
$$\xi \in [a,b]$$

$$\Longrightarrow |E_2(f)| \leq \frac{(b-a)^5}{2880} M_4$$

$$M_4 = \max_{x\in[a,b]} |f^{(4)}(x)|$$

As we've observed, large polynomial degree on uniform nodes leads to the Runge phenomenon, degrading interpolation, and thus Newton–Coles integration/quadrature.

We thus use composite rules, which cut the interval $[a,b]$ into smaller, equal size subintervals, and use Newton-Cotes on each subinterval.



on each interval use trapezoidal rule: $\qquad x_i = a + ih \quad i=0,1,\dots m$

$$h = \frac{b-a}{m}$$

$$\int_{x_{i-1}}^{x_i} f(x) \approx \frac{1}{2} h \left(f(x_{i-1}) + f(x_i) \right)$$

$$\Longrightarrow \int_a^b f(x)\,dx \approx h \left[\frac{1}{2} f(x_0) + f(x_1) + \dots + f(x_{m-1}) + \frac{1}{2} f(x_m) \right]$$

"composite trapezoidal rule"

$$\mathcal{E}_1(f) = \int_a^b f(x)\,dx - h \left[\frac{1}{2} f(x_0) \dots\dots + f(x_{m-1}) + \frac{1}{2} f(x_m) \right]$$

$$= \sum_{i=1}^{m} \left[\int_{x_{i-1}}^{x_i} f(x)\,dx - \frac{1}{2} h \left(f(x_{i-1}) + f(x_i) \right) \right]$$

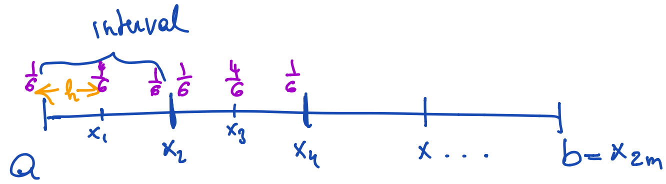
$$\Longrightarrow |\mathcal{E}_1(f)| \leq \frac{h^3}{12} \sum_{i=1}^{m} \max_{x \in [x_{i-1}, x_i]} |f''(x)|$$

$$\leq \frac{(b-a)^3}{m^3 \cdot 12} m \cdot M_2$$

$$M_2 = \max_{\xi \in [a,b]} |f''(\xi)|$$

$$= \frac{(b-a)^3}{12 m^2} M_2$$

Composite Simpson rule:



$$h = \frac{b-a}{2m}$$

$$\int_a^b f(x) \approx \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots + f(x_{2m})\right]$$

$$|\mathcal{E}_2(f)| = \left| \int_a^b f(x)\,dx - \text{Simpson} \right| \leq \frac{(b-a)^5}{2880\,m^4} M_4$$

---

§9 Polynomial approximation in the 2-norm

Overview: We will introduce an inner product on $\mathbb{P}_n$, the space of polynomials of degree $\leq n$. This allows us to define angles and in particular, when polynomials are orthogonal to each other.

Inner product on a linear space V over $\mathbb{R}$ is a map $\langle \cdot, \cdot \rangle \longrightarrow \mathbb{R}$ that satisfies for all $f, g, h \in V$, $\lambda \in \mathbb{R}$:

- $\langle f+g, h \rangle = \langle f, h \rangle + \langle g, h \rangle$
- $\langle \lambda f, g \rangle = \lambda \langle f, g \rangle$
- $\langle f, g \rangle = \langle g, f \rangle$
- $\langle f, f \rangle > 0$ if $f \neq 0$

We say that f, g are orthogonal if $\langle f, g \rangle = 0$

Inner products induce norms through:

$$\|f\| = \sqrt{\langle f, f \rangle}$$

<u>Examples:</u> 1.) $V = \mathbb{R}^n$ with $\langle x, y\rangle = x^T y = \sum_{i=1}^n x_i y_i$

Induced norm is

$$\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{\sum_{i=1}^n x_i^2} = \|x\|_2$$

2-norm

Note that while the 2-norm is induced from an inner product, the 1-norm and the ∞-norm are <u>not</u>.

2.) The space of continuous functions over $[a,b]$ is a function space with inner product

$$\langle f, g\rangle = \int_a^b f(x)\, g(x)\, dx$$

As example for orthogonal functions with respect to this inner product on $[-\pi, \pi]$, consider $P_{2k}(x) = \cos(kx)$, $P_{2k+1}(x) = \sin(kx)$ $\quad k = 0, 1, 2, \ldots$

These are orthogonal since

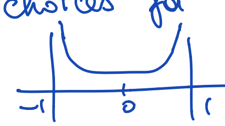
$$\int_{-\pi}^{\pi} P_l(x) P_k(x)\, dx = \begin{cases} 0 & \text{if } l \neq k \\ (P_k, P_k) > 0 & \text{if } l = k \end{cases}$$

3.) We can consider a weighted inner product between functions

$$\langle f, g\rangle = \int_a^b w(x)\, f(x)\, g(x)\, dx \quad \text{with} \quad w : (a,b) \to \mathbb{R}, \; w \geq 0.$$

For $w(x) = 1$ for all $x \in [a,b]$ this reduces to the 2. Example. Other choices for w are:

$w(x) = \frac{1}{\sqrt{1+x^2}}$ on $[-1, 1]$:



The introduction of w is also important on unbounded intervals $I \subset \mathbb{R}$, e.g. $I = \mathbb{R}_{\geq 0}$.

Best approximation in L_2-norm:

Given $f : [a,b] \longrightarrow \mathbb{R}$, find $p_n \in \mathbb{P}_n$ such that

$$\|f - p_n\|_2 = \inf_{q \in \mathbb{P}_n} \|f - q\|_2 ,$$

where

$\|\cdot\|_2$ is induced by (weighted) inner product between functions on $[a,b]$.

Graphically: Find p_n such that

$$\int_a^b (f - p_n)^2 (x) \quad \text{is minimal}$$

quantity for the difference between f and p_n (in)