Hermite interpolation:

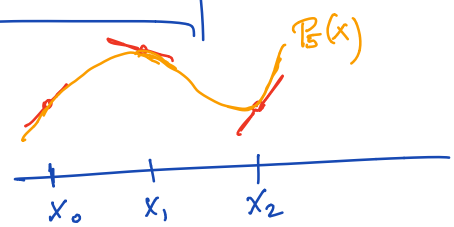
$x_0, x_1, \dots, x_n \in \mathbb{R}$

$y_0, \dots, y_n, z_0, \dots, z_n \in \mathbb{R}$

Find $p_{2n+1} \in \mathcal{P}_{2n+1}$ such that $p_{2n+1}(x_i) = y_i \quad i = 0, \dots, n$

$p'_{2n+1}(x_i) = z_i \quad i = 0, \dots, n$

$$p_{2n+1} = \boxed{\sum_{k=0}^{n} H_k(x)\, x_k + K_k(x)\, z_k}$$ is Hermite interpolation polynomial.



$$H_k(x_i) = \begin{cases} 1 & k = i \\ 0 & \text{else} \end{cases} \qquad i, k = 0, \dots, n$$

$$H'_k(x_k) = 0$$

$$K_k(x_i) = 0$$

$$K'_k(x_i) = \begin{cases} 1 & i = k \\ 0 & \text{else} \end{cases}$$

$$H_k(x), K_k(x) \in \mathcal{P}_{2n+1}$$

Theorem: $n \geq 0$, $f : [a,b] \to \mathbb{R}$ cont'ous, its $2n+2$nd derivative is continuous, Then the Hermite interpolant of f satisfies

$$f(x) - p_{2n+1}(x) = \frac{f^{(2n+2)}(\xi)}{(2n+2)!} \left(\pi_{n+1}(x)\right)^2, \quad \xi \in [a,b]$$

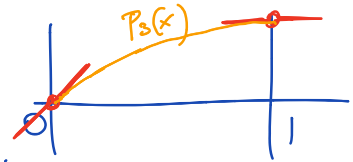
$$\pi_{n+1}(x) = \prod_{k=0}^{n} (x - x_k)$$

$$\boxed{M_{2n+2} = \max_{z \in [a,b]} |f^{(2n+2)}(z)|}$$

$$\Longrightarrow |f(x) - p_{2n+1}(x)| \leq \frac{M_{2n+2}}{(2n+2)!} \left(\pi_{n+1}(x)\right)^2$$

Example: Find $p_3 \in \mathcal{P}_3$ s.t.

$$p_3(0) = 0, \quad p_3(1) = 1, \quad p_3'(0) = 1, \quad p_3'(1) = 0$$

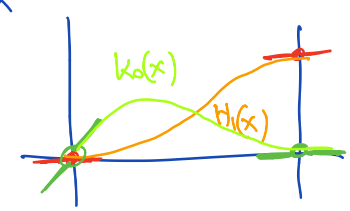


$$p_3(x) = H_1(x) + K_0(x)$$

$$H_1(x) = L_1(x)^2\left(1 - 2L_1'(x_1)(x - x_1)\right) = x^3(3-2x)$$

$$K_0(x) = L_0(x)^2(x - x_0) = (1-x)^2 x$$

$$\boxed{p_3(x) = -x^3 + x^2 + x}$$



Alternative way using monomial basis $\{1, x, x^2, x^3\}$

$$p_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$\left.\begin{aligned} p_3(0) &= a_0 = 0 \\ p_3(1) &= a_0 + a_1 + a_2 + a_3 = 1 \\ p_3'(0) &= a_1 = 1 \\ p_3'(1) &= a_1 + 2a_2 + 3a_3 = 0 \end{aligned}\right\}$$

4 equations, 4 unknowns

$\Longrightarrow$ has a solution since matrix is invertible.

§7 Numerical integration / quadrature (§7.1 - 7.4)

Simple integrals can be computed exactly, e.g.

$$\int_0^1 e^x \, dx \quad , \quad \int_0^{\pi} \cos(x)\, dx$$

But, most integrals <u>cannot</u> be computed analytically, e.g.

$$\int_0^1 e^{x^2} dx \quad \text{or} \quad \int_0^{\pi} \cos x^2 \, dx$$

We want to compute integrals numerically and estimate the approximation error.

Newton - Cotes formulae

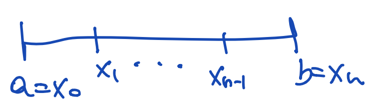
Consider $\int_a^b f(x)\, dx$

Since polynomials are easy to integrate, the idea is to replace f with an interpolating p and integrate the polynomial exactly.

$$\int_a^b f(x)\, dx \approx \int_a^b p_n(x)\, dx$$

For simplicity, assume uniform points $x_i = a + ih \quad i = 0, \dots, n$

$$h = \frac{b-a}{n}$$



Lagrange interpolation:

$$p_n(x) = \sum_{k=0}^{n} L_k(x)\, f(x_k)$$

$$L_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i}$$

$$\Longrightarrow \int_a^b f(x)\, dx \approx \int_a^b \sum_{k=0}^{n} L_k(x) f(x_k)\, dx = \sum_{k=0}^{n} \overbrace{\left(\int_a^b L_k(x)\, dx \right)}^{w_k} f(x_k)$$

$$= \sum_{k=0}^{n} w_k f(x_k)$$

w_k ... quadrature weights

x_k ... quadrature nodes

For each n, we obtain a quadrature rule, also called

<u>Newton-Cotes formula</u>

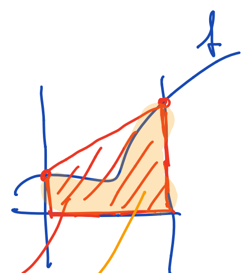
<u>$n=1$</u> Trapezoidal rule

$x_0 = a, \quad x_1 = b$

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{1} w_i f(x_i)$$

$$w_0 = \int_a^b L_0(x)\,dx = \int_a^b \frac{x-b}{a-b}\,dx = \frac{b-a}{2}$$

$$w_1 = \int_a^b L_1(x)\,dx = \frac{b-a}{2}$$



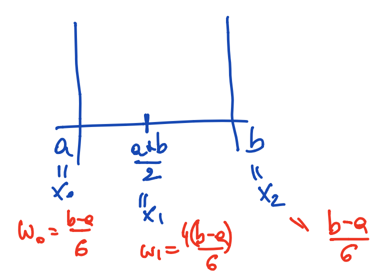
$$\Longrightarrow \int_a^b f(x)\,dx \approx \frac{b-a}{2}\left[f(a) + f(b)\right]$$

<u>$n=2$</u> Simpson's rule

$$p_2(x) = \sum_{k=0}^{2} L_k(x) f(x_k) \qquad w_k = \int_a^b L_k(x)\,dx = \begin{cases} \frac{b-a}{6} & k=0,2 \\ \frac{4(b-a)}{6} & k=1 \end{cases}$$

$$\Longrightarrow \int_a^b f(x)\,dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Rules work well for $n = 1, .., 5, 6$ but are problematic for larger n.



Error estimates: Study the error

$$E_n(f) := \int_a^b f(x)\,dx - \sum_{k=0}^{n} w_k f(x_k)$$

Theorem: $n \geq 1$, $f : [a,b] \to \mathbb{R}$, $f^{(n+1)}$ continuous on $[a,b]$.

Then

$$|E_n(f)| \leq \frac{M_{n+1}}{(n+1)!} \int_a^b |\Pi_{n+1}(x)|\,dx$$

$$M_{n+1} = \max_{x \in [a,b]} |f^{(n+1)}(x)|$$

Proof: $$E_n(f) = \int_a^b [f(x) - p_n(x)]\,dx$$

estimate from poly. interpolation

$$\Longrightarrow |E_n(f)| \leq \int_a^b |f(x) - p(x)|\,dx \leq \frac{M_{n+1}}{(n+1)!} \int_a^b |\Pi_{n+1}(x)|\,dx$$

$\square$.

Error for trapezoidal rule:

$$|E_1(f)| \leq \frac{M_2}{2!} \int_a^b |(x-a)(x-b)|\,dx = \frac{M_2}{2} \int_a^b (x-a)(b-x)\,dx$$

$$= \frac{(b-a)^3}{12} M_2$$

for Simpson's rule: $$|E_2(f)| \leq \frac{(b-a)^4}{196} M_3$$