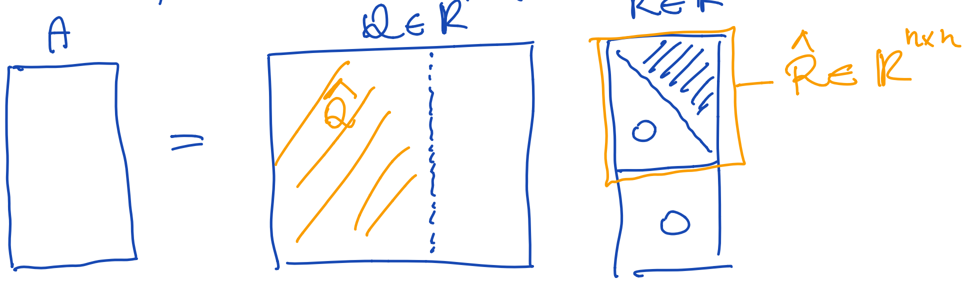
$A \in \mathbb{R}^{m \times n}$, $m \geq n$



$$\begin{bmatrix} * \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$ QR- factorization

$$= \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$ $\hat{Q}\hat{R}$ factorization

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|_2^2$$

$A = QR$

$Q^T A = R$

$$= (Ax-b)^T (Ax-b)$$
$$= (Ax-b)^T Q Q^T (Ax-b)$$
$$= \|Q^T(Ax-b)\|_2^2$$
$$= \|Rx - Q^T b\|_2^2$$

$b \in \mathbb{R}^m$, $b_1 \in \mathbb{R}^n$, $b_2 \in \mathbb{R}^{m-n}$

$$= \left\| \begin{pmatrix} \hat{R}x \\ 0 \end{pmatrix} - \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right\|_2^2$$

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = Q^T b = \begin{pmatrix} \hat{Q}^T \\ \hline //// \end{pmatrix} b$$

$$= \|\hat{R}x - b_1\|_2^2 + \|b_2\|_2^2$$

$\Longrightarrow$ Solve $\hat{R}x = b_1$, $\hat{R} = \begin{pmatrix} \diagdown\!\!\!/\!/\!/ \\ 0 \end{pmatrix}$

$\hat{R}$ invertible $\Longleftrightarrow$ rank$(A) = n$, $A \in \mathbb{R}^{m \times n}$

$A = \boxed{}$, Normal equations for $\min \|Ax-b\|_2^2$

$$A^TAx = A^Tb$$

2-norm of matrix

$$\max_{i=1}^{n} \sqrt{|\lambda_i(A^TA)|} \underset{\text{A symm.}}{=} \max_{i=1}^{n} |\lambda_i(A)|$$

Power method & inverse iteration

Find eigenvalues & eigenvectors of $A \in \mathbb{R}^{n\times n}$

Start with $x_0 \in \mathbb{R}^n$, compute

$$x_{k+1} = Ax_k \qquad k = 0,1,2,3,\dots$$

Thm: (Power method) λ_1 simple eigenvalue of A, $A = A^T$ with $|\lambda_1| > |\lambda_2| \geq \dots \geq |\lambda_n|$, Let $x_0 \in \mathbb{R}^n$ which is not orthogonal to the eigenspace of λ_1. Then $y_k = \dfrac{x_k}{\|x_k\|}$ with $x_{k+1} = Ax_k$ converges to the normalized eigenvector of A corresponding to λ_1.

Remarks: Only for largest eigenvalue, $\lambda_1 = \dfrac{(Ay)^T y}{y^T y}$

y eigenvector

Convergence speed is $\left|\dfrac{\lambda_2}{\lambda_1}\right|$, linear convergence

Inverse power method:

$A \in \mathbb{R}^{n\times n}$, consider $\theta \in \mathbb{R}$ estimate for eigenvalue, close to λ_i, λ_i simple

Then $(A - \Theta I)^{-1}$ has as largest eigenvalue

$\frac{1}{\lambda_i - \Theta}$ $\Longrightarrow$ use power method

$$x_{k+1} = (A - \Theta I)^{-1} x_k \qquad k = 0, 1, 2, \dots$$

$$\Longleftrightarrow (A - \Theta I)\, x_{k+1} = x_k$$

$\longrightarrow$ solve system in each iteration