§1 ITERATIVE SOLUTION OF EQUATIONS

Only few equations have "closed form" solutions, i.e. "analytic solution". E.g. $x^2 - 5x - 7 = 0$ has a closed form solution.

Polynomials of order ≥ 5 have no closed form solutions, e.g.

$$f(x) = x^5 - 4x - 2 = 0$$

has no closed form solution.

$\longrightarrow$ Study existence of solutions for

$$f(x) = 0$$

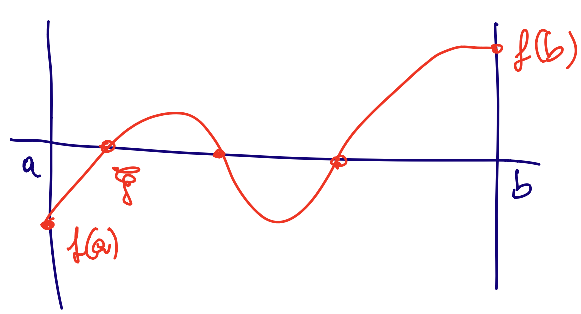
and methods to find solutions ξ.

§1.2: SIMPLE / FIXED POINT ITERATION

Consider $f : [a,b] \to \mathbb{R}$, $a < b$

<u>Thm:</u> f CONTINUOUS, $f(a) \leq 0$, $f(b) \geq 0$ (or vice versa)

$\Longrightarrow \exists\, \xi \in [a,b] : f(\xi) = 0$



<u>Proof:</u> Intermediate value theorem. □

Alternative fixed point formulation

$$g(x) = x$$

[this is equivalent with $f(x)=0$, since $\underbrace{f(x)+x}_{g(x)} = x$]

<u>Thm</u> (Brouwer's fixed point, holds in more general settings)

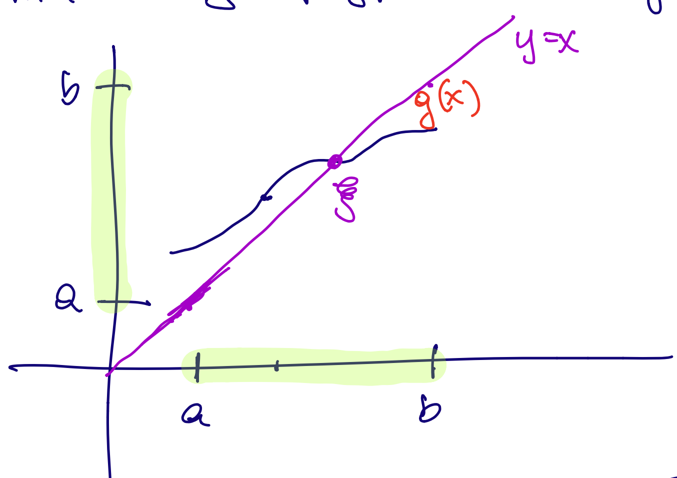
$g: [a,b] \to \mathbb{R}$ continuous, $g(x) \in [a,b]$ $\forall x \in [a,b]$

$\Longrightarrow \exists\, \xi$ in $[a,b]$: $\xi = g(\xi)$.

<u>Proof:</u> $f(x) := x - g(x)$, $f(a) = a - g(a) \le 0$

$f(b) = b - g(b) \ge 0$

$\xrightarrow[\text{Thm}]{\text{previous}} \exists\, \xi : f(\xi) = 0 \Longleftrightarrow g(\xi) = \xi$. $\square$



<u>EXAMPLE:</u> $f(x) = e^x - 2x - 1 \qquad x \in [1,2]$

has a root since $f(1) < 0$, $f(2) > 0$.

1.) $g(x) = e^x - x - 1$ THEN $g(x) = x \Longleftrightarrow f(x) = 0$

$g(x) \in [1,2]$ $\forall x$? No, since $g(1) < 1$

2.) $g(x) = \ln(2x+1)$

$g(x) = x \Longleftrightarrow \ln(2x+1) = x$

$\Longleftrightarrow 2x+1 = e^x \Longleftrightarrow f(x) = 0$

$g(x) \in [1,2] \ \forall x$?

$g(1) = \ln(3) \approx 1.09\ldots \in [1,2]$

$g(2) = \ln(5) \approx 1.7\ldots \in [1,2]$

$\rightarrow g(x) \in [1,2]$ since $\ln$ is monotone

$\Longrightarrow$ Brouwer's thm applies & proves existence of a fixed point.

FIXED POINT ITERATION

$g : [a,b] \to \mathbb{R}$, $g(x) \in [a,b] \ \forall x$, g continuous

(*) Given $x_0 \in [a,b]$, compute

$$\boxed{x_{k+1} = g(x_k) \qquad k = 0,1,2,3,\ldots}$$

ITERATES INDEX

IF CONVERGING, $x_k \to \xi$

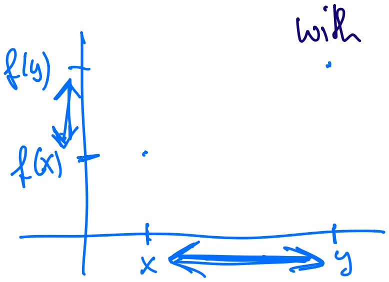
$$\xi = \lim_{k\to\infty} x_k = \lim_{k\to\infty} g(x_{k-1}) = g\left(\lim_{k\to\infty} x_{k-1}\right) = g(\xi)$$

Will it converge?

Def: (contraction) $g : [a,b] \to \mathbb{R}$ is CONTRACTIVE if

$$|g(x) - g(y)| \le L\,|x-y| \quad \forall x,y \in [a,b]$$

with $L \in (0,1)$ "Lipschitz-continuous with constant <1"



Thm: $g : [a,b] \to \mathbb{R}$, $g(x) \in [a,b]$ $\forall x \in [a,b]$

CONTINUOUS, CONTRACTION

$\Longrightarrow \exists!$ FIXED POINT $\xi \in [a,b]$ and the

there exists one and only one

ITERATES x_k FROM $(*)$ CONVERGE TO ξ.

Proof: EXISTENCE ✓

UNIQUINESS: Let ξ, η FIXED POINTS

$$|\xi - \eta| = |g(\xi) - g(\eta)| \le L\,|\xi - \eta|, \quad L < 1$$

$$\Longrightarrow |\xi - \eta| = 0 \Longrightarrow \xi = \eta \ ✓$$

CONVERGENCE:

$x_0 \in [a,b]$

$$|x_k - \xi| = |g(x_{k-1}) - g(\xi)| \le L\,|x_{k-1} - \xi|$$

$$\le L^2\,|x_{k-2} - \xi| \le \dots$$

$$\le L^k\,|x_0 - \xi|$$

$$L < 1 \Longrightarrow L^k \to 0 \text{ as } k \to \infty$$

$$\Longrightarrow x_k \longrightarrow \xi \text{ as } k \to \infty. \qquad \square$$