On a Model of Rotating Superfluids

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Abstract

We consider an energy-functional describing rotating superfluids at a rotating velocity ω, and prove similar results as for the Ginzburg-Landau functional of superconductivity : mainly the existence of branches of solutions with vortices, the existence of a critical ω above which energy-minimizers have vortices, evaluations of the minimal energy as a function of ω, and the derivation of a limiting free-boundary problem.

I Introduction

I.1 The energy functional

The aim of this paper is to study a question that was asked by Yves Pomeau, concerning a model of rotating superfluids. The evolution of a superfluid, such as superfluid helium II at zero temperature, is generally modelled (after some rescaling) by the following nonlinear Schrödinger equation, called the Gross-Pitaevskii equation :

$$-i\hbar\frac{\partial u}{\partial t} = \hbar^2\Delta u + u(1-|u|^2). \tag{I.1}$$

The Gross-Pitaevskii equation is also used to model the evolution of Bose-Einstein condensates. Here u is a complex-valued function characterizing the local state of the superfluid (it is a pseudo wave-function and $0 \le |u| \le 1$). If the superfluid is in a cylindrical bucket of two-dimensional section Ω, smooth, bounded and simply connected, and rotating around a vertical axis at the angular velocity ω; then, its energy, written in the rotating frame, taking into account the Coriolis force, is

$$\int_\Omega \hbar^2|\nabla u + iu\omega\times x|^2 + \frac{1}{2}(1-|u|^2)^2,$$

supplemented with the boundary condition $u=0$ on $\partial\Omega$. Here $x=(x_1,x_2)\in\Omega$ with the origin set at the rotation axis, and $\times$ is the vectorial product in $\mathbb{R}^3$. By $iu\omega\times x$ we mean

the complex-valued vector $iu\omega(x_2, x_1)$, then considering ω as a positive real number. This model could also serve to describe Bose-Einstein condensates, whose evolution is given by the Gross-Pitaevskii equation. For a rotating Bose-Einstein condensate trapped in a harmonic potential, a more realistic model includes a term $\int_\Omega (a(x) - |u|^2)^2$ where $a(x)$ is a quadratic function vanishing on $\partial\Omega$ instead of $\int_\Omega (1 - |u|^2)^2$, (see [CD] and [Af]), but would also lead to the same kind of analysis.
We replace the study of (I.1) by the study of

$$(\mathrm{I.2}) \qquad J(u) = \frac{1}{2} \int_\Omega |\nabla u + iu\omega \times x|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2,$$

over $H_0^1(\Omega, \mathbb{C})$, where ε is a small parameter. If we expand the first term, we obtain

$$(\mathrm{I.3}) \qquad J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2 + \frac{\omega^2}{2} \int_\Omega |u|^2 |x|^2 + \omega \int_\Omega (iu, x_2 u_{x_1} - x_1 u_{x_2}).$$

Here $(.,.)$ denotes the scalar product in $\mathbb{R}^2$, where complex numbers are seen as belonging to $\mathbb{R}^2$. Another minimization problem which can be considered to derive this is the following: minimize a Hamiltonian of the form

$$H = \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2,$$

where $u \in H_0^1(\Omega, \mathbb{C})$, with fixed angular momentum

$$M = \int_\Omega (iu, x \times \nabla u) = \int_\Omega (iu, x_1 u_{x_2} - x_2 u_{x_1}).$$

(The Hamiltonian H and the momentum M are quantities that are conserved in time for the evolution of the type (I.1).) Using a Lagrange multiplier λ, this is equivalent to minimizing

$$H - \lambda M = \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2 - \lambda \int_\Omega (iu, x_1 u_{x_2} - x_2 u_{x_1}).$$

Up to the term $\frac{1}{2}\omega^2 \int_\Omega |u|^2 |x|^2$, this is the same expression as (I.3) for $\omega = \lambda$. Thus, the rotation velocity ω can be seen as the Lagrange multiplier in the previous problem. On the other hand, we shall see that if ω is sufficiently small compared to $\frac{1}{\varepsilon}$, the term $\frac{1}{2}\omega^2 \int_\Omega |u|^2 |x|^2$ is of lower order in the energy, hence can be neglected, since, up to slight adjustments in our proofs, it would lead to the same qualitative results.

Another question that physicists consider is to minimize an energy of the type J or $H - \lambda M$ with a fixed "number of particles" $N = \int_\Omega |u|^2$. Again, this can be taken into account through a Lagrange multiplier. It adds a term which is also negligible when ε is small and ω not too large. Thus, we reduce to the study of J given by the expressions (I.2) or (I.3).

As already mentioned, ε is a small parameter, we will actually make it tend to zero. This corresponds to the case where the characteristic scale of the phenomenon ε, is small compared to the scale of the domain, which is relevant for usual sizes of domains, and is a limit often considered by physicists (see for example [F]). In the physics of Bose-Einstein condensates, ε small corresponds to the "Thomas-Fermi" approximation (see [CD, Af]).

The question is, of course, to find steady states (or critical points) for this energy in the rotating frame, and to describe them. The main feature of rotating superfluids is that, for certain velocities, they exhibit *vortices* : u has some isolated zeros in Ω, and $\frac{u}{|u|}$ has a nonvanishing (topological) degree around these zeros. More precisely, consider a a point where u vanishes and $r > 0$ small such that u does not vanish on $\partial B(a,r)$, then $\frac{u}{|u|}$ is a mapping from $\partial B(a,r)$ to S^1, hence it has a *topological degree*, or winding number (which is the number of turns of the phase of u). This is what is called the degree of the isolated zero. The characteristic scale of the phenomenon is thus ε, the scale of a vortex. In experiments, there can be up to thousands of vortices in the domain. For more details on the physical aspects, one can refer to the physical litterature ([TT, F] for example).

This behaviour of superfluids is very similar to the behaviour of superconductors in an external magnetic field. Actually, we prove here that there is a total analogy between this model and the Ginzburg-Landau model of superconductivity, and that we can adjust our results on the Ginzburg-Landau energy to this functional. The Ginzburg-Landau functional for superconductors is

$$G(u,A) = \int_\Omega \frac{1}{2}|\nabla u - iAu|^2 + \frac{1}{2}|curl\ A - h_{ex}|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2, \tag{I.4}$$

where h_{ex} is the intensity of the applied magnetic field, $A = (A_1, A_2) \in \mathbb{R}^2$ is the vector potential of the magnetic field, and $h = curl\ A$ the induced magnetic field in the material. The first term $\int_\Omega |\nabla u + iu\omega \times x|^2$ is very similar to the term $\int_\Omega |\nabla u - iAu|^2$ in the Ginzburg-Landau functional. Actually, J is even simpler, it only depends on one function, and, as we shall see, the role of the external field h_{ex} is replaced by the angular velocity ω.

In [S1, S2, S3, SS1, SS2, SS3], we studied in details the functional (I.4) and its minimizers, and proved that they exhibited a vortex-structure when $H_{c_1} \leq h_{ex} \leq H_{c_2}$, where H_{c_1} and H_{c_2} are critical values depending on ε. Here, we adjust these results and obtain very similar ones.

Let us emphasize that the main difference between the two problems is the *boundary condition* : here $u = 0$ on $\partial\Omega$ whereas, for Ginzburg-Landau, all functions in H^1 were admissible, so no boundary conditions were imposed. This condition $u = 0$ induces a cost of $\frac{C}{\varepsilon}$ at least in the energy, because u has to be small on a layer of size of the order of ε near $\partial\Omega$. This cost is very large compared to the Ginzburg-Landau energy G. Hence, if we make comparisons with test maps, all the fine information on the behaviour of u in Ω will be hidden by the energetic cost of the boundary layer. The method for solving this problem was suggested to us by Itai Shafrir, and is one that has been introduced by Lassoued and Mironescu [LM] and also used by André and Shafrir in [AS]. It consists in dividing u by ρ, the real-valued function which vanishes at $\partial\Omega$ and minimizes J over the space of real-valued functions. Then, we can prove that J splits as

$$J(u) = J(\rho) + \int_\Omega \frac{\rho^2}{2}|\nabla v|^2 + \frac{\rho^4}{4\varepsilon^2}(1-|v|^2)^2 + \omega \int_\Omega \rho^2(iv, x_2 v_{x_1} - x_1 v_{x_2}), \tag{I.5}$$

where $v = \frac{u}{\rho}$. $J(\rho)$ contains the boundary layer contribution while $J(u) - J(\rho) \ll J(\rho)$ can be studied as the Ginzburg-Landau functional. Let us emphasize again that the ideas of the results are not new, but borrowed from those of [S1, S2, S3, SS1, SS2, AS, LM], and that this paper consists in showing that these ideas remain valid and can be adjusted to this new problem.

I.2 Notations

We study J on $H_0^1(\Omega, \mathbb{C})$. Critical points of J are solutions of the following associated Euler equation :

$$(G.P)\begin{cases} -\Delta u = \frac{u}{\varepsilon^2}(1-|u|^2) + 2i\nabla u.\omega \times x - \omega^2 r^2 u & \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{cases}$$

that we call the Gross-Pitaevskii equation. By the maximum principle, a solution u of (G.P) satisfies $|u| \leq 1$. We write $x = (x_1, x_2)$ and $r = |x|$, $\times$ is the vector product in $\mathbb{R}^3$ while $(,)$ is the scalar product on $\mathbb{R}^2$. ω denotes the rotation-vector perpendicular to Ω in $\mathbb{R}^3$ in the expression $\omega \times x$, otherwise its norm. We write $\nabla^\perp f = (-f_{x_2}, f_{x_1})$. F will denote the functional studied in [BBH], i.e.

$$F(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2.$$

For any subset $V \subset \Omega$, J_V or F_V will denote the energy-functionals restricted to V. The domain $D_{\mathcal{M}}$ over which we perform a local minimization of J, corresponds roughly to the $u \in H_0^1$ for which $F(u) \leq \mathcal{M}|\log\varepsilon|$. $\mathcal{R}$ will denote the space of Radon measures on Ω.

I.3 Statement of the results of existence of branches of solutions

We prove the following results, where the notion of "vortex" will be specified later. In all the paper, ω is considered as a function of ε such that $\omega\varepsilon \to 0$ as $\varepsilon \to 0$, and C denotes some positive constant, independent of ε.

Theorem 1 *Suppose $\Omega = B_R = B(0,R)$. Defining a rotation velocity ω_1 by*

$$\omega_1 = \frac{|\log\varepsilon|}{R^2}, \tag{I.6}$$

there exist $k(\varepsilon) = O(1)$, $k'(\varepsilon) = O(|\log|\log\varepsilon||)$, and $\varepsilon_0(\mathcal{M})$ such that for $\varepsilon < \varepsilon_0$, the following holds :
- if and only if $\omega \leq \omega_1 - k(\varepsilon)$, the minimum of J is $J(\rho) - o(1)$ and if $\omega \leq \omega_1 - k'(\varepsilon)$ any minimizer is vortex-less.
- if $\omega_1 + k(\varepsilon) \leq \omega \leq \omega_1 + O(1)$, there exists a minimizer of J over $D_{\mathcal{M}}$ which is a solution of (G.P). In addition, it has exactly one vortex a of degree one, and $|a| \to 0$ as $\varepsilon \to 0$.

This theorem which is similar to Theorem 1 of [S1], shows that there exists a critical value of ω above which vortices become energetically favourable. The expression of ω_1, equivalent to that of H_{c_1} in [S1], is an explicit function of the size of the domain, and corresponds to the expressions found in physics literature (see [F]).

Theorem 2 *Suppose $\Omega = B_R$, and ω is any function of ε such that $\omega \to +\infty$ as $\varepsilon \to 0$, and $\omega \le C\varepsilon^{-\alpha}$ for some small $\alpha > 0$; then $\forall n \in \mathbb{N}^*$ such that $n < \frac{\mathcal{M}}{\pi}$, and $\forall \varepsilon < \varepsilon_0$, there exists a branch of stable solutions of (G.P) such that :*
1) u has exactly n vortices of degree 1, located at a_i^ε.
2) $|a_i^\varepsilon| \to 0$ as $\varepsilon \to 0$, and if we set $\tilde{a}_i = a_i\sqrt{\omega}$, the $\tilde{a}_i$'s tend to minimize

$$w(x_1, \cdots, x_n) = -\pi \sum_{i \neq j} \log |x_i - x_j| + \pi \sum_i |x_i|^2$$

so that $|a_i| \le \frac{C}{\sqrt{\omega}}$, and $|a_i - a_j| \ge \frac{C}{\sqrt{\omega}}$.
3)

$$J(u) = J(\rho) + \pi n \left(|\log \varepsilon| - R^2\omega\right) + \frac{\pi}{2}(n^2 - n)\log \omega + w(\tilde{a}_1, \cdots, \tilde{a}_n) + Q_n + o(1).$$

The solution with n vortices minimizes J in $D_{\mathcal{M}}$ exactly for $\omega_n \le \omega \le \omega_{n+1}$, where ω_n has an expression of the form

$$\omega_n = \frac{|\log \varepsilon|}{R^2} + \frac{n-1}{R^2}|\log |\log \varepsilon|| + O(1). \tag{I.7}$$

The result can also be reformulated as follows : $\forall n \in \mathbb{N}$, there exists $\varepsilon_0(n)$ such that $\forall \varepsilon < \varepsilon_0(n)$, there exists a branch of stable solutions of (G.P) satisfying 1), 2) and 3).

This theorem is the analogue of Theorem 2 of [S3]. It proves, in the case of a disc, the existence of *branches of stable solutions with n vortices of degree 1.* These solutions coexist for a wide range of ω, their energy follows a simple explicit formula. In addition, they are globally minimizing, i.e. they achieve the minimum over all H_0^1, for $\omega_n \le \omega \le \omega_{n+1}$; this has been proved for the Ginzburg-Landau energy in a forthcoming paper [SS4].
What seems most interesting to us is the minimization of w : this says that we can replace the minimization of J over H_0^1 by the minimization of the explicit function w over Ω^n. After rescaling, the positions of the vortices of our branches of solutions tend to minimize w. Then, the natural question is to ask what minimizers of w look like. This is not so easy to calculate. I. Shafrir and S. Gueron have worked on this problem (see [GS]). They prove that for $n \le 6$, the regular polygons centered at the origin are local (and very likely) global minimizers, for $4 \le n \le 6$ there are other stable critical shapes : the regular "stars" which are regular polygons centered at the origin plus the origin. For $7 \le n \le 11$, they are again local minimizers (and probably global). For higher n, numerically, the minimizers look like series of concentric polygons and then triangular lattices, first concentrated around the origin, then scattered all over Ω, as n increases. Observations have been made (since the 70's) on the vortices in rotating superfluid helium II, which show pictures of vortices

which are exactly the ones described for the minimizers of w : i.e. regular polygons, stars, lattices. One can refer to [TT] for pictures.

Thus, our results agree with the physical observations and theoretical predictions (see [F] for superfluids), and particularly with those found in [BR, CD] on Bose-Einstein condensates. Moreover, they state precise values of the ω_n for which the nth vortex becomes energetically favourable, which seems to be a new result, they say that the vortices are concentrated around 0 at a scale $\frac{C}{\sqrt{\omega}}$ and prove the multiplicity of stable solutions for a given ω around ω_1.

I.4 Methods of the proofs

As the proofs are borrowed from other papers, we only explain their main step s. For Theorems 1 and 2, let us just say that the method consists in splitting J as (I.3) and then splitting $J - J(\rho)$ similarly as in [S1]. The term

$$\int_{B_R} \frac{\rho^2}{2}|\nabla v|^2 + \frac{\rho^4}{4\varepsilon^2}(1-|v|^2)^2,$$

can be replaced by

$$\int_{B_{R'}} \frac{1}{2}|\nabla v|^2 + \frac{1}{4\varepsilon^2}(1-|v|^2)^2 = F_{B_{R'}}(v)$$

where $R' = R - \varepsilon^\beta$ $(0 < \beta < 1)$ and F is the functional studied in [BBH]. Then, we prove that if u is a configuration with a bounded number of vortices a_i of degree d_i, then the angular momentum M can be expressed as :

$$M = \int_{B_R} (iu, x_1 u_{x_2} - x_2 u_{x_1}) \simeq -\int_{B_R} (iv, dv \wedge dX) \simeq -2\pi \sum_i d_i X(a_i) \simeq \pi \sum_i d_i (R^2 - |a_i|^2),$$

where $X = \frac{|x|^2 - R^2}{2}$. Here X plays the same role as ξ in [S1, S2, S3], hence we can perform the same analysis to evaluate the cost and gain of each vortex, and see that vortices will tend to the point of minimum of X (which is the origin). To find our branches of n-vortices solutions, we perform a local minimization exactly as in [S3], over domains of the type

$$U_n = \left\{ u \in H_0^1(B_R, \mathbb{C}) / n|\log\,\varepsilon| < F\left(\frac{u}{\rho}\right) < \left(n + \frac{1}{2}\right)|\log\,\varepsilon| \right\}$$

and prove that it yields a **solution** of (G.P) which has n vortices.

I.5 Statement of the results on global minimizers

The following results are the analogues of those of [SS2] and [SS3] on the Ginzburg-Landau functional.

We assume that $\omega(\varepsilon)$ is such that $\omega \ll \frac{1}{\varepsilon^2}$ and that

$$\lambda = \lim_{\varepsilon \to 0} \frac{|\log\,\varepsilon|}{\omega} \qquad \text{(I.8)}$$

exists and is finite. Then, for any λ, we define the limiting functional E as :

$$E(f) = \frac{\lambda}{2} \int_\Omega |\Delta f + 2| + \frac{1}{2} \int_\Omega |\nabla f|^2, \tag{I.9}$$

over

$$\{f \in H_0^1(\Omega) / \Delta f + 2 \in \mathcal{R}\},$$

where $\mathcal{R}$ is the space of bounded Radon measures on ω.
We study any family (u_ε) of global minimizers of J over $H_0^1(\Omega)$. Such a u_ε is solution of (G.P), therefore one can check that it satisfies

$$div((iu, \nabla u) - \omega \nabla^\perp X) = 0$$

where $X = \frac{|x|^2}{2}$. We will see that we can find a unique $U_\varepsilon \in H_0^1$ such that

$$\nabla^\perp U = (iu, \nabla u) - \omega \nabla^\perp X. \tag{I.10}$$

This equation is the analogue of the second Ginzburg-Landau equation. It yields a relevant quantity U which plays the same role as the induced magnetic field h for Ginzburg-Landau. We shall see how U is related to the total vorticity of u.

Theorem 3 *1) Assume λ exists and is finite, $\omega \ll \frac{1}{\varepsilon^2}$, u_ε minimizes J and U_ε is associated to u_ε by (I.10). Then, as $\varepsilon \to 0$,*

$$\frac{U_\varepsilon}{\omega} \rightharpoonup U_* \quad \textit{weakly in } H_0^1(\Omega),$$

where U_* is the unique minimizer of E, and solution of the following obstacle problem :

$$\begin{cases} U_* = 0 & \text{on } \partial\Omega \\ U_* \le \dfrac{\lambda}{2} & \text{in } \Omega \\ (\Delta U_* + 2)\left(U_* - \dfrac{\lambda}{2}\right) = 0 & \text{in } \Omega \\ \Delta U_* + 2 \ge 0. \end{cases} \tag{I.11}$$

In addition $U_* \in C^{1,\alpha}(\Omega), \forall \alpha < 1$. Moreover,

$$\min J \sim_{\varepsilon \to 0} F(\rho) + \omega^2 E(h_*). \tag{I.12}$$

2) If $\lambda = 0$, then $U_* = 0$, and the convergence is strong in H_0^1. If $\lambda > 0$, for $\varepsilon < \varepsilon_0$, we can find a family of balls $(B_i)_{i \in I_\varepsilon} = (B(a_i, r_i))_{i \in I_\varepsilon}$ such that

$$\left\{x, dist(x, \partial\Omega) \ge \varepsilon^\beta / ||v|(x) - 1| \ge \frac{1}{|\log\, \varepsilon|}\right\} \subset \cup_{i \in I_\varepsilon} B_i, \tag{I.13}$$

$$\sum_{i \in I_\varepsilon} r_i \le \frac{1}{|\log\, \varepsilon|^6}, \tag{I.14}$$

$$\forall i \in I_\varepsilon, \quad \frac{1}{2} \int_{B_i} |\nabla U|^2 \ge \pi |d_i| |\log\, \varepsilon| (1 - o(1)), \tag{I.15}$$

where $d_i = deg(u, \partial B_i)$.
For any such family, if we define $\mu_\varepsilon = \frac{2\pi}{\omega} \sum_{i \in I_\varepsilon} d_i \delta_{a_i}$, *we have*

$$\mu_\varepsilon \rightharpoonup \mu_* = \Delta U_* + 2$$

and

$$\frac{2\pi}{\omega} \sum_{i \in I_\varepsilon} |d_i| \delta_{a_i} \rightharpoonup \mu_*$$

in the sense of measures.
3) If we set $\mathcal{U}_\lambda = \{x \in \Omega / U_*(x) = \frac{\lambda}{2}\}$, *we have* $\mu_* = 2\mathbf{1}_{\mathcal{U}_\lambda}$, *where* $\mathbf{1}_{\mathcal{U}_\lambda}$ *denotes the characteristic function of* $\mathcal{U}_\lambda$. $\mathcal{U}_\lambda = \emptyset \Leftrightarrow \lambda \geq 2max\ \xi_0$ *where* ξ_0 *is the solution of*

$$\text{(I.16)} \qquad \begin{cases} -\Delta \xi_0 = 2 & \text{in } \Omega \\ \xi_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

This theorem is mostly relevant in the intermediate case $\omega = O(|\log \varepsilon|)$ corresponding to $\lambda > 0$. We then isolate the zeroes of u_ε (which are not too close to $\partial\Omega$) in vortex balls B_i and define a vorticity measure μ_ε, proved to be closely related to U_ε. μ_ε converges weakly to μ_* which is a uniform measure of density 2 on a subset $\mathcal{U}_\lambda$ of Ω. Thus, qualitatively, we expect u_ε to have vortices of positive degrees, regularly scattered over $\mathcal{U}_\lambda$ with a density $\sim 2\omega$ when ε is sufficiently small. $\mathcal{U}_\lambda$ is determined by (I.7) which is a free boundary problem. It is a classical obstacle problem (see [R]). If $\partial\mathcal{U}_\lambda$ is smooth (which is not always the case, but is the case at least for almost every value of λ from a result of [BM]), then (I.7) can be rewritten more simply :

$$\begin{cases} U_* = 0 & \text{on } \partial\Omega \\ -\Delta U_* = 2 & \text{in } \Omega \backslash \mathcal{U}_\lambda \\ U_* = \frac{\lambda}{2} & \text{on } \partial\mathcal{U}_\lambda \\ \frac{\partial U_*}{\partial n} = 0 & \text{on } \partial\mathcal{U}_\lambda. \end{cases}$$

The size of the vortex-region $\mathcal{U}_\lambda$ depends on λ. If λ is very large (corresponding to small ω's), then $\mathcal{U}_\lambda = \emptyset$. More precisely, if $\omega \leq \omega_1 \sim \frac{|\log \varepsilon|}{2\max\ \xi_0}$, then $\mathcal{U}_\lambda = \emptyset$, and following [SS1], we could have proved rigorously that u_ε has no vortex in this case. Thus, some $\omega_1 \sim \frac{|\log \varepsilon|}{2\max\ \xi_0}$ or $\lambda = 2\max\ \xi_0$ corresponds to a critical value (first critical velocity), and is compatible with the result of Theorem 1. Indeed, if Ω is a ball $B(0,R)$, then $\xi_0 = \frac{R^2 - |x|^2}{2}$, and thus $2\max\ \xi_0 = R^2$. This theorem generalizes the result of Theorem 1 to arbitrary simply connected geometries.

If $\omega \geq \omega_1$, then $\mathcal{U}_\lambda$ is nonempty and minimizers have vortices. $\mathcal{U}_\lambda$ increases as λ decreases (i.e. as ω increases), until, for $\lambda = 0$, corresponding to $\omega \gg |\log \varepsilon|$, $\mathcal{U}_\lambda = \Omega$, and the vortices fill all Ω. The main difference compared to the result of [SS3] on the Ginzburg-Landau functional is that the limiting measure μ_* always has density 2, whereas in Ginzburg-Landau it had a density $1 - \frac{\lambda}{2}$, thus depending on λ and on the applied field.

(I.12) provides an asymptotic expansion of the minimal energy, in which $F(\rho)$ carries the boundary layer cost of any configuration due to the boundary condition $u = 0$. Indeed,

$F(\rho)=\frac{1}{2}\int_\Omega|\nabla\rho|^2+\frac{1}{2\varepsilon^2}(1-\rho^2)^2$ is of the order of $\frac{\sqrt{2}}{3\varepsilon}l(\partial\Omega)$ as we shall see in Section II, and $F(\rho)=J(\rho)-\frac{1}{2}\int_\Omega\omega^2r^2\rho^2$, hence, as soon as $\omega^2\ll\frac{1}{\varepsilon}$, $F(\rho)\sim J(\rho)$ is the term of highest order in (I.12). In the case of $\omega\gg|\log\varepsilon|$ i.e. when $\lambda=0$, then this theorem only states that $\min J\sim F(\rho)$, and $\frac{U_\varepsilon}{\omega}\to 0$. We are in fact able to get more precise results (adjusted from [SS2]) in the following theorems :

Theorem 4 *Assume $|\log\varepsilon|\ll\omega\ll\frac{1}{\varepsilon}$. Then*

$$J(\rho)-\int_\Omega\frac{\omega^2}{2}r^2+\omega|\Omega|\log\frac{1}{\varepsilon\sqrt{\omega}}(1-o(1))\leq\min_{H_0^1(\Omega,\mathbb{C})}J\leq F(\rho)+\omega|\Omega|\log\frac{1}{\varepsilon\sqrt{\omega}}+O(\omega),$$

where $|.|$ denotes the volume.
If in addition $\omega\leq\frac{C}{\varepsilon^{4/5}}$, then

$$\min_{H_0^1(\Omega,\mathbb{C})}J=F(\rho)+\omega|\Omega|\log\frac{1}{\varepsilon\sqrt{\omega}}(1+o(1)).$$

Theorem 5 *Let $|\log\varepsilon|\ll\omega\leq\frac{C}{\varepsilon^{4/5}}$, and u_ε be a corresponding minimizer of J. Then, for $\varepsilon<\varepsilon_0$, there exists a family of disjoint disks (B_i^ε) with radii each less than $\frac{1}{\sqrt{\omega}}$ and sum less than $|\Omega|\sqrt{\omega}$, such that $|u_\varepsilon|\geq\frac{1}{2}$ on ∂B_i^ε and, if a_i^ε is the center of B_i^ε and $d_i^\varepsilon=deg(\frac{u_\varepsilon}{|u_\varepsilon|},\partial B_i^\varepsilon)$, then*

$$\mu_\varepsilon=\frac{2\pi}{\omega}\sum_i d_i^\varepsilon\delta_{a_i^\varepsilon}\underset{\varepsilon\to 0}{\longrightarrow}2\,dx$$

in the weak sense of measures, where dx is the Lebesgue measure on $\mathbb{R}^2$ restricted to Ω.
Moreover,

$$\pi\sum_i|d_i^\varepsilon|\simeq\pi\sum_i d_i^\varepsilon\simeq\omega|\Omega|,$$

and most of the vortex-energy is concentrated in the balls, i.e.

$$J_{\Omega\setminus\cup_i B_i^\varepsilon}(u_\varepsilon)-F(\rho)=o(J(u_\varepsilon)-F(\rho)).$$

Of course, for any value of ω, we have the trivial solution $u\equiv 0$ which has an energy $\frac{|\Omega|}{4\varepsilon^2}$. We believe that, for ω higher than some critical value $\omega\geq\frac{C}{\varepsilon}$, it becomes minimal.

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II The splitting of the energy

We introduce ρ_ε, for a general domain Ω. It is defined to be the minimizer of the following problem :

$$\text{(II.1)}\qquad\min_{H_0^1(\Omega,\mathbb{R})}\frac{1}{2}\int_\Omega|\nabla\rho|^2+\frac{1}{2\varepsilon^2}(1-\rho^2)^2+\omega^2r^2\rho^2.$$

We will often drop the subscript and write ρ instead of ρ_ε.

Lemma II.1 *ρ_ε satisfies the following :*

$$\rho \in C^\infty \quad 0 \le \rho_\varepsilon \le 1, \qquad |\nabla \rho| \le \frac{C}{\varepsilon} \tag{II.2}$$

(II.3) if $\Omega = B_R$, then ρ is radial and is a solution of $(G.P)$,

$$-\Delta \rho_\varepsilon + \omega^2 r^2 \rho_\varepsilon = \frac{1}{\varepsilon^2}\rho_\varepsilon(1-\rho_\varepsilon^2) \tag{II.4}$$

$$1-\rho_\varepsilon \le Ce^{-\frac{\delta(x)}{2\varepsilon}} + O(\varepsilon^2\omega^2) \qquad \text{where } \delta(x) = \mathrm{dist}(x, \partial\Omega) \tag{II.5}$$

$$J(\rho) \le \frac{\sqrt{2}}{3\varepsilon} l(\partial\Omega) + \frac{\omega^2}{2}\int_\Omega |x|^2 + O(1), \tag{II.6}$$

where $l(\partial\Omega)$ denotes the length of $\partial\Omega$.

$$\int_\Omega (1-\rho^2)^2 \le C(\varepsilon + \omega^4\varepsilon^4). \tag{II.7}$$

Proof : It is well-known since the work of Brezis and Oswald [BO] that, as soon as ε is small enough, there exists a positive minimizer for the functional (II.1), and that it is the only positive solution of

$$\begin{cases} -\Delta\rho + \omega^2 r^2 \rho = \frac{1}{\varepsilon^2}\rho(1-\rho^2) & \text{in } \Omega \\ \rho = 0 & \text{on } \partial\Omega. \end{cases} \tag{II.8}$$

It is also standard that $\rho \le 1$ and $|\nabla\rho| \le \frac{C}{\varepsilon}$. If $\Omega = B_R$, the fact that ρ is radial comes from the uniqueness of the positive solution, and the fact that ρ satifies (G.P) comes from (II.8).

We then prove (II.5). The proof is similar to that of Proposition 2.1 of [AS]. Consider x_0 such that $\delta(x) :=$dist$(x_0, \partial\Omega) > K\varepsilon$, for some K to be determined afterwards. Let ϕ_1 be a positive eigenfunction corresponding to the first eigenvalue λ_1 of $-\Delta$ on $B(0,1)$:

$$\begin{cases} -\Delta\phi_1 = \lambda_1\phi_1 & \text{in } B(0,1) \\ \phi_1 = 0 & \text{on } \partial B(0,1), \end{cases}$$

and satisfying $\phi_1 \le \frac{1}{2}$ on $B(0,1)$.
Let us write $\phi(x) = \phi_1\left(\frac{x-x_0}{K\varepsilon}\right)$, then $\Delta\phi = \frac{\lambda_1}{K^2\varepsilon^2}\phi$ on $B(x_0, K\varepsilon)$. If K is chosen large enough (independent from ε), then

$$\frac{\lambda_1}{K^2}\phi \le \phi(1-\phi^2) - \omega^2\varepsilon^2 r^2\phi \qquad \text{in } B(x_0, K\varepsilon),$$

for small ε, since $\omega\varepsilon \to 0$. Hence,

$$-\Delta\phi \le \frac{\phi}{\varepsilon^2}(1-\phi^2) - \omega^2 r^2 \phi,$$

and thus ϕ is a subsolution for (II.8), implying

$$\rho \ge \phi \quad \text{in } B(x_0, K\varepsilon).$$

Therefore, there exists $1 > a > 0$, independent from ε, such that

$$\rho \geq \ \phi \geq a > 0 \quad \text{in } B(x_0, K\varepsilon).$$

Hence

$$\rho \geq a > 0 \quad \text{in } \tilde{\Omega} := \{x/\delta(x) > \frac{K\varepsilon}{2}\}. \tag{II.9}$$

Now, as in [AS], it is enough to prove the estimate (II.5) on $\tilde{\Omega}$. We prove it by using suitable subsolutions. Consider again $x_0 \in \tilde{\Omega}$, and let $\mu = \mathrm{dist}(x_0, \partial\tilde{\Omega})$. On $B(x_0, \mu)$, we consider w_1, the subsolution of [AS], defined by :

$$w_1(\eta) = th\left(th^{-1}a + \frac{\mu^2 - \eta^2}{3\mu\varepsilon}\right), \quad \text{where } \eta = |x - x_0|.$$

As in [AS], we have $w_1 \geq a$ and

$$-\Delta w_1 \leq \frac{8}{9\varepsilon^2}(1 - w_1^2)w_1 + \frac{4}{3\mu\varepsilon}(1 - w_1^2).$$

As previously, we may consider only $\mu \geq \frac{24\varepsilon}{a}$, then $\frac{4}{3\mu\varepsilon} \leq \frac{a}{18\varepsilon^2} \leq \frac{w_1}{18\varepsilon^2}$ and

$$-\Delta w_1 \leq \left(\frac{8}{9} + \frac{1}{18}\right)\frac{1}{\varepsilon^2}w_1(1 - w_1^2). \tag{II.10}$$

We then define w_2 by $w_2 = w_1 - M\omega^2\varepsilon^2$. From (II.10), as $w_2 \leq w_1$,

$$\begin{aligned}
-\Delta w_2 - \frac{1}{\varepsilon^2}w_2(1 - w_2^2) + \omega^2 r^2 w_2 \ &\leq\ \frac{17}{18\varepsilon^2}(1 - w_1^2)w_1 - \frac{1}{\varepsilon^2}w_2(1 - w_2^2) + \omega^2 r^2 w_2 \\
&\leq\ \frac{17}{18\varepsilon^2}(1 - w_2^2)(w_2 + M\omega^2\varepsilon^2) - \frac{1}{\varepsilon^2}w_2(1 - w_2^2) + \omega^2 r^2 w_2 \\
&\leq\ -\frac{1}{18\varepsilon^2}w_2(1 - w_2^2) + \frac{17}{18\varepsilon^2}(1 - w_2^2)M\omega^2\varepsilon^2 + \omega^2 r^2 w_2.
\end{aligned}$$

But, for ε small enough, $w_2 \geq \frac{a}{2}$, hence

$$\begin{aligned}
-\Delta w_2 - \frac{1}{\varepsilon^2}w_2(1 - w_2^2) + \omega^2 r^2 w_2 \ &\leq\ -\frac{w_2}{\varepsilon^2}\left(\frac{1 - w_2^2}{18} - \frac{17}{18}(1 - w_2^2)\frac{2}{a}M\varepsilon^2\omega^2 - \omega^2\varepsilon^2 r^2\right) \\
&\leq\ -\frac{w_2}{\varepsilon^2}\left(\frac{1 - w_2^2}{18}(1 - o(1)) - \omega^2\varepsilon^2 r^2\right). \qquad \text{(II.11)}
\end{aligned}$$

On the other hand,

$$1 - w_2^2 \geq \frac{1}{2}(1 - w_2) \geq \frac{M}{2}\omega^2\varepsilon^2$$

for ε small enough. Then,

$$\frac{1 - w_2^2}{18}(1 - o(1)) - \omega^2\varepsilon^2 r^2 \geq \frac{M}{40}\omega^2\varepsilon^2 - r^2\omega^2\varepsilon^2 > 0$$

if M is chosen large enough compared to $\max_\Omega r$. Therefore, for a suitable choice of M,

$$-\Delta w_2 - \frac{1}{\varepsilon^2} w_2(1-w_2^2) + \omega^2 r^2 w_2 \le 0 \quad \text{on } B(x_0,\mu),$$

and $w_2 \le a \le \rho$ on $\partial B(x_0,\mu)$, hence w_2 is a subsolution for (II.8) and we deduce that $\rho \ge w_2 \ge w_1 - M\omega^2\varepsilon^2$ on $B(x_0,\mu)$, and, as in [AS],

$$1-\rho \le C\varepsilon^{-\frac{\mu}{2\varepsilon}} + O(\omega^2\varepsilon^2) \quad \text{on } B(x_0,\mu).$$

As $\mu = \delta(x_0) - \frac{1}{2}K\varepsilon$, we obtain that

$$1-\rho(x_0) \le C\varepsilon^{-\frac{\delta(x_0)}{2\varepsilon}} + O(\varepsilon^2\omega^2),$$

and finally, changing C if necessary, this estimate is true on all Ω, which proves (II.5).

For (II.6), it is well-known (see [AS] or [ORS]) that

$$\min_{H_0^1(\Omega,\mathbb{R})} \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-u^2)^2 \le \frac{\sqrt{2}}{3\varepsilon} l(\partial\Omega) + O(1).$$

Hence, by definition of ρ,

$$J(\rho) \le \frac{\sqrt{2}}{3\varepsilon} l(\partial\Omega) + \frac{\omega^2}{2}\int_\Omega |x|^2 + O(1).$$

This implies that

$$\frac{1}{4\varepsilon^2}\int_\Omega (1-\rho^2)^2 + \frac{\omega^2}{2}\int_\Omega r^2\rho^2 \le \frac{C}{\varepsilon} + \frac{\omega^2}{2}\int_\Omega r^2,$$

thus

$$\frac{1}{4\varepsilon^2}\int_\Omega (1-\rho^2)^2 \le \frac{C}{\varepsilon} + \frac{\omega^2}{2}\int_\Omega r^2(1-\rho^2) \le \frac{C}{\varepsilon} + C\frac{\omega^2}{2}\left(\int_\Omega (1-\rho^2)^2\right)^{\frac{1}{2}},$$

from which we deduce (II.7). □

Thus, in the case of a disc domain, ρ_ε is a vortex-less solution of (G.P). As explained in the introduction, the fact that $u=0$ on $\partial\Omega$ induces a cost of $\frac{C}{\varepsilon}$ in the energy. That cost can be, as in [AS], removed by considering $v = \frac{u}{\rho}$. Then, $v \simeq u$ except near the boundary, and the boundary cost is "carried" by ρ. This can be proved by using the fact that the energy splits very conveniently under the decomposition $u = \rho v$, exactly as in [LM] or [AS]. More precisely, we have the following lemma, in which $H^1_{\rho^2}$ denotes the H^1 space with respect to the measure $\rho^2 dx$, and the same for $L^2_{\rho^2}$.

Lemma II.2 *Let $u \in H_0^1(\Omega,\mathbb{C})$. $\exists \varepsilon_0$, $\forall \varepsilon < \varepsilon_0$, $v = \frac{u}{\rho}$ is well-defined, belongs to $H^1_{\rho^2}$ and*

$$J(u) = J(\rho) + \int_\Omega \frac{\rho^2}{2}|\nabla v|^2 + \frac{\rho^4}{4\varepsilon^2}(1-|v|^2)^2 + \omega\int_\Omega \rho^2(iv, x_2 v_{x_1} - x_1 v_{x_2}). \tag{II.12}$$

Proof : Let dx denote the Lebesgue measure on $\mathbb{R}^2$.

Step 1 : We prove that $v \in H^1_{\rho^2}$.
ρ only vanishes on $\partial\Omega$, hence $v = \frac{u}{\rho}$ is well-defined on Ω. Furthermore,

$$\int_\Omega |v|^2 \rho^2 dx = \int_\Omega |u|^2 dx < \infty$$

hence $v \in L^2_{\rho^2}$. Then, $\nabla v = \dfrac{\nabla u}{\rho} - \dfrac{u}{\rho^2}\nabla\rho$. As $u \in H^1_0(\Omega, \mathbb{C})$, $\dfrac{|\nabla u|^2}{\rho^2}$ is $\rho^2 dx$ integrable. On the other hand, we can say from (II.5) that

$$\exists \lambda > 0, \quad \delta(x) \geq \lambda\varepsilon \Longrightarrow \rho_\varepsilon(x) \geq \frac{1}{2}. \tag{II.13}$$

Hence, we have

$$\int_{\{x/\delta(x)\geq\lambda\varepsilon\}} \left|\frac{u}{\rho^2}\nabla\rho\right|^2 \rho^2 \leq C\|\nabla\rho\|^2_{L^\infty} \int_\Omega |u|^2 < \infty, \tag{II.14}$$

while, with (II.5),

$$\begin{aligned}
\int_{\{x/\delta(x)\leq\lambda\varepsilon\}} \left|\frac{u}{\rho^2}\nabla\rho\right|^2 \rho^2 &= \int_{\{x/\delta(x)\leq\lambda\varepsilon\}} \frac{|u|^2|\nabla\rho|^2}{\rho^2} \\
&\leq \|\nabla\rho\|^2_{L^\infty} \int_{\{x/\delta(x)\leq\lambda\varepsilon\}} C\varepsilon^2 \frac{|u|^2}{\delta(x)^2} \\
&\leq C\varepsilon^2 \|\nabla\rho\|^2_{L^\infty} \int_\Omega |\nabla u|^2 < \infty
\end{aligned}$$

where we have used the Hardy inequality.
Hence, we deduce that $\nabla v \in L^2_{\rho^2}$ and $v \in H^1_{\rho^2}$ with $\|v\|_{H^1_{\rho^2}} \leq C(\varepsilon)\|u\|_{H^1_0}$.

Step 2 : We prove the splitting of the energy. This proof is very similar to that of [LM] and [AS], and was suggested by I. Shafrir. For any $t > 0$, we denote $\Omega_t = \{x \in \Omega/\delta(x) > t\}$. For any $t > 0$ sufficiently small, we have

$$\begin{aligned}
\int_{\Omega_t} \frac{1}{2}|\nabla(\rho v)|^2 + \frac{1}{4\varepsilon^2}(1-\rho^2|v|^2)^2 + \frac{\omega^2 r^2}{2}\rho^2|v|^2 &= \int_{\Omega_t} \frac{1}{2}|v|^2|\nabla\rho|^2 + \frac{1}{4}\nabla\rho^2.\nabla(|v|^2-1) + \frac{1}{2}\rho^2|\nabla v|^2 \\
&\quad + \int_{\Omega_t} \frac{1}{4\varepsilon^2}(1-\rho^2+\rho^2(1-|v|^2))^2 + \frac{\omega^2 r^2}{2}\rho^2 + \frac{\omega^2 r^2}{2}\rho^2(|v|^2-1) \\
&= \int_{\Omega_t} \frac{1}{2}|\nabla\rho|^2 + \frac{1}{4\varepsilon^2}(1-\rho^2)^2 + \frac{\omega^2 r^2}{2}\rho^2 + + \frac{1}{4\varepsilon^2}\rho^4(1-|v|^2)^2 + \frac{1}{2}\rho^2|\nabla v|^2 \\
&\quad + \int_{\Omega_t} (|v|^2-1)\left(-\frac{1}{4}\Delta(\rho^2) + \frac{\omega^2 r^2}{2}\rho^2 - \frac{1}{2\varepsilon^2}\rho^2(1-\rho^2) + \frac{1}{2}|\nabla\rho|^2\right) + \frac{1}{2}\int_{\partial\Omega_t} \rho\frac{\partial\rho}{\partial n}(|v|^2-1).
\end{aligned}$$

Now, since ρ satisfies (II.4), we have

$$-\Delta\rho^2 = 2\rho(-\Delta\rho) - 2|\nabla\rho|^2 = -2\omega^2 r^2\rho^2 + \frac{2}{\varepsilon^2}(1-\rho^2)\rho^2 - 2|\nabla\rho|^2,$$

so that

$$\begin{aligned}\int_{\Omega_t} \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2 + \frac{\omega^2 r^2}{2}|u|^2 &= \int_{\Omega_t} \frac{1}{2}|\nabla\rho|^2 + \frac{1}{4\varepsilon^2}(1-\rho^2)^2 + \frac{\omega^2 r^2}{2}\rho^2 \\ &+ \int_{\Omega_t} \frac{1}{2}\rho^2|\nabla v|^2 + \frac{\rho^4}{4\varepsilon^2}(1-|v|^2)^2 + \frac{1}{2}\int_{\partial\Omega_t} \rho\frac{\partial\rho}{\partial n}(|v|^2-1).\end{aligned} \tag{II.15}$$

But, from the properties stated on ρ, we have

$$\left|\int_{\partial\Omega_t} \rho\frac{\partial\rho}{\partial n}|v|^2\right| \leq \int_{\partial\Omega_t} \left|\frac{\partial\rho}{\partial n}\right| \frac{|u|^2}{\rho} \leq C\varepsilon\|\nabla\rho\|_{L^\infty}\int_{\partial\Omega_t} \frac{|u|^2}{\delta(x)}.$$

From the Hardy inequality,

$$\int_\Omega \frac{|u|^2}{\delta(x)^2} \leq C\int_\Omega |\nabla u|^2 < \infty,$$

hence, we can find a sequence $t_n \to 0$ such that

$$\int_{\partial\Omega_{t_n}} \frac{|u|^2}{\delta(x)^2} \leq \frac{C}{t_n|\log\, t_n|},$$

therefore,

$$\int_{\partial\Omega_{t_n}} \frac{|u|^2}{\delta(x)} = \int_{\partial\Omega_{t_n}} \frac{|u|^2}{t_n} \leq \frac{C}{|\log\, t_n|} \longrightarrow 0 \quad \text{as } n\to\infty.$$

On the other hand,

$$\lim_{t\to 0}\int_{\partial\Omega_t} \rho\frac{\partial\rho}{\partial n} = 0.$$

Applying (II.15) to $t = t_n$, and passing to the limit $n\to\infty$, we obtain

$$\int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2 + \frac{\omega^2 r^2}{2}|u|^2 = J(\rho) + \int_\Omega \frac{1}{2}\rho^2|\nabla v|^2 + \frac{\rho^4}{4\varepsilon^2}(1-|v|^2)^2.$$

There remains to deal with

$$\int_\Omega (iu, x_2 u_{x_1} - x_1 u_{x_2}).$$

But replacing u by ρv, we obtain that this term is equal to

$$\int_\Omega \rho^2(iv, x_2 v_{x_1} - x_1 v_{x_2}) + \int_\Omega \rho(iv,v)(x_2\rho_{x_1} - x_1\rho_{x_2}),$$

where the second term vanishes identically. Hence we have the desired result. $\square$

III Branches of vortex solutions in the case of the disc

In this section $\Omega = B(0,R) = B_R$. We consider rotation speeds

$$\omega \leq C\varepsilon^{-\alpha}$$

for α sufficiently small, to be specified in the proof, and obtain similar results as those of [S1, S2, S3] concerning branches of solutions of (G.P).
As mentioned in the introduction, we cannot study zeros of u close to $\partial\Omega$ because $|u|$ vanishes at the boundary and is smaller than $\frac{1}{2}$ on a layer of size $\lambda\varepsilon$ near the boundary. On the opposite, we can study vortices of v, which does not have to vanish on $\partial\Omega$, and vortices of v are vortices of u. But it is difficult to get information on v and its vortices near the boundary, and anyway, it is not very relevant, since u has something like a layer of vortices of size ε near the boundary. This is why we restrict to studying v on the domain $\{x \in \Omega, \delta(x) \geq \varepsilon^{\beta}\}$ where β is some constant < 1 and close to 1. Furthermore, there are no boundary conditions on v, hence we can adjust the techniques of [S1, S2, S3] to v. The rest of this section is just these adjustments.

III.1 Defining the domains of minimization

We perform, as in [S1, S2, S3], local minimizations of J over well-chosen domains.
First, Lemma II.2 has allowed us to separate the very strongly-divergent part $J(\rho_\varepsilon)$ (in $\frac{C}{\varepsilon}$) from the rest which is very similar to the Ginzburg-Landau energy functional J of [S1, S2, S3], and diverges at most in $C\omega^2$. We use here the notation of [S1] :

$$F(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{1}{2\varepsilon^2}(1-|v|^2)^2 \tag{III.1}$$

for the energy functional studied in [BBH]. We shall also write F_V for the functional F restricted to any subdomain V of Ω.
Then, we denote

$$G(v) = \frac{1}{2}\int_\Omega \rho^2|\nabla v|^2 + \frac{\rho^4}{2\varepsilon^2}(1-|v|^2)^2, \tag{III.2}$$

for the "weighted" BBH-functional that appears naturally in the splitting of J.
We then define the following mappings :

$$\begin{aligned} H^1 &\longrightarrow H^1_{\rho^2} \\ u &\longmapsto v = \frac{u}{\rho} \end{aligned} \tag{III.3}$$

is a continuous mapping, as proved in section II. So is

$$\begin{aligned} H^1_{\rho^2} &\longrightarrow H^1_{\rho^2} \\ v &\longmapsto T(v) = \left| \begin{array}{l} \dfrac{v}{\rho|v|} \text{ if } |v| \geq \dfrac{1}{\rho} \\ v \text{ otherwise.} \end{array} \right. \end{aligned} \tag{III.4}$$

In addition, we have the following lemma, whose proof is postponed to the end of this section :

Lemma III.1 *1) For any* $u \in H_0^1(\Omega, \mathbb{C})$,

$$\text{(III.5)} \qquad |T(v)| \leq \frac{1}{\rho}.$$

2) There exists $\alpha > 0$ *such that, for any* u *satisfying* $J(u) \leq J(\rho)$, *if* $\omega \leq C\varepsilon^{-\alpha}$,

$$\text{(III.6)} \qquad J(\rho T(v)) \leq J(\rho v) + o(1) \quad \text{as } \varepsilon \to 0.$$

This lemma means that, if we replace u by $\rho T(v)$, we get a function with values in $\overline{B(0,1)}$ and with a lower energy than u, up to a small error term. Hence, we can make this replacement to find local minimizers.

We now define the domains of minimization. A real (large) positive constant $\mathcal{M} \notin \pi\mathbb{N}$ $\quad(\mathcal{M} > \pi)$ being set, we define (as in [S1]),

$$\text{(III.7)} \qquad D_{\mathcal{M}} = \left\{ u \in H_0^1(B_R, \mathbb{C}) / G \circ T\left(\frac{u}{\rho}\right) < \mathcal{M}|\log \varepsilon| \right\}.$$

This is going to be our largest domain of minimization. We shall also use smaller domains of minimization, of the form

$$D_{a,b} = \left\{ u \in H_0^1(B_R, \mathbb{C}) / a < \frac{G \circ T\left(\frac{u}{\rho}\right)}{|\log \varepsilon|} < b \right\},$$

where $a, b < \mathcal{M}$ but may depend on ε. $D_{\mathcal{M}}$ and $D_{a,b}$ are open domains in H_0^1, from the continuity of the mappings (III.3) and (III.4).

III.2 Definition of the regularized map and its vortices

We wish to study minimizers of J in $D_{\mathcal{M}}$ or $D_{a,b}$. Considering any u in one of these domains, we write $v = T(\frac{u}{\rho})$, so that $|v| \leq \frac{1}{\rho}$ and, thanks to Lemma III.1, we can study v instead of $\frac{u}{\rho}$.

As in [S1, S2, S3], we need to define vortices of v for any $u \in D_{\mathcal{M}}$. But, exactly as in these papers, this is impossible to do directly because u is not a priori solution of (G.P) hence there is no upper bound of the type $|\nabla u| \leq \frac{C}{\varepsilon}$ on its gradient. As in [S1, S2, S3], in order to define vortices of v, we replace it, following the method of [AB], by a regularized map v^γ which has well-defined vortices. First, we remove the boundary of the domain, by setting

$$\text{(III.8)} \qquad B' = B_R \backslash \{x / \delta(x) \leq \varepsilon^\beta\},$$

where β is some constant $\in]0,1[$. From (II.5),

$$0 \leq 1-\rho \leq C\varepsilon^{1-\beta} + O(\varepsilon^2\omega^2) = o(1) \text{ in } B', \tag{III.9}$$

hence we can consider ρ as being equal to 1 in B'. Then, our regularized map v^γ is defined from v to be the solution of the following problem :

$$\min_{H^1(B',\mathbb{C})} \int_{B'} \frac{1}{2}|\nabla w|^2 + \frac{(1-|w|^2)^2}{4\varepsilon^2} + \frac{|v-w|^2}{2\varepsilon^{2\gamma}}, \tag{III.10}$$

where γ is some constant in $]0,1[$.
Exactly as in [S1, S2, S3] and [AB], this v^γ has the same behaviour as v at scales larger than ε^γ (it is a parabolic regularization of v), hence its vortex-structure and behaviour with respect to J are going to be almost the same as those of v, as we shall prove.

Lemma III.2 *If $u \in D_{\mathcal{M}}$ and $v = T(\frac{u}{\rho})$, then v^γ satisfies*

$$|\nabla v^\gamma| \leq \frac{C}{\varepsilon} \qquad |v^\gamma| \leq \frac{1}{\rho},$$

$$F_{B'}(v^\gamma) \leq F_{B'}(v) \leq G(v) + o(1). \tag{III.11}$$

Proof : The first assertions are standard (recall that $|v| \leq \frac{1}{\rho}$). For the second result, take v as a test-map in (III.10). Then, observe that

$$\begin{aligned} F_{B'}(v) - G(v) &\leq \int_{B'} \frac{1}{2}(1-\rho^2)|\nabla v|^2 + \frac{(1-\rho^4)}{4\varepsilon^2}(1-|v|^2)^2 \\ &\leq C\|1-\rho\|_{L^\infty(B')}G(v) \\ &\leq C(\varepsilon^{1-\beta} + \varepsilon^{1-\alpha})|\log\varepsilon| = o(1) \end{aligned}$$

where we have used (III.9) and the fact that $u \in D_{\mathcal{M}}$. □
Then, as in [S1], Proposition III.2, we deduce from the analysis of [AB], that we can define vortices of size σ of v^γ satisfying the following properties

Lemma III.3 *Let $0 < \gamma < \beta < \mu < 1$, with $\overline{\mu} = \mu^{N+1} > \beta$; for ε sufficiently small, we may find a bounded number of balls $(B(a_i,\sigma))_{i\in\mathcal{J}}$ such that the following properties hold*

$$\begin{aligned} &\lambda\varepsilon \leq \varepsilon^\mu \leq \sigma \leq \varepsilon^{\overline{\mu}} < \varepsilon^\beta, \\ &|v^\gamma(x)| \geq \frac{1}{2} && \text{if } x \in B' \setminus \textstyle\bigcup_{i\in\mathcal{J}} B(a_i,\sigma), \\ &|v^\gamma(x)| \geq 1 - \frac{2}{|\log\varepsilon|^2} && \text{if } x \in \partial B(a_i,\sigma), \quad \text{for } i \in \mathcal{J}, \\ &\int_{\partial B(a_i,\sigma)} e_\varepsilon(v^\gamma) \leq \frac{C(\beta,\mu)}{\sigma} && \text{for } i \in \mathcal{J}, \\ &|a_i - a_j| \geq 8\sigma && \text{for any } i \neq j \in \mathcal{J}. \end{aligned}$$

We shall also write

$$d_i = \deg\,(v^\gamma, \partial B(a_i,\sigma)) \neq 0. \tag{III.12}$$

u or v will be called "vortex-less" if $\mathcal{J} = \varnothing$ i.e. if $|v^\gamma| \geq \frac{1}{2}$ in B'. It is proved in [S1] Proposition VI.2, that this implies (if $J(u) \leq J(\rho)$) that $|v| \geq \frac{1}{2}$ in B'.
Similarly, if v itself has well-defined vortices, then there is a close link between them and those of v^γ, see [S1], Proposition III.3.
From now on, we denote $((a_i, d_i))_{i=1}^{l}$ the vortices of v^γ, also called "vortices of u" or "vortices of v" by extension. We have the following lower bound, borrowed from [S1], Proposition V.1 and Lemma V.1,

Proposition III.1

$$F_{B'}(v^\gamma) \geq \pi \sum_{i\in\mathcal{J}} |d_i| \log \frac{\sigma}{\varepsilon} + O(1).$$

$$F_{B'}(v^\gamma) \geq \pi \sum_{i\in\mathcal{J}} d_i^2 |\log \sigma| + \pi \sum_{i\in\mathcal{J}} |d_i| \log \frac{\sigma}{\varepsilon} + W((a_1, d_1), \cdots, (a_l, d_l)) + O(1),$$

where

$$W = -\pi \sum_{i\neq j} d_i d_j \log |a_i - a_j| - \pi \sum_{i=1}^{l} d_i R_0(a_i),$$

and R_0 is the solution of

$$\begin{cases} \Delta R_0 = 0 & \text{in } \Omega \\ R_0 = -\pi \sum_{i=1}^{l} d_i \log |x - a_i| & \text{on } \partial\Omega. \end{cases}$$

III.3 Expanding the energy-functional

Having a definition of vortices, we use it to expand the energy functional and get an expansion which is totally similar to that of [S1].
First, we have the following lemma, the proof of which is left to the end of the section.

Lemma III.4 *If $\omega \leq C\varepsilon^{-\alpha}$, for some $\alpha > 0$ sufficiently small, we have*

$$\omega \int_{B_R} (iu, x_2 u_{x_1} - x_1 u_{x_2}) = \omega \int_{B_R} \rho^2 (iv, x_2 v_{x_1} - x_1 v_{x_2})$$
$$= \omega \int_{B_R} \rho^2 (iv, dv \wedge dX) = 2\pi\omega \sum_{i\in\mathcal{J}} d_i X(a_i) + o(1) \quad \text{as } \varepsilon \to 0,$$

where X is the function defined on B_R as

(III.13) $$X(x) = \frac{|x|^2 - (R - \varepsilon^\beta)^2}{2},$$

so that

(III.14) $$X = 0 \quad \text{on } \partial B'.$$

Once this lemma is proved, the expansion of the energy follows very easily from (II.12) :

Proposition III.2 *There exists $\alpha > 0$ such that, if $\omega \leq C\varepsilon^{-\alpha}$, $u \in D_{\mathcal{M}}$ and $|u| \leq 1$,*

$$J(u) = J(\rho) + G(v) + 2\pi\omega \sum_{i\in\mathcal{J}} d_i X(a_i) + o(1) \quad \text{as } \varepsilon \to 0,$$

where G was defined in (III.2).

It is easy to deduce from this proposition, as in [S1], that the minimal energy over vortex-less configurations is $J(\rho) + o(1)$ as $\varepsilon \to 0$. Actually, it is smaller than $J(\rho)$, and it should be $J(\rho)$ since ρ is a solution of (G.P). One could hope to prove, following the method of [S2], the uniqueness of a vortex-less solution, hence the minimality of ρ among them, but the problem is that the notion of "vortex-less" is difficult to define up to the boundary of the domain. Hence it remains an open problem to know whether ρ is minimal for $\omega \leq \omega_1$.

III.4 The critical velocity

We are now in a position to deduce the critical velocity, that we denote ω_1, by analogy with the first critical field H_{c_1} for superconductors in [S1].
ω_1 is defined to be the largest value of ω below which the minimum of J is larger than $J(\rho) - o(1)$. For $\omega \geq \omega_1$, minimizers of J will have vortices. We will look for them in $D_{\mathcal{M}}$, domain which corresponds roughly (in view of Proposition III.1) to configurations with less than $\frac{\mathcal{M}}{\pi}$ vortices.

We get the following theorem, which is completely analogous to Theorem 1 of [S1] :

Theorem 1 *Defining a rotation velocity ω_1 by*

$$\omega_1 = \frac{|\log \varepsilon|}{R^2}, \tag{III.15}$$

there exist $k(\varepsilon) = O(1)$, $k'(\varepsilon) = O(|\log |\log \varepsilon||)$, and $\varepsilon_0(\mathcal{M})$ such that for $\varepsilon < \varepsilon_0$, the following holds :
- if and only if $\omega \leq \omega_1 - k(\varepsilon)$, the minimum of J is $J(\rho) - o(1)$ and if $\omega \leq \omega_1 - k'(\varepsilon)$ any minimizer satisfies $|v| \geq \frac{1}{2}$ on B'. ($v = T(\frac{u}{\rho})$)
- if $\omega_1 + k(\varepsilon) \leq \omega \leq \omega_1 + O(1)$, there exists a minimizer of J over $D_{\mathcal{M}}$ which is a solution of (G.P). In addition, "it" has exactly one vortex a of degree one, and $|a| \to 0$ as $\varepsilon \to 0$.

Proof : Step 1 : Upper bound for the minimal energy.
We start from Proposition III.2 and

$$J(u) = J(\rho) + G(v) + 2\pi\omega \sum_{i\in\mathcal{J}} d_i X(a_i) + o(1).$$

We can construct a configuration the following way : we choose v such that $|v| \leq 1$, v has a zero of degree one at the center of B_R, and

$$F(v) = \frac{1}{2}\int_{B_R} |\nabla v|^2 + \frac{1}{2\varepsilon^2}(1 - |v|^2)^2 \leq \pi|\log \varepsilon| + O(1).$$

This is possible exactly as in [S1], proof of Proposition VI.1. Then, defining $u = \rho v$, it is easy to check that $u \in D_{\mathcal{M}}$ and

$$J(u) \leq J(\rho) + F(v) + 2\pi\omega X(0) + o(1)$$

$$J(u) \leq J(\rho) + \pi|\log\,\varepsilon| - \pi\omega R^2 + O(1). \tag{III.16}$$

in view of the definition (III.13).
Hence, we have

$$\min_{D_{\mathcal{M}}} J \leq \min(J(\rho), J(\rho) + \pi|\log\,\varepsilon| - \pi\omega R^2 + O(1)), \tag{III.17}$$

and

$$\min_{D_{\mathcal{M}}} J < J(\rho) - 1$$

if

$$\frac{|\log\,\varepsilon|}{R^2} + k(\varepsilon) \leq \omega \leq C\varepsilon^{-\alpha},$$

for some $k(\varepsilon) = O(1)$.
$min_{D_{\mathcal{M}}} J < J(\rho)$ for $\omega \geq C\varepsilon^{-\alpha}$ will be proved in the next section.

Step 2 : Study of the minimizers of J around ω_1.
Let u achieving $min\ J$ over $\overline{D_{\mathcal{M}}}$ (such a u exists), and let $v = T\left(\frac{u}{\rho}\right)$, v^γ its regularized map. From Lemma III.1, we have

$$J(\rho v) \leq \min_{\overline{D_{\mathcal{M}}}} J + o(1) \leq J(\rho) + o(1). \tag{III.18}$$

On the other hand, thanks to Proposition III.2, and (III.11),

$$J(u) = J(\rho) + G(v) + 2\pi\omega \sum_i d_i X(a_i) + o(1) \tag{III.19}$$

$$\geq J(\rho) + F_{B'}(v^\gamma) + 2\pi\omega \sum_i d_i X(a_i) + o(1), \tag{III.20}$$

For $\omega \leq \omega_1 + O(1)$, we have by minimality of u in $\overline{D_{\mathcal{M}}}$,

$$J(u) \leq \min_{D_{\mathcal{M}}} J \leq min(J(\rho), J(\rho) + \pi|\log\,\varepsilon| - \pi\omega R^2 + O(1)) \tag{III.21}$$

hence

$$F_{B'}(v^\gamma) + 2\pi\omega \sum_i d_i X(a_i) \leq min(o(1), \pi|\log\,\varepsilon| - \pi\omega R^2 + O(1)) \leq O(1). \tag{III.22}$$

Comparing this to Proposition III.1, which asserts that

$$\begin{aligned} F_{B'}(v^\gamma) &\geq \pi \sum_i |d_i| \log\frac{\sigma}{\varepsilon} + O(1) \\ &\geq \pi(1-\mu) \sum_i |d_i||\log\,\varepsilon| + O(1), \end{aligned}$$

we deduce (exactly as in [S1], proof of Lemma VI.1), that

$$d_i > 0 \quad \forall i \in \mathcal{J}, \tag{III.23}$$

because $min\ X = X(0) = -\frac{R^2}{2} + o(1)$.
In view of the expression of W, exactly as in [S1], this implies that $W \geq O(1)$, and that the following lower bounds hold :

$$F_{B'}(v^\gamma) \;\geq\; \pi\left(\sum_i |d_i|^2 \log\frac{1}{\sigma} + \sum_i |d_i| \log\frac{\sigma}{\varepsilon}\right) + O(1) \tag{III.24}$$

$$F_{B'}(v^\gamma) \;\geq\; \pi\sum_i |d_i| \log\frac{1}{\varepsilon} + O(1). \tag{III.25}$$

For some $k(\varepsilon) = O(1)$, if $\omega \leq \omega_1 + k(\varepsilon)$, combining this with (III.20) and (III.21) implies that $\mathcal{J} = \emptyset$, hence that $|v^\gamma| \geq \frac{1}{2}$ on B'. From the analysis of [S1], (see Proposition VI.2), this allows to prove that $|v| \geq \frac{1}{2}$ on B', i.e. that v is vortex-less on B'.
On the other hand, if $\omega \geq \omega_1 + O(1)$, the comparison of (III.24) and (III.22) yields $d_i = 1$, $\forall i \in \mathcal{J}$, and $X(a_i) \to min\ X$, hence v^γ only has vortices of degree 1, tending to the origin. But then, if Card $\mathcal{J} > 1$, i.e. if v^γ has more than one vortex, its vortices repell one another because $W \to +\infty$ as $|a_i - a_j| \to 0$, and we are in this case since all the vortices tend to 0. Then, using Proposition III.1,

$$F_{B'}(v^\gamma) \geq \pi\sum_i |d_i| \log\frac{1}{\varepsilon} + W + O(1), \tag{III.26}$$

where $W \to +\infty$ if Card $\mathcal{J} \geq 2$. This is in contradiction with (III.20) and (III.18) if $\omega \leq \omega_1 + O(1)$. Hence, for $\omega \leq \omega_1 + O(1)$, any minimizer in $\overline{D_{\mathcal{M}}}$ has exactly one vortex of degree one tending to 0.

Step 3 : We can easily adjust the method of [SS1] to prove that, for $\omega \leq \omega_1 - k'(\varepsilon)$, a global minimizer of J is vortex-less, and belongs to $D_{\mathcal{M}}$.

Step 4 : There remains to prove that a minimizer of J in $\overline{D_{\mathcal{M}}}$ is a solution of (G.P) if $\frac{\mathcal{M}}{\pi} \notin \mathbb{N}$. This can be done exactly as in the forthcoming proof of Lemma III.6, therefore we omit the details. □

III.5 Branches of n-vortices solutions

In this subsection, we adjust the results of [S3] concerning the existence of branches of stable solutions with n vortices, for any $n < \frac{\mathcal{M}}{\pi}$.

For any $n \geq 1$, we consider the domain

$$U_n = \left\{ u \in H_0^1(B_R, \mathbb{C}) / n|\log\,\varepsilon| + K < G\left(T\left(\frac{u}{\rho}\right)\right) < \left(n + \frac{1}{2}\right)|\log\,\varepsilon| \right\},$$

where K is a constant, to be set later. This domain is some $D_{a,b}$, but for simplicity, we denote it U_n as in [S3]. We begin with two lemmas.

Lemma III.5 *Suppose ω is any function of ε tending to $+\infty$ as $\varepsilon \to 0$, and such that $\omega \leq C\varepsilon^{-\alpha}$. We have*

(III.27)
$$\underset{\overline{U_n}}{inf}\, J = J(\rho) + \pi n \left(|\log \varepsilon| - R^2\omega\right) + \frac{\pi}{2}(n^2 - n)\log \omega + w(\tilde{a_1}, \cdots, \tilde{a_n}) + Q_n + o(1),$$

where $\tilde{a}_i = a_i\sqrt{\omega}$, the a_i's being the vortices of a minimizer, w is defined as

$$w(x_1, \cdots, x_n) = -\pi \sum_{i\neq j} \log |x_i - x_j| + \pi \sum_i |x_i|^2, \tag{III.28}$$

and Q_n is a constant depending only on n.

Proof : This lemma is exactly similar to Lemma III.1 of [S3]. We only give the main ideas of the proof. (III.27) relies on the expansion of Proposition III.2, from which we have obtained the lower bound (III.20) :

$$J(u) \geq J(\rho) + F_{B'}(v^\gamma) + 2\pi\omega \sum_i d_i X(a_i) + o(1). \tag{III.29}$$

Then, $F_{B'}$ is bounded from below exactly as in [S1] using Proposition III.1, and W is expanded.
Actually, we construct a test configuration as in the proof of Theorem 1, for which the equality holds in (III.27) and that has n vortices of degree 1, located on a regular polygon of edge-size $\frac{1}{\sqrt{\omega}}$, centered at 0. Using the explicit expression of X and expanding W, we get the expression (III.27) as an upper bound for $inf_{\overline{U_n}} J$.

For the lower bound, we begin by noticing that the upper bound implies $d_i = 1, \forall i \in \mathcal{J}$, and $a_i \to 0$ as for Theorem 1, for the vortices of a minimizing map in $\overline{U_n}$. Then, the lower bound is obtained from (III.29), expanding X and W as previously, and performing the rescaling $\tilde{a}_i = a_i\sqrt{\omega}$ on the vortices. □

Lemma III.6 *If ε is sufficiently small, $\omega \to +\infty$ and $\omega \leq C\varepsilon^{-\alpha}$, $min_{\overline{U_n}} J$ is achieved, and it is not achieved on ∂U_n.*

Proof : It is exactly the same as the proof of Lemma III.2 in [S3]. We consider u_k a minimizing sequence in U_n and $v_k = T\left(\frac{u_k}{\rho}\right)$. We regularize them into v_k^γ with vortices $(a_i(k), d_i(k))$. Then, thanks to Proposition III.2,

$$\begin{aligned} J(\rho v_k) &= J(\rho) + G(v) + 2\pi\omega \sum_i d_i(k) X(a_i(k)) + o_\varepsilon(1) \\ &\leq \underset{\overline{U_n}}{inf}\, J + o_\varepsilon(1) + o_k(1). \end{aligned}$$

Then,

$$\text{(III.30)} \qquad \underset{\overline{U_n}}{inf}\, J \geq J(\rho) + F_{B'}(v_k^\gamma) + \pi\omega \sum_i d_i(k)\left(|a_i(k)|^2 - R^2\right) + o_\varepsilon(1) + o_k(1).$$

Comparing this with the expression (III.27) for $inf_{\overline{U_n}} J$, we deduce that, for small ε, all $d_i(k)$ are positive, that $\sum_i |d_i(k)| = n$ (we already knew that $\sum_i |d_i(k)| \leq n$ since $u_k \in \overline{U_n}$), and that $\sum_i |a_i(k)|^2 \to 0$ as $\varepsilon \to 0$.
Up to the extraction of a subsequence, $u_k \rightharpoonup u$ in H_0^1 and, by lower semi-continuity, $J(u) \leq inf_{\overline{U_n}} J$.

There remains to prove that $u \in \overline{U_n}$. This can be proved as in [S3], using the idea that the weak H^1 convergence preserves the vortices. One proves that, if $v = T\left(\frac{u}{\rho}\right)$, and (b_i, q_i) are the vortices of v^γ, then

(III.31)

$$\pi \sum_i d_i(k)(|a_i(k)| - R^2) \simeq \int_{B'} (iv_k, dv_k \wedge dX) \underset{k\to\infty}{\longrightarrow} \int_{B'} (iv, dv \wedge dX) \simeq \pi \sum_i q_i(|b_i|^2 - R^2).$$

On the one hand, we have, by definition of U_n,

$$G(v) \leq \underset{k\to+\infty}{liminf}\, G\left(T\left(\frac{u_k}{\rho}\right)\right) \leq \pi\left(n + \frac{1}{2}\right)|\log\, \varepsilon|,$$

leading to $\sum_i |q_i| \leq n$. On the other hand, (III.31) and the previous results on $(a_i(k), d_i(k))$ yield $\sum_{q_i>0} q_i \geq n$, hence $\sum_i |q_i| = n$ and

$$G(v) \geq F_{B'}(v) - o(1) \geq F_{B'}(v^\gamma) - o(1) \geq \pi n|\log\, \varepsilon| + O(1).$$

This proves, if K is chosen small enough, that $u \in \overline{U_n}$ for small ε. Therefore, $inf_{\overline{U_n}} J$ is achieved.

Step 2 : We prove that the minimum is not achieved on ∂U_n. We follow the proof of Theorem 2 in [S3]. By contradiction, suppose it is, then there is a minimizer u such that

$$\text{(III.32)} \qquad G\left(T\left(\frac{u}{\rho}\right)\right) = \pi\left(n + \frac{1}{2}\right)|\log\, \varepsilon|,$$

or

$$\text{(III.33)} \qquad G\left(T\left(\frac{u}{\rho}\right)\right) = \pi n|\log\, \varepsilon| + K.$$

If (III.32) holds, then, with the usual notations, $\sum_i |d_i| \leq n$ and

$$\begin{aligned}
\underset{\overline{U_n}}{inf}\, J &= J(\rho) + G\left(T\left(\frac{u}{\rho}\right)\right) + 2\pi\omega \sum_i d_i X(a_i) + o(1) \\
&\geq J(\rho) + \pi\left(n + \frac{1}{2}\right)|\log\, \varepsilon| - \pi\omega \sum_i |d_i| R^2 + o(1) \\
&\geq J(\rho) + \pi n\left(|\log\, \varepsilon| - \omega R^2\right) + \frac{\pi}{2}|\log\, \varepsilon| + o(1).
\end{aligned}$$

Comparing this to (III.27), we are led to

$$\frac{\pi}{2}(n^2-n)\log\,\omega+O(1)\geq\frac{\pi}{2}|\log\,\varepsilon|,$$

which is a contradiction as soon as $\omega\leq C\varepsilon^{-\alpha}$ for α sufficiently small compared to $\mathcal{M}$. The second possibility is (III.33). From the proof of the first step of the lemma, we know that, since u minimizes J in $\overline{U_n}$, we must have $\sum_i|d_i|=n$ (and $\sum_i|a_i|^2\to 0$). But then,

$$\begin{aligned}F_{B'}(v) &\geq \pi n|\log\,\varepsilon|+O(1)\\ &> \pi n|\log\,\varepsilon|+K,\end{aligned}$$

if K is chosen small enough. Hence we get that (III.33) is impossible, and this concludes the proof. $\square$

We are now able to derive from these lemmas the following theorem, analogous to Theorem 2 in [S2].

Theorem 2 *Suppose ω is any function of ε such that $\omega\to+\infty$ as $\varepsilon\to 0$, and $\omega\leq C\varepsilon^{-\alpha}$, then $\forall\varepsilon<\varepsilon_0$, $\forall n\in\mathbb{N}^*$ such that $n<\frac{\mathcal{M}}{\pi}$, there exists a branch of stable solutions of (G.P) such that :*
1) u (or v^γ) has exactly n vortices of degree 1, located at a_i^ε.
2) $|a_i^\varepsilon|\to 0$ as $\varepsilon\to 0$, and if we set $\tilde{a}_i=a_i\sqrt{\omega}$, the $\tilde{a}_i$'s tend to minimize

$$w(x_1,\cdots,x_n)=-\pi\sum_{i\neq j}\log\,|x_i-x_j|+\pi\sum_i|x_i|^2$$

so that $|a_i|\leq\frac{C}{\sqrt{\omega}}$, and $|a_i-a_j|\geq\frac{C}{\sqrt{\omega}}$.
3) As $\varepsilon\to 0$,

$$J(u)=\min_{\overline{U_n}}J=J(\rho)+\pi n\left(|\log\,\varepsilon|-R^2\omega\right)+\frac{\pi}{2}(n^2-n)\log\,\omega+w(\tilde{a}_1,\cdots,\tilde{a}_n)+Q_n+o_\varepsilon(1).$$

The solution with n vortices minimizes J in $D_{\mathcal{M}}$ exactly for $\omega_n\leq\omega\leq\omega_{n+1}$, where ω_n has an expression of the form

$$\omega_n=\frac{|\log\,\varepsilon|}{R^2}+\frac{n-1}{R^2}|\log\,|\log\,\varepsilon||+O(1). \tag{III.34}$$

Proof : The solutions are given by Lemma III.5 and III.6. Consider $n<\frac{\mathcal{M}}{\pi}$ and u a minimizer of J in U_n as given by Lemma III.6. Then, $u\in U_n$ which is open in H_0^1, hence it is a local minimizer of the energy, therefore it is a stable solution of (G.P). The characteristics of its vortices and its energy are easily derived from the proof of Lemma III.5. There remains to check that its range of minimality is $[\omega_n,\omega_{n+1}]$. It amounts to computing when $inf_{\overline{U_n}}J<inf_{\overline{U_k}}J,\forall k\neq n$. Comparing the expressions (III.27) for different values of n (exactly as in [S2]), we get the desired result. $\square$

III.6 Proof of Lemma III.1

The first assertion follows readily from the definition of T. For the second one, we begin by expanding $J(u)$ as

$$
\begin{aligned}
J(u) &= \int_{B_R} \frac{1}{2}|\nabla u + iu\omega \times x|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2 \\
&= \int_{B_R} \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2 + \frac{\omega^2}{2}r^2|u|^2 + \omega(iu, x_2 u_{x_1} - x_1 u_{x_2}). \qquad \text{(III.35)}
\end{aligned}
$$

Next, we notice that $\rho T(v) = u$ if $|u| \leq 1$ and $\rho T(v) = \frac{u}{|u|}$ if $|u| \geq 1$, hence obvioulsy

$$\int_{B_R} |\nabla(\rho T(v))|^2 \leq \int_{B_R} |\nabla u|^2,$$

and

$$\int_{B_R} (1-|\rho T(v)|^2)^2 \leq \int_{B_R} (1-|u|^2)^2,$$

hence

$$F(\rho T(v)) \leq F(u). \qquad \text{(III.36)}$$

Moreover, as in [S1], proof of Lemma II.2,

$$\Delta := \left| \int_{B_R} (i\rho T(v), x_2(\rho T(v))_{x_1} - x_1(\rho T(v))_{x_2}) - (iu, x_2 u_{x_1} - x_1 u_{x_2}) \right|$$

$$
\begin{aligned}
= \Big| \int_{|u|\geq 1} &\left(i\frac{u}{|u|}, x_2(\frac{u}{|u|})_{x_1} - x_1(\frac{u}{|u|})_{x_2} \right) - |u|^2 \left(i\frac{u}{|u|}, x_2(\frac{u}{|u|})_{x_1} - x_1(\frac{u}{|u|})_{x_2} \right) \\
&- \left(iu, \frac{u}{|u|}(x_2|u|_{x_1} - x_1|u|_{x_2}) \right) \Big|
\end{aligned}
$$

where the last term vanishes, hence

$$\Delta = \left| \int_{|u|\geq 1} (1-|u|^2) \left(i\frac{u}{|u|}, x_2(\frac{u}{|u|})_{x_1} - x_1(\frac{u}{|u|})_{x_2} \right) \right|.$$

But, we consider u such that $J(u) \leq J(\rho)$, hence, arguing as in the proof of (II.7),

$$\int_{B_R} (1-|u|^2)^2 \leq C\varepsilon^4\omega^4 + C\varepsilon, \qquad \text{(III.37)}$$

and by Cauchy-Schwarz's inequality,

$$\Delta \leq C \left(\int_{B_R} (1-|u|^2)^2 \right)^{\frac{1}{2}} \left(\int_{|u|\geq 1} \left| \nabla \frac{u}{|u|} \right|^2 \right)^{\frac{1}{2}}. \qquad \text{(III.38)}$$

On the other hand, $J(u) \le J(\rho)$ also implies that

$$\begin{aligned}
\int_{B_R} \frac{\rho^2}{2}|\nabla v|^2 + \frac{\rho^4}{4\varepsilon^2}(1-|v|^2)^2 &\le \omega \left| \int_{B_R} \rho^2 (iv, x_2 v_{x_1} - x_1 v_{x_2}) \right| \\
\int_{B_R} \frac{\rho^2}{2}|\nabla v|^2 &\le \omega \int_{B_R} \rho^2 |v||\nabla v| \\
&\le \omega \|u\|_{L^2(B_R)} \|\rho \nabla v\|_{L^2(B_R)}. \qquad \text{(III.39)}
\end{aligned}$$

But, using (III.37),

$$\int_{B_R} |u|^2 = \int_{B_R} |u|^2 - 1 + \int_{B_R} 1 \le C + C\left(\int_{B_R} (1-|u|^2)^2\right)^{\frac{1}{2}} \le C.$$

Therefore, (III.39) implies that

$$\int_{B_R} \rho^2 |\nabla v|^2 \le C\omega^2. \qquad \text{(III.40)}$$

From (III.40), we deduce

$$\int_{|u|\ge 1} \left|\nabla \frac{u}{|u|}\right|^2 \le \int_{\rho|v|\ge 1} \left|\nabla \frac{v}{|v|}\right|^2 \le \int_{\rho|v|\ge 1} \rho^2 |v|^2 \left|\nabla \frac{v}{|v|}\right|^2 \le \int_{B_R} \rho^2 |\nabla v|^2 \le C\omega^2. \qquad \text{(III.41)}$$

Therefore, combining (III.41) and (III.37), (III.38) becomes

$$\omega \Delta \le C\omega^2 (\varepsilon^{\frac{1}{2}} + \omega^2 \varepsilon^2) = o(1),$$

as soon as $\omega \ll \varepsilon^{-\frac{1}{4}}$. Combined with (III.35) and (III.36), this allows us to conclude that $J(\rho T(v)) \le J(\rho v) + o(1)$ for ω satisfying $\omega \le C\varepsilon^{-\alpha}$ if we choose $0 < \alpha < \frac{1}{4}$.

III.7 Proof of Lemma III.4

The first step is to prove that

$$I := \int_{B_R} (iu, dX \wedge du) \simeq \int_{B'} (iv^\gamma, dX \wedge dv^\gamma).$$

Writing $u = \rho v$,

$$\int_{B_R} (iu, dX \wedge du) = \int_{B_R} \rho^2 (iv, dX \wedge dv) + \int_{B_R} (iv, v) dX \wedge d\rho,$$

where the second term vanishes identically. Then, recalling that we assume $\rho|v| \le 1$,

$$\begin{aligned}
\left| \int_{B_R \setminus B'} \rho^2 (iv, dX \wedge dv) \right| &\le \int_{B_R \setminus B'} \rho^2 |v||\nabla v||\nabla X| \\
&\le C \left(\int_{B_R} |\nabla v|^2 \rho^2\right)^{\frac{1}{2}} \left(\int_{B_R \setminus B'} \rho^2 |v|^2\right)^{\frac{1}{2}} \\
&\le C\|v\|_{H^1_{\rho^2}} (vol(B_R \setminus B'))^{\frac{1}{2}} \\
&\le C|\log \varepsilon|^{\frac{1}{2}} \varepsilon^{\frac{\beta}{2}},
\end{aligned}$$

where we have used the fact that $u \in D_{\mathcal{M}}$. This term is thus a $o(\frac{1}{\omega})$ as soon as $\alpha < \frac{\beta}{2}$. Hence we can replace I by $\int_{B'} \rho^2 (iv, dX \wedge dv)$. Furthermore, as $\rho \geq 1 - o(1)$ on B',

$$\left| \int_{B'} (\rho^2 - 1)(iv, dX \wedge dv) \right| \leq C \left(\int_{B_R} (1-\rho^2)^2 \right)^{\frac{1}{2}} \|\nabla v\|_{L^2(B')} \leq C |\log \varepsilon|^{\frac{1}{2}} (\varepsilon^{\frac{1}{2}} + \omega^2 \varepsilon^2) = o(\frac{1}{\omega})$$

if α is well-chosen. Thus, we can assert that

$$I = \int_{B'} (iv, dX \wedge dv) + o(\frac{1}{\omega}).$$

Then, we deduce, following exactly the proof of [S1], Lemmas IV.2 and IV.3, that

$$I = \int_{B'} (iv^\gamma, dX \wedge dv^\gamma) + o(\frac{1}{\omega}) = 2\pi \sum_i d_i X(a_i) + o(\frac{1}{\omega}),$$

because we have chosen $X = 0$ on $\partial B'$. This completes the proof.

IV Study of global minimizers of J for $\omega \gg |\log \varepsilon|$

In this section, we return to the general case of Ω and prove that results similar to those of [SS2] can be obtained for the functional J. We do not use the approach of local minimization any longer, but study minimizers of J over all H_0^1 and evaluate their energy as a function of ω. We start with the particular case $\lambda = 0$, i.e. $\omega \gg |\log \varepsilon|$. The case $\lambda > 0$ is treated in Section V, and can be read independently.

IV.1 An upper bound for the minimal energy

Following exactly the same method as in [SS2], we prove the following proposition

Proposition IV.1 *Assume $\omega \to +\infty$ and $\omega \ll \frac{1}{\varepsilon}$. Then, as $\varepsilon \to 0$,*

$$\min_{H_0^1(\Omega, \mathbb{C})} J \leq F(\rho) + \omega |\Omega| \log \frac{1}{\varepsilon \sqrt{\omega}} + O(\omega),$$

where $|.|$ denotes the volume. If we only assume $\omega \ll \frac{1}{\varepsilon^2}$, then as $\varepsilon \to 0$, $\min_{H_0^1(\omega, \mathbb{C})} J \leq F(\rho) + o(\omega^2)$.

The proof consists, as in [SS2], in constructing a test configuration and evaluating its energy. We only state the main steps.
The idea of the proof comes from the following facts. We notice that, if u is solution of (G.P), then the following equation is satisfied, letting $u = \eta e^{i\varphi}$,

$$(\text{IV.1}) \qquad (-\Delta u, iu) = \left(\frac{u}{\varepsilon^2}(1 - |u|^2) + 2i \nabla u . \omega \times x - \omega^2 r^2 u, iu \right),$$

thus, if we define this time X by

$$X = \frac{|x|^2}{2},$$

we have

$$\operatorname{div}(\nabla u, iu) = \omega \nabla \eta^2 . \nabla^\perp X,$$

$$\operatorname{div}\left(\eta^2\left(\nabla \varphi - \omega \nabla^\perp X\right)\right) = 0, \tag{IV.2}$$

where $\eta^2 \nabla \varphi$ is to be understood as $(\nabla u, iu)$ hence has a meaning everywhere. Thus, as Ω is simply connected, we can write

$$\eta^2(\nabla \varphi - \omega \nabla^\perp X) = \nabla^\perp U, \tag{IV.3}$$

for some function $U \in H^1(\Omega, \mathbb{C})$. Actually, the proof can be adjusted to non-simply connected domains as in [SS2]. Moreover, as $\eta = 0$ on $\partial\Omega$, $\nabla U = 0$ on $\partial\Omega$, hence U is constant on ∂U, and since it is defined up to a constant, we can choose to take $U = 0$ on $\partial\Omega$:

$$\begin{cases} \eta^2(\nabla \varphi - \omega \nabla^\perp X) = \nabla^\perp U & \text{in } \Omega \\ U = 0 & \text{on } \partial\Omega \end{cases} \tag{IV.4}$$

Here, U plays a role similar to that of the induced magnetic field h in [SS2]. Then, one notices, as in [SS2], that

$$\begin{aligned} \int_\Omega |\nabla u + i\omega \times xu|^2 &= \int_\Omega |\nabla \eta|^2 + \eta^2 |\nabla \varphi - \omega \nabla^\perp X|^2 \\ &= \int_\Omega |\nabla \eta|^2 + \frac{|\nabla U|^2}{\eta^2} \geq \int_\Omega |\nabla \eta|^2 + |\nabla U|^2. \end{aligned} \tag{IV.5}$$

The idea of the proof is to construct U as in [SS2], and to deduce the phase φ by (IV.4), so that u has vortices of degree 1, regularly set on a lattice in Ω.

Proof of the proposition :

Step 1 : Let K be a square centered at 0, of edge-size $\sqrt{\frac{\pi}{\omega}}$. Defining

$$\begin{cases} \mu = \frac{2}{\varepsilon^2} & \text{in } B(0,\varepsilon) \subset K \\ \mu = 0 & \text{otherwise in } K, \end{cases} \tag{IV.6}$$

we choose U to be the solution of

$$\begin{cases} \Delta U = \mu - 2\omega & \text{in } K \\ \frac{\partial U}{\partial n} = 0 & \text{on } \partial K \end{cases} \tag{IV.7}$$

This problem has a solution because

$$\int_K (\mu - 2\omega) = 2\pi - 2\omega |K| = 0. \tag{IV.8}$$

The solution can indeed be found by minimizing

$$\int_K |\nabla U|^2 + \int_K (\mu - 2\omega) U$$

over all U such that $\int_K U = 0$. As in [SS2], in order to compute the energy of U, we use the following comparison map in $D = B(0, \sqrt{\frac{\pi}{\omega}})$:

$$\text{(IV.9)} \qquad \begin{cases} \Delta f = \mu - 2\omega & \text{in } D \\ f = 0 & \text{on } \partial D \end{cases}$$

As in [SS2], we prove that f is radial, and $\forall r \in \left[\varepsilon, \sqrt{\frac{\pi}{\omega}}\right]$,

$$f'(r) 2\pi r = \int_{\partial B(0,r)} \frac{\partial f}{\partial n} = \int_{B(0,r)} \Delta f = \int_{B(0,r)} \mu - 2\omega = 2\pi - 2\omega\pi r^2,$$

so that

$$\text{(IV.10)} \qquad f'(r) = \frac{1}{r} - \omega r,$$

and similarly, for $r \le \varepsilon$, in view of (IV.6),

$$\text{(IV.11)} \qquad f'(r) = \frac{r}{\varepsilon^2} - \omega r.$$

From (IV.10) and (IV.11), we check that

$$\text{(IV.12)} \qquad \int_D |\nabla f|^2 \le C + 2\pi \log \frac{1}{\varepsilon\sqrt{\omega}}.$$

Step 2 : Comparison of U and f.
Defining $g = U - f$, we have

$$\text{(IV.13)} \qquad \begin{cases} \Delta g = 0 & \text{in } K \\ \frac{\partial g}{\partial n} = -\frac{\partial f}{\partial n} & \text{on } \partial K \end{cases}$$

Exactly the same proof as in [SS2], Lemma III.2 can be reproduced (it uses essentially the scaled trace theorem and Poincaré-Wirtinger inequality), to obtain that $\|\nabla g\|_{L^2(K)} \le C$. Then, obviously,

$$\text{(IV.14)} \qquad \int_K |\nabla U|^2 \le C + 2\pi \log \frac{1}{\varepsilon\sqrt{\omega}}.$$

Step 3 : Extension of U and definition of u.
As in [SS2], we cover Ω with a square lattice of period $\sqrt{\frac{\pi}{\omega}}$, and we extend U and μ by

periodicity with respect to this lattice. We denote a_i the centers of the squares. There are $\simeq \frac{|\Omega|\omega}{\pi}$ of them. This leads to a $U \in H^1$ because $\frac{\partial U}{\partial n} = 0$ on ∂K, and

$$\text{(IV.15)} \qquad \frac{1}{2}\int_\Omega |\nabla U|^2 \leq |\Omega|\omega \log \frac{1}{\varepsilon\sqrt{\omega}} + O(\omega).$$

Then, we define

$$\text{(IV.16)} \qquad \nabla\varphi = \nabla^\perp U + \omega\nabla^\perp X \quad \text{in } \Omega\backslash \cup_i B(a_i,\varepsilon),$$

and choose η that satisfies

$$\text{(IV.17)} \qquad \begin{cases} \eta \equiv 0 \text{ in } B(a_i,\varepsilon) \\ \eta \equiv 1 \text{ in } \Omega\backslash \cup_i B(a_i,2\varepsilon) \\ \eta \leq 1 \\ \displaystyle\frac{1}{2}\int_{B(a_i,2\varepsilon)} |\nabla\eta|^2 + \frac{1}{2\varepsilon^2}(1-\eta^2)^2 \leq C \\ \displaystyle\int_{B(a_i,2\varepsilon)} (1-\eta^2) \leq C\varepsilon^2. \end{cases}$$

u is finally defined by

$$u = \rho\eta e^{i\varphi}.$$

This has a meaning because, considering V, any subset of $\Omega\backslash \cup_i B(a_i,\varepsilon)$,

$$\begin{aligned} \int_{\partial V} \frac{\partial\varphi}{\partial\tau} &= \int_{\partial V} \frac{\partial U}{\partial n} + \omega\frac{\partial X}{\partial n} \\ &= \int_V \Delta U + \omega\Delta X = \int_V \Delta U + 2\omega = \int_V \mu \in 2\pi\mathbb{Z}, \end{aligned}$$

in view of the definition of μ, (IV.6), and (IV.7). In addition, in $\cup_i B(a_i,\varepsilon)$, $\eta \equiv 0$ so u is well-defined there too.

Step 4 : Energy of u.
We have all the elements to bound $J(u)$ from above. We denote as usual $v = \frac{u}{\rho} = \eta e^{i\varphi}$. Starting from (II.12), and translating the lattice as in [SS2] if necessary, we have, using $\rho \leq 1$, (IV.16) and (IV.17),

$$\begin{aligned} J(u) &= J(\rho) + \int_\Omega \frac{\rho^2}{2}|\nabla v|^2 + \frac{\rho^4}{4\varepsilon^2}(1-|v|^2)^2 + \int_\Omega \omega\rho^2(iv, dv \wedge dX) \\ &= J(\rho) - \frac{\omega^2}{2}\int_\Omega \rho^2 r^2 |v|^2 + \frac{1}{2}\int_\Omega \rho^2|\nabla v - i\omega v\nabla^\perp X|^2 + \int_\Omega \frac{\rho^4}{4\varepsilon^2}(1-|v|^2)^2 \\ &= J(\rho) - \frac{\omega^2}{2}\int_\Omega \rho^2 r^2 + \frac{1}{2}\int_\Omega \rho^2\left(|\nabla\eta|^2 + \eta^2|\nabla U|^2\right) + \frac{\rho^4}{4\varepsilon^2}(1-\eta^2)^2 + \frac{\omega^2}{2}\rho^2 r^2(1-\eta^2) \end{aligned}$$

Since there are $O(\omega)$ squares, from (IV.17), we have

$$\frac{1}{2}\int_\Omega |\nabla \eta|^2 + \frac{1}{2\varepsilon^2}(1-\eta^2)^2 \leq C\omega$$

and $\int_\Omega (1-\eta) \leq C\varepsilon^2\omega$. Thus

$$\left|\frac{\omega^2}{2}\int_\Omega \rho^2 r^2 (1-\eta^2)\right| \leq C\varepsilon^2\omega^3 = o(\omega)$$

since $\varepsilon\omega \to 0$. Thus, using (IV.15),

$$\begin{aligned} J(u) &\leq J(\rho) - \frac{\omega^2}{2}\int_\Omega \rho^2 r^2 + \frac{1}{2}\int_\Omega |\nabla U|^2 + \int_\Omega \frac{1}{2}|\nabla \eta|^2 + \frac{1}{4\varepsilon^2}(1-\eta^2)^2 + O(\omega^3\varepsilon^2) \\ &\leq J(\rho) - \frac{\omega^2}{2}\int_\Omega \rho^2 r^2 + |\Omega|\omega \log \frac{1}{\varepsilon\sqrt{\omega}} + O(\omega). \end{aligned}$$

Furthermore, as

$$J(\rho) = \frac{1}{2}\int_\Omega |\nabla \rho|^2 + \frac{1}{2\varepsilon^2}(1-\rho^2)^2 + \omega^2 r^2 \rho^2 = F(\rho) + \frac{1}{2}\omega^2 \int_\Omega r^2\rho^2,$$

we can also write

$$J(u) \leq F(\rho) + |\Omega|\omega \log \frac{1}{\varepsilon\sqrt{\omega}} + O(\omega).$$

If one removes the assumption $\omega \ll \frac{1}{\varepsilon}$, and replaces it by $\omega \ll \frac{1}{\varepsilon^2}$, we get at least $J(u) \leq F(\rho) + o(\omega^2)$. This proves the proposition. □

Proposition IV.2 *If* $|\log \varepsilon| \ll \omega \ll \frac{1}{\varepsilon}$, *then*

$$J(\rho) - \frac{\omega^2}{2}\int_\Omega r^2 + \omega|\Omega| \log \frac{1}{\varepsilon\sqrt{\omega}}(1+o(1)) \leq \min_{H_0^1(\Omega,\mathbb{C})} J \leq F(\rho) + \omega|\Omega| \log \frac{1}{\varepsilon\sqrt{\omega}}(1+o(1)).$$

Proof : This proof is essentially the same as that of Section IV in [SS2]. We explain how to obtain it by modifying step by step the proof of [SS2]. h_{ex} will always be replaced by ω. We consider u a minimizer of J, we then know that $|u| \leq 1$ by the maximum principle.

Step 1 : We split the energy as in Lemma II.2, using $v = \frac{u}{\rho}$:

(IV.18) $$J(u) = J(\rho) + \int_\Omega \frac{\rho^2}{2}|\nabla v|^2 + \frac{\rho^4}{4\varepsilon^2}(1-|v|^2)^2 - \omega \int_\Omega \rho^2 (iv, \nabla v.\nabla^\perp X).$$

Writing as usual $v = |v|e^{i\varphi}$, (IV.18) can be transformed into

(IV.19) $$J(u) = J(\rho) + \frac{1}{2}\int_\Omega \rho^2 |\nabla v - i\omega v \nabla^\perp X|^2 + \frac{\rho^4}{4\varepsilon^2}(1-|v|^2)^2 - \frac{\omega^2}{2}\int_\Omega \rho^2 |v|^2 r^2.$$

Using that $\rho|v| = |u| \le 1$, we may write

$$J(u) \ge J(\rho) - \int_\Omega \frac{\omega^2}{2} r^2 + \frac{1}{2}\int_\Omega \rho^2 |\nabla v - i\omega v \nabla^\perp X|^2 + \frac{\rho^4}{4\varepsilon^2}(1-|v|^2)^2. \tag{IV.20}$$

We are going to study the last integral as in [SS2]. Setting as previously $\Omega' = \Omega \backslash \{x/\delta(x) \le \varepsilon^\beta\}$, where $\beta \in]0,1[$, we have, from (II.5),

$$0 \le 1 - \rho \le o(1) \qquad \text{in } \Omega'. \tag{IV.21}$$

We now restrict to studying

$$j(u) := \int_{\Omega'} \frac{1}{2}|\nabla v - i\omega v \nabla^\perp X|^2 + \frac{1}{4\varepsilon^2}(1-|v|^2)^2, \tag{IV.22}$$

because we can assert that

$$J(u) \ge J(\rho) - \int_\Omega \frac{\omega^2}{2} r^2 + (1 - o(1)) j(u). \tag{IV.23}$$

Step 2 : As in [SS2], we partition Ω' into disjoint squares K whose centers are on a square lattice, of size $\delta(\varepsilon) = o(1)$, to be chosen as in [SS2] in order to satisfy

$$L \ll \omega\delta^2 \ll \min(\omega, L^2) \qquad \text{with } L := \log \frac{1}{\varepsilon\sqrt{\omega}}. \tag{IV.24}$$

We distinguish between the "good squares" on which

$$j_K(u) \le 2\delta^2 \omega L, \tag{IV.25}$$

and the "bad squares" on which

$$j_K(u) > 2\delta^2 \omega L. \tag{IV.26}$$

j_K here denotes the restriction of j to K.
We need to prove a lower bound on the good squares only, since we already have one on the bad squares. From now on, K denotes a "good square".

Step 3 : We denote $\Omega_t = \{x \in K/|v(x)| < t\}$, $\gamma_t = \partial\Omega_t$. Using $\int_K |\nabla v - iv\omega\nabla^\perp X|^2 = \int_K |\nabla|v||^2 + |v|^2|\nabla\varphi - \omega\nabla^\perp X|^2$ and applying the co-area formula as in [SS2], section IV.2, we obtain the following result,

$$j_K(u) \ge \int_0^\infty a(t) + 2tb(t)\, dt \ge \int_0^1 a(t) + 2tb(t)\, dt \tag{IV.27}$$

$$a(t) := \int_{\gamma_t \cap K} \frac{|\nabla|v||}{2} + \frac{(1-t^2)^2}{4\varepsilon^2 |\nabla|v||} \tag{IV.28}$$

$$b(t) := \frac{1}{2}\int_{K\backslash\overline{\Omega_t}} |\nabla\varphi - \omega\nabla^\perp X|^2. \tag{IV.29}$$

This will provide the desired lower bound.
Lemma IV.2 of [SS2] can be reproduced without change. We will show in Steps 5 and 6 that the result of Lemma IV.3 of [SS2] remains valid.

Step 4 : End of the proof of the proposition.
We deduce as in section IV.3 of [SS2] the similar lower bound on our good square :

$$j_K(u) \geq \omega\delta^2 L(1-o(1)). \tag{IV.30}$$

This lower bound is also true on the bad squares from (IV.26). If $N(\varepsilon)$ is the total number of squares included in Ω', $N(\varepsilon)\delta^2 \to |\Omega'|$ as $\varepsilon \to 0$, since $\delta \to 0$. Hence, adding up all the lower bounds, we are led to

$$j(u) \geq |\Omega'|\omega L(1-o(1)),$$

and as $|\Omega'| = |\Omega| - o(1)$, we deduce with (IV.23),

$$J(u) \geq J(\rho) - \int_\Omega \frac{\omega^2}{2} r^2 + (1-o(1))|\Omega|\omega \log \frac{1}{\varepsilon\sqrt{\omega}},$$

which proves Proposition IV.2 and Theorem 3. The conclusion in the case $\omega \leq \frac{1}{\varepsilon^{\frac{4}{5}}}$ follows from (II.7).

Step 5 : We show how to adjust the proof of Lemma IV.3 of [SS2]. First, we need to adjust the proof of Proposition IV.1 in [SS2], which relies on Lemma IV.4. For this lemma, juste replace the calculation by the following :

$$\begin{aligned} \forall t \in [r,R] \quad 2\pi d = \int_{\partial B_t} \frac{\partial\varphi}{\partial\tau} &= \int_{\partial B_t} \frac{\partial\varphi}{\partial\tau} - \omega\frac{\partial X}{\partial n} + \int_{\partial B_t} \omega\frac{\partial X}{\partial n} \\ &= \int_{\partial B_t} \frac{\partial\varphi}{\partial\tau} - \omega\frac{\partial X}{\partial n} + \int_{B_t} 2\omega, \end{aligned} \tag{IV.31}$$

since $\Delta X = 2$. Now, using the Cauchy-Schwarz inequality

$$\begin{aligned} \int_{\partial B_t} \left|\frac{\partial\varphi}{\partial\tau} - \omega\frac{\partial X}{\partial n}\right|^2 &\geq \frac{1}{2\pi t}\left(\int_{\partial B_t} \frac{\partial\varphi}{\partial\tau} - \omega\frac{\partial X}{\partial n}\right)^2 \\ &\geq \frac{1}{2\pi t}(2\pi d - 2\pi\omega t^2)^2, \end{aligned}$$

with (IV.31). Using the notation $e(t) = \frac{1}{2}\int_{\partial B_t} \left|\frac{\partial\varphi}{\partial\tau} - \omega\frac{\partial X}{\partial n}\right|^2$ as in [SS2], we get that

$$e(t) \geq \frac{\pi|d|}{t} - 2\pi|d|\omega t,$$

which leads to the same result as in [SS2] if we replace $f(r,R)$ by $\pi\left(\log\frac{R}{r} - \omega(R^2 - r^2)\right)$. This new f also satisfies the results of Lemma IV.5 of [SS2]. The rest of the proof of Proposition IV.1 is identical and yields to balls B_i, with lower bounds

$$\frac{1}{2}\int_{B_i\setminus V} |\nabla\varphi - \omega\nabla^\perp X|^2 \geq \pi|d_i|\left(\log\frac{\sigma}{r(V)} - C\right)_+, \tag{IV.32}$$

with the same notations as in Proposition IV.1 of [SS2], except that ω is replaced by V.

- *Step 6 :* We adjust the rest of the proof of Lemma IV.3. The lower bound (IV.22) on $b(t)$ is still true (with ω instead of h_{ex}). "Step 1 : an auxiliary field" has to be suppressed and "Step 2 : estimating d_t" to be modified as follows. As in (IV.4), we can define a function U on our square K such that

$$\eta^2(\nabla\varphi - \omega\nabla^\perp X) = \nabla^\perp U.$$

As in (IV.5), since we are on a good square

$$\begin{aligned} 4\omega^2\delta^2 L \geq 2j_K(u) &\geq \int_\Omega |\nabla|v||^2 + |v|^2|\nabla\varphi - \omega\nabla^\perp X|^2 \\ &\geq \int_\Omega |\nabla|v||^2 + \frac{|v|^2}{\eta^4}|\nabla U|^2 \\ &= \int_\Omega |\nabla|v||^2 + \frac{1}{\eta^2\rho^2}|\nabla U|^2 \\ &\geq \int |\nabla\eta|^2 + |\nabla U|^2, \end{aligned} \tag{IV.33}$$

hence we have a good control on U. Therefore, replacing $\overline{h}$ by U, we can find as in [SS2] some t_0 satisfying the equivalent of (IV.31) :

$$\partial K_{t_0} \cap \cup_i B_i = \varnothing \qquad \text{and} \qquad \int_{\partial K_{t_0}} \left|\frac{\partial U}{\partial n}\right|^2 < \frac{8\omega\delta^2}{\alpha t_0^2} L. \tag{IV.34}$$

Consequently, as in (IV.32) in [SS2],

$$\begin{aligned} 2\pi d_t \geq 2\pi \sum_i d_{B_i} &= \int_{\partial K_{t_0}} \frac{\partial\varphi}{\partial\tau} = \int_{\partial K_{t_0}} \frac{1}{\eta^2}\frac{\partial U}{\partial n} + \omega\frac{\partial X}{\partial n} \\ &= \int_{K_{t_0}} 2\omega + \int_{\partial K_{t_0}} \frac{1}{\eta^2}\frac{\partial U}{\partial n}. \end{aligned} \tag{IV.35}$$

The second term can be bounded from above, using the fact that $\partial K_{t_0} \cap \cup_i B_i = \varnothing$, hence $\eta > t$ on ∂K_{t_0} :

$$\left|\int_{\partial K_{t_0}} \frac{1}{\eta^2}\frac{\partial U}{\partial n}\right| \leq \frac{1}{t^2}\left(\int_{\partial K_{t_0}} \left|\frac{\partial U}{\partial n}\right|^2\right)^{\frac{1}{2}} \sqrt{\delta} \leq \frac{C\delta^{\frac{3}{2}}}{t^2}\left(\frac{\omega L}{\alpha}\right)^{\frac{1}{2}},$$

using (IV.34). The first term is exactly $2\omega t_0^2$, and using the estimates on t_0, we obtain the exact analogue of (IV.33) of [SS2] :

$$2\pi d_t \geq 2\omega\delta^2\left(1 - \frac{4\alpha}{\delta} - \frac{C}{t}\left(\frac{L}{\omega\delta\alpha}\right)^{\frac{1}{2}}\right). \tag{IV.36}$$

We are led to

$$2\pi d_t \geq 2\omega\delta^2 \left(1 - \frac{C}{t}\Delta\right)_+ ,$$

then, the rest of the proof can be exactly reproduced. Notice that the only change is the factor 2 coming from ΔX that appears in (IV.36) above and leads to twice the lower bound of [SS2]. This completes the proof of the proposition and of Theorem 3. $\square$

We recall Theorem 3 :

Theorem 3 *Assume* $|\log \varepsilon| \ll \omega \ll \frac{1}{\varepsilon}$. *Then*

$$J(\rho) - \int_\Omega \frac{\omega^2}{2} r^2 + \omega|\Omega| \log \frac{1}{\varepsilon\sqrt{\omega}}(1 - o(1)) \leq \min_{H_0^1(\Omega,\mathbb{C})} J \leq F(\rho) + \omega|\Omega| \log \frac{1}{\varepsilon\sqrt{\omega}} + O(\omega),$$

where $|.|$ *denotes the volume.*
If in addition $\omega \leq \frac{C}{\varepsilon^{4/5}}$, *then*

$$\min_{H_0^1(\Omega,\mathbb{C})} J = F(\rho) + \omega|\Omega| \log \frac{1}{\varepsilon\sqrt{\omega}}(1 + o(1)).$$

Reproducing the proof of section V of [SS2], we can obtain the same result concerning the existence of large vortex-balls in Ω' :

Theorem 4 *Let* $|\log \varepsilon| \ll \omega \leq \frac{C}{\varepsilon^{4/5}}$, *and* u_ε *be a corresponding minimizer of* J. *Then, for* $\varepsilon < \varepsilon_0$, *there exists a family of disjoint disks* (B_i^ε) *with radii each less than* $\frac{1}{\sqrt{\omega}}$ *and sum less than* $|\Omega|\sqrt{\omega}$, *such that* $|u_\varepsilon| \geq \frac{1}{2}$ *on* ∂B_i^ε *and, if* a_i^ε *is the center of* B_i^ε *and* $d_i^\varepsilon = deg(\frac{u_\varepsilon}{|u_\varepsilon|}, \partial B_i^\varepsilon)$, *then*

$$\mu_\varepsilon = \frac{2\pi}{\omega} \sum_i d_i^\varepsilon \delta_{a_i^\varepsilon} \underset{\varepsilon\to 0}{\longrightarrow} 2\, dx$$

in the weak sense of measures, where dx *is the Lebesgue measure on* $\mathbb{R}^2$ *restricted to* Ω.

Moreover,

$$\pi \sum_i |d_i^\varepsilon| \simeq \pi \sum_i d_i^\varepsilon \simeq \omega|\Omega|,$$

and most of the vortex-energy is concentrated in the balls, i.e.

$$J_{\Omega \setminus \cup_i B_i^\varepsilon}(u_\varepsilon) - F(\rho) = o(J(u_\varepsilon) - F(\rho)).$$

V Global minimizers in the general case : the free-boundary model

In this section, we will consider ω's such that $\omega \leq O(|\log \varepsilon|)$ i.e. the intermediate case for which the results of Section IV are not relevant. More precisely, similarly as in [SS3], we assume that

$$\lambda = \lim_{\varepsilon \to 0} \frac{|\log\, \varepsilon|}{\omega(\varepsilon)} \tag{V.1}$$

exists and is finite, positive. We recall that $\mathcal{R}$ denotes the set of Radon measures on Ω, and we define

$$E(f) = \frac{\lambda}{2} \int_\Omega |\Delta f + 2| + \frac{1}{2} \int_\Omega |\nabla f|^2 \tag{V.2}$$

over the set

$$V = \{f \in H_0^1(\Omega) / \Delta f \in \mathcal{R}\}. \tag{V.3}$$

V.1 Upper bound for the minimal energy

We have the following

Proposition V.1 *Let $\omega(\varepsilon)$ be such that $\lambda > 0$ and u_ε be a minimizer of J, then*

$$\limsup_{\varepsilon \to 0} \frac{J(u_\varepsilon) - F(\rho)}{\omega^2} \le \min_V E.$$

We shall see later that $\min_V E$ is uniquely achieved by some function U_* for which $\mu_* = \Delta U_* + 2$ is a positive measure, and an $L^\infty(\Omega)$ function.
We do not write the proof of this proposition in full, because it is mainly the same as in [SS3]. One only has to replace the definition of G by the Green kernel

$$\begin{cases} -\Delta_x G(x,y) = \delta_y & \text{in } \Omega \\ G(x,y) = 0 & \text{on } \partial\Omega. \end{cases} \tag{V.4}$$

We construct as in Proposition II.2 of [SS3] a family of $O(\omega)$ points a_i^ε, and a family of measures μ_ε supported on the disjoint circles $C(a_i^\varepsilon, \varepsilon)$, such that μ_ε tends to μ_* in $\mathcal{R}$, where μ_* is the measure associated to the minimizer U_* of E. We define a test-function U_ε by

$$\begin{cases} \Delta U_\varepsilon = \omega(\mu_\varepsilon - 2) & \text{in } \Omega \\ U_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{V.5}$$

Then, it is easy to adjust the proof of [SS3] in order to get

$$\begin{aligned} \frac{1}{2\omega^2} \int_\Omega |\nabla U_\varepsilon|^2 &= \frac{1}{2} \int_{\Omega \times \Omega} G(x,y) d(\mu_\varepsilon - 2)(x) d(\mu_\varepsilon - 2)(y) \\ &\le \frac{\lambda}{2} \int_\Omega \mu_* + \frac{1}{2} \int_{\Omega \times \Omega} G(x,y) d(\mu_* - 2)(x) d(\mu_* - 2)(y) + o(1) \\ &\le \frac{\lambda}{2} \int_\Omega \mu_* + \frac{1}{2} \int_\Omega |\nabla U_*|^2 + o(1). \end{aligned}$$

Thus,

$$\text{(V.6)} \qquad \frac{1}{2\omega^2}\int_\Omega |\nabla U_\varepsilon|^2 \le E(U_*) + o(1).$$

Then, as for Proposition IV.1, we construct our test-function by choosing

$$\text{(V.7)} \qquad \nabla\varphi = \nabla^\perp U_\varepsilon + \omega\nabla^\perp X \quad \text{in } \Omega\setminus\cup_i B(a_i^\varepsilon,\varepsilon).$$

The measures μ_ε were defined so that the mass of $\omega\mu_\varepsilon$ restricted to each circle $C(a_i,\varepsilon)$ is exacly 2π ; therefore, if V is any regular subset of $\Omega\setminus\cup_i B(a_i,\varepsilon)$, we have

$$\begin{aligned}\int_{\partial V}\frac{\partial\varphi}{\partial\tau} &= \int_{\partial V}\frac{\partial U_\varepsilon}{\partial n} + \omega\frac{\partial X}{\partial n} = \int_V \Delta U_\varepsilon + \omega\Delta X = \int_V \Delta U + 2\omega \\ &= \int_V \omega\mu_\varepsilon \in 2\pi\mathbb{Z}.\end{aligned}$$

Hence, $e^{i\varphi}$ can be well-defined on $\Omega\setminus\cup_i B(a_i,\varepsilon)$. Then, we choose η satisfying the same conditions as in (IV.17), and define the test-function u to be $u=\rho\eta e^{i\varphi}$. Again, as in (IV.19), denoting $v=\frac{u}{\rho}=\eta e^{i\varphi}$,

$$J(u) = J(\rho) + \frac{1}{2}\int_\Omega \rho^2|\nabla v - i\omega v\nabla^\perp X|^2 + \frac{\rho^4}{4\varepsilon^2}(1-|v|^2)^2 - \frac{\omega^2}{2}\int_\Omega \rho^2|v|^2r^2.$$

Hence, using $\rho\le 1$, $\rho\eta\le 1$, and (V.7),

$$\begin{aligned}J(u) &\le J(\rho) - \frac{\omega^2}{2}\int_\Omega \rho^2r^2 + \frac{1}{2}\int_\Omega |\nabla\eta|^2 + |\nabla U|^2 + \frac{1}{2\varepsilon^2}(1-\eta^2)^2 + \frac{\omega^2}{2}\int_\Omega \rho^2r^2(1-|v|^2) \\ &\le F(\rho) + \frac{1}{2}\int_\Omega |\nabla U|^2 + \frac{1}{2}\int_\Omega |\nabla\eta|^2 + \frac{1}{2\varepsilon^2}(1-\eta^2)^2 + \frac{\omega^2}{2}\int_\Omega \rho^2r^2(1-|v|^2).\end{aligned}$$

From the conditions (IV.17), since there are $O(\omega)$ balls, $\frac{1}{2}\int_\Omega|\nabla\eta|^2 + \frac{1}{2\varepsilon^2}(1-\eta^2)^2 \le O(\omega)$, and $\omega^2\int_\Omega(1-|v|^2)\le\varepsilon^2\omega^3$. Hence, using (V.6),

$$J(u) \le F(\rho) + \omega^2 E(U_*) + O(\omega),$$

and the proposition is proved. □

V.2 Constructing vortex-balls

From now on, we consider u_ε minimizing J, and we associate to it U_ε defined by (I.10). Here we prove the following proposition stated in the introduction :

Proposition V.2 *If $\lambda > 0$, there exists ε_0 such that for $\varepsilon < \varepsilon_0$, if u_ε minimizes J, and U_ε is associated to u_ε by (I.10), then there exists a family of balls (depending on ε) $(B_i)_{i\in I_\varepsilon} = (B(a_i, r_i))_{i\in I_\varepsilon}$ satisfying*

$$\left\{x \in \Omega'/||v|-1| \geq \frac{1}{|\log\,\varepsilon|}\right\} \subset \cup_{i\in I_\varepsilon} B_i \tag{V.8}$$

$$\sum_{i\in I_\varepsilon} r_i \leq \frac{1}{|\log\,\varepsilon|^6} \tag{V.9}$$

$$\frac{1}{2}\int_{B_i} |\nabla U_\varepsilon|^2 \geq \pi |d_i||\log\,\varepsilon|(1-o(1)) \tag{V.10}$$

where $d_i = deg(u, \partial B_i)$.

As in Section III, Ω' denotes $\{x \in \Omega/\delta(x) \geq \varepsilon^\beta\}$.
Proof : As in Proposition IV.2, we define $v = \frac{u}{\rho}$, $\Omega_t = \{x \in \Omega'/|v(x)| < t\}$, and $\gamma_t = \partial\Omega_t$. From the a-priori upper bound of Proposition V.1, $J(u) - F(\rho) \leq C\omega^2$, hence

$$\frac{1}{2}\int_\Omega \rho^2|\nabla v|^2 + \frac{\rho^4}{2\varepsilon^2}(1-|v|^2)^2 \leq C\omega^2$$

and

$$\frac{1}{2}\int_{\Omega'} |\nabla v|^2 + \frac{1}{2\varepsilon^2}(1-|v|^2)^2 \leq C\omega^2. \tag{V.11}$$

As in [SS3], Proposition I.1, applying the coarea formula to (V.11), we can find $t \in \left[1 - \frac{1}{|\log\,\varepsilon|}, 1 + \frac{1}{|\log\,\varepsilon|}\right]$, such that $r(\gamma_t) \leq C\varepsilon|\log\,\varepsilon|^3$. (As in [SS2], $r(\gamma_t)$ is defined as the infimum over all finite coverings of γ_t by balls, of the sum of the radii of the balls.) Arguing as for (IV.32) with $\sigma = \frac{1}{|\log\,\varepsilon|^6}$ and $V = \Omega_t$, we can find balls B_i covering Ω_t such that

$$\frac{1}{2}\int_{B_i\backslash\Omega_t} |\nabla\varphi - \omega\nabla^\perp X|^2 \geq \pi|d_i||\log\,\varepsilon|(1-o(1)). \tag{V.12}$$

Hence,

$$\begin{aligned}\frac{1}{2}\int_{B_i\backslash\Omega_t} |\nabla U|^2 = \quad & \frac{1}{2}\int_{B_i\backslash\Omega_t} \eta^4|\nabla\varphi - \omega\nabla^\perp X|^2 = \\ & \frac{1}{2}\int_{B_i\backslash\Omega_t} |\nabla\varphi - \omega\nabla^\perp X|^2 + \frac{1}{2}\int_{B_i\backslash\Omega_t} (\eta^4 - 1)|\nabla\varphi - \omega\nabla^\perp X|^2.\end{aligned} \tag{V.13}$$

Since $0 \leq 1 - \rho \leq o(1)$ on Ω' and $|1 - |v|| \leq \frac{1}{|\log\,\varepsilon|}$ on $\Omega'\backslash\Omega_t$, (V.13) becomes

$$\begin{aligned}\frac{1}{2}\int_{B_i\backslash\Omega_t} |\nabla U|^2 \;\; &\geq \;\; \frac{1}{2}\int_{B_i\backslash\Omega_t} |\nabla\varphi - \omega\nabla^\perp X|^2(1-o(1)) \\ &\geq \;\; \pi|d_i||\log\,\varepsilon|(1-o(1)).\end{aligned}$$

□

From this construction, we can define the vorticity measures μ_ε

$$\mu_\varepsilon = \frac{2\pi}{\omega} \sum_i d_i \delta_{a_i}, \tag{V.14}$$

where the (a_i, d_i)'s are given by Proposition V.2.

Lemma V.1 *Under the same hypotheses, we can extract a sequence $\varepsilon_n \to 0$ such that*

$$\frac{U_{\varepsilon_n}}{\omega} \rightharpoonup U_0 \quad \textit{weakly in } H_0^1(\Omega),$$
$$\mu_{\varepsilon_n} \to \mu_0 \quad \textit{in } \mathcal{R}.$$

In addition,

$$\Delta U_0 = \mu_0 - 2. \tag{V.15}$$

Proof :
Step 1 : From the upper bound of Proposition V.1,

$$J(u_\varepsilon) - F(\rho) \le C\omega^2. \tag{V.16}$$

On the other hand, as in (IV.20),

$$J(u_\varepsilon) \ge J(\rho) - \int_\Omega \frac{\omega^2 r^2}{2} + \frac{1}{2}\int_\Omega \rho^2 |\nabla v - i\omega v \nabla^\perp X|^2. \tag{V.17}$$

In addition,

$$J(\rho) - \int_\Omega \frac{\omega^2 r^2}{2} = F(\rho) + \int_\Omega \frac{\omega^2 r^2}{2}(\rho^2 - 1).$$

Since we know that $\frac{1}{\varepsilon^2}\int_\Omega (1-\rho^2)^2 \le \frac{C}{\varepsilon}$, we have $|\int_\Omega \frac{1}{2}\omega^2 r^2(\rho^2-1)| \le C\omega^2 \varepsilon^{\frac{1}{2}} = o(1)$ because we are in the case $\omega = O(|\log \varepsilon|)$. Thus, we can transform (V.17) into

$$J(u_\varepsilon) \ge F(\rho) + \frac{1}{2}\int_\Omega \rho^2 |\nabla |v||^2 + \frac{\rho^2 |v|^2}{\eta^4}|\nabla U|^2 - o(1).$$

Since $\eta \le 1$,

$$J(u_\varepsilon) - F(\rho) \ge \frac{1}{2}\int_\Omega \frac{|\nabla U|^2}{\eta^2} - o(1) \ge \frac{1}{2}\int_\Omega |\nabla U|^2 - o(1). \tag{V.18}$$

Comparing to (V.16) and (V.18), we see that $\int_\Omega |\nabla U|^2 \le C\omega^2$, hence $\frac{U}{\omega}$ is bounded in H_0^1, and extracting a subsequence $\varepsilon_n \to 0$, we can assume that it converges weakly to some U_0

in H_0^1.
Step 2 : We prove the same result for μ_ε. From (V.18),

$$\begin{aligned} C\omega^2 \geq J(u) - F(\rho) &\geq \frac{1}{2}\sum_{i\in I_\varepsilon}\int_{B_i} |\nabla U|^2 - o(1) \\ &\geq \pi \sum_{i\in I_\varepsilon} |d_i||\log\varepsilon|(1-o(1)) \\ &\geq \left(\frac{1}{2}|\mu_\varepsilon|\right)\omega|\log\varepsilon|(1-o(1)). \end{aligned} \tag{V.19}$$

Hence, $\int_\Omega |\mu_\varepsilon| \leq \frac{C\omega}{|\log\varepsilon|} \leq \frac{C}{\lambda}$. Since we have assumed $\lambda > 0$, (μ_ε) is bounded in the sense of measures; hence, extracting again if necessary, we can assume that it converges weakly to some Radon measure μ_0.
Step 3 : There remains to compare U_0 and μ_0. For that purpose we use the following lemma, in the spirit of Lemma III.4.

Lemma V.2

$$\frac{1}{\omega}\mathrm{curl}\ (iu, \nabla u) - \mu_\varepsilon \to 0 \text{ in } \mathcal{D}'.$$

Proof : Consider $\xi \in C_0^\infty(\Omega)$. Since ξ has compact support included in Ω, for ε sufficiently small, its support is included in $\Omega' = \{x \in \Omega/\delta(x) \leq \varepsilon^\beta\}$. Hence, for ε small enough,

$$\int_\Omega \frac{\xi}{\omega}\mathrm{curl}\ (iu, \nabla u) = \int_{\Omega'} \frac{\xi}{\omega}\mathrm{curl}\ (iu, \nabla u),$$

and then we can proceed as in [SS1], Lemma II.3 (or [ASS] Lemma II.2), and obtain

$$\int_\Omega \frac{\xi}{\omega}\mathrm{curl}\ (iu, \nabla u) = \frac{2\pi}{\omega}\sum_{i\in I_\varepsilon} d_i\xi(a_i) + o(1)\|\nabla\xi\|_{L^\infty}.$$

□

Returning to the proof of the proposition, since $\mu_\varepsilon \to \mu_0$ in $\mathcal{R}$, Lemma V.2 implies that $\frac{1}{\omega}\mathrm{curl}\ (iu, \nabla u) \to \mu_0$ in $\mathcal{D}'$. Back to (I.10), we have

$$\nabla^\perp U = (iu, \nabla u) - |u|^2\omega\nabla^\perp X. \tag{V.20}$$

Let again ξ be some test-function in $C_0^\infty(\Omega)$.

$$\left|\int_\Omega (1-|u|^2)\nabla^\perp X.\nabla^\perp\xi\right| \leq C\int_\Omega |1-|u|^2| \leq C\left(\varepsilon^2 J(u)\right)^{\frac{1}{2}} \leq o(1),$$

from the a priori bound $J(u) \leq F(\rho) + O(\omega^2) \leq \frac{C}{\varepsilon}$. Hence, curl $((1-|u|^2)\nabla^\perp X) \rightharpoonup 0$ in $\mathcal{D}'$. Then, we divide (V.20) by ω and take the curl :

$$\begin{aligned} \frac{\Delta U}{\omega} &= \frac{\mathrm{curl}\ (iu, \nabla u)}{\omega} + \mathrm{curl}\ ((1-|u|^2)\nabla^\perp X) - \Delta X \\ \frac{\Delta U}{\omega} &= -2 + \frac{\mathrm{curl}\ (iu, \nabla u)}{\omega} + \mathrm{curl}\ ((1-|u|^2)\nabla^\perp X). \end{aligned} \tag{V.21}$$

We saw that curl $((1-|u|^2)\nabla^\perp X) \to 0$ in $\mathcal{D}'$, $\frac{\text{curl }(iu,\nabla u)}{\omega} \to \mu_0$ in $\mathcal{D}'$, and $\frac{\Delta U}{\omega} \to \Delta U_0$ in $\mathcal{D}'$, hence passing to the limit in (V.21), we are led to

$$\Delta U_0 = \mu_0 - 2. \tag{V.22}$$

□

V.3 Lower bound for the energy

We are now in a position to deduce

Lemma V.3 *Under the same hypotheses,*

$$\liminf_{n\to\infty} \frac{J(u_{\varepsilon_n}) - F(\rho)}{\omega^2} \geq \frac{\lambda}{2}\int_\Omega |\mu_0| + \frac{1}{2}\int_\Omega |\nabla U_0|^2 + E(U_0).$$

Proof : In the spirit of [SS3], we start with the lower bound of (V.18):

$$J(u_\varepsilon) - F(\rho) \geq \frac{1}{2}\int_\Omega |\nabla U|^2 - o(1),$$

and split the energy between that contained in the vortex-balls B_i and that contained outside the B_i's:

$$J(u_\varepsilon) - F(\rho) \geq \sum_{i\in I_\varepsilon} \frac{1}{2}\int_{B_i} |\nabla U|^2 + \int_{\Omega\setminus\cup_i B_i} |\nabla U|^2 - o(1). \tag{V.23}$$

Arguing as in [SS3], we can extract from ε_n a subsequence if necessary and define

$$\mathcal{A}_N = \cup_{n>N} \cup_{i\in I_{\varepsilon_n}} B_i$$

such that meas$\mathcal{A}_N \to 0$ as $N \to \infty$, and $\mathcal{A}_N$ contains all the balls. Then, by weak convergence of $\frac{U}{\omega}$ to U_0,

$$\liminf_{n\to\infty} \frac{1}{\omega^2}\int_{\Omega\setminus\cup_i B_i} |\nabla U|^2 \geq \liminf_{n\to\infty} \frac{1}{\omega^2}\int_{\Omega\setminus\mathcal{A}_N} |\nabla U|^2 \geq \int_{\Omega\setminus\mathcal{A}_N} |\nabla U_0|^2. \tag{V.24}$$

On the other hand, from (V.18),

$$\sum_{i\in I_{\varepsilon_n}} \frac{1}{2}\int_{B_i} \frac{|\nabla U|^2}{\omega^2} \geq \frac{|\log \varepsilon_n|}{2\omega}\int_\Omega |\mu_{\varepsilon_n}|,$$

and, by convexity of $\mu \mapsto \int_\Omega |\mu|$ and weak convergence of μ_{ε_n},

$$\liminf_{n\to\infty} \frac{1}{2\omega^2} \sum_{i\in I_{\varepsilon_n}} \frac{1}{2}\int_{B_i} |\nabla U|^2 \geq \frac{\lambda}{2}. \tag{V.25}$$

Combining (V.23)—(V.25), we are led to

$$\liminf \frac{J(u)-F(\rho)}{\omega^2} \geq \frac{\lambda}{2}\int_\Omega |\mu_0| + \frac{1}{2}\int_{\Omega\setminus\mathcal{A}_N} |\nabla U_0|^2.$$

Since this is true for all N, passing to the limit as $N\to\infty$, we obtain

$$\liminf \frac{J(u)-F(\rho)}{\omega^2} \geq \frac{\lambda}{2}\int_\Omega |\mu_0| + \frac{1}{2}\int_{\Omega} |\nabla U_0|^2.$$

$\square$

Therefore, we have the lower bound

$$\liminf_{n\to\infty} \frac{J(u_{\varepsilon_n})-F(\rho)}{\omega^2} \geq \inf E,$$

corresponding to the upper bound of Proposition V.1. There only remains to check that $\min E$ is achieved, and what the minimizer is.

Proposition V.3 *For any $\lambda \geq 0$, $\min_V E$ is uniquely achieved by a $U_* \in H^1_0 \cap C^{0,\gamma}(\forall \gamma < 1)$, solution of the following obstacle problem (we write $\mu_* = \Delta U_* + 2$) :*

$$\text{(V.26)} \qquad \begin{cases} U_* = 0 & \text{on } \partial\Omega \\ U_* \leq \frac{\lambda}{2} & \text{in } \Omega \\ (\Delta U_* + 2)(U_* - \frac{\lambda}{2}) = 0 & \text{in } \Omega. \end{cases}$$

Moreover $\mu_* \geq 0$ and $\mu_* \in L^\infty(\Omega)$. In addition, if we define the set $\mathcal{U}_\lambda$ to be $\mathcal{U}_\lambda =$ Supp $\mu_* = \{x \in \Omega / U_*(x) = \frac{\lambda}{2}\}$, then $\mathcal{U}_\lambda = \varnothing \Leftrightarrow \lambda > 2\max \xi_0$ where ξ_0 is given by (I.16).

Proof : The minimum of E is achieved by some U_* by weak lower semi-continuity, and this minimizer is unique since E is convex. We observe that for any $f \in H^1_0$ with $\mu = \Delta f + 2 \in \mathcal{R}$,

$$\text{(V.27)} \qquad f(y) = -\int_\Omega G(x,y) d(\mu(x) - 2) \quad \text{a.e. in } y.$$

(We recall that $-\Delta_x G = \delta_y$.) Thus

$$\text{(V.28)} \qquad E(f) = \frac{\lambda}{2}\int_\Omega |\mu| + \frac{1}{2}\int_{\Omega\times\Omega} G(x,y) d(\mu(x)-2) d(\mu(y)-2).$$

Making variations in μ around $\mu_* = \Delta U_* + 2$ as in [SS3], we obtain that

$$\text{(V.29)} \qquad \frac{\lambda}{2}|\mu_*| - U_*\mu_* = 0,$$

$$\text{(V.30)} \qquad -\frac{\lambda}{2} \leq U_* \leq \frac{\lambda}{2} \quad \text{a.e.}$$

From (V.29), we deduce, denoting by μ_*^+ and μ_*^- the positive and negative parts of μ_* ($\mu_* = \mu_*^+ - \mu_*^-$), that

$$\begin{cases} U_* = \frac{\lambda}{2} & \mu_*^+ \text{ a.e.} \\ U_* = -\frac{\lambda}{2} & \mu_*^- \text{ a.e.} \end{cases}$$

Denoting by U_*^- the negative part of U_*,

$$\text{(V.31)} \qquad \int_\Omega \mu_* U_*^- = \int_\Omega (\Delta U_* + 2) U_*^- = -\int_\Omega |\nabla U_*|^2 + 2\int_\Omega U_* \leq 0.$$

On the other hand,

$$\text{(V.32)} \qquad \int_\Omega \mu_* U_* = \int_\Omega U_*^- \mu_*^+ - \int_\Omega U_*^- \mu_*^-.$$

Since $U_* = \frac{\lambda}{2}$, μ_*^+- almost everywhere, there remains

$$\int_\Omega \mu_* U_*^- = \frac{\lambda}{2} \int_\Omega \mu_*^- \geq 0.$$

Combining this with (V.31), we have $\int_\Omega |\mu_*^-| = 0$ hence $\mu_* = \mu_*^+$ is a positive measure. Thus, U_* satisfies

$$\begin{cases} U_* \leq \frac{\lambda}{2} \\ U_* = 0 & \text{on } \partial\Omega \\ \mu_* \geq 0 \\ \mu_*(U_* - \frac{\lambda}{2}) = 0. \end{cases}$$

The proof of [SS3] adjusts also easily to state that μ_* is absolutely continuous with respect to the Lebesgue measure, and in fact L^∞, hence $U_* \in C^{0,\gamma}(\forall \gamma < 1)$.

If we denote by $\mathcal{U}_\lambda$ the "vortex-region", ie the support of μ_*, or the set $\{x \in \Omega / U_*(x) = \frac{\lambda}{2}\}$, we prove that $\mathcal{U}_\lambda = \varnothing \Leftrightarrow \lambda > 2\max \xi_0$. This comes from the maximum principle : recall that ξ_0 is the solution of

$$\begin{cases} -\Delta \xi_0 = 2 & \text{in } \Omega \\ \xi_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

Observe that $\xi_0 = \frac{R^2}{2} - \frac{|x|^2}{2}$ if $\Omega = B(0,R)$. $\mu_* \geq 0$ means that $\Delta U_* \geq 2$ hence $-\Delta(\xi_0 - U_*) \geq 0$. In addition, $\xi_0 - U_* = 0$ on $\partial\Omega$; consequently, by the maximum principle, $\xi_0 \geq U_*$ in Ω. Now if $\mathcal{U}_\lambda \neq \varnothing$, there exists $x \in \Omega$ such that $U_*(x) = \frac{\lambda}{2}$ and $\xi_0(x) \geq \frac{\lambda}{2}$. Therefore, $\lambda \leq 2\max \xi_0$. On the other hand, if $\mathcal{U}_\lambda = \varnothing$, then the equation for U_* reduces to

$$\begin{cases} U_* \leq \frac{\lambda}{2} \\ U_* = 0 & \text{on } \partial\Omega \\ \Delta U_* + 2 = 0 & \text{in } \Omega, \end{cases}$$

Hence $U_* = \xi_0$ and $\xi_0 \leq \frac{\lambda}{2}$ and $\lambda \geq 2\max \xi_0$. $\square$

Comparing the upper bound of Proposition V.1 to the lower bound of Lemma V.3, we have

$$E(U_*) \geq \liminf_{n\to\infty} \frac{J(u_{\varepsilon_n}) - F(\rho)}{\omega^2} \geq E(U_0) \geq E(U_*).$$

Hence, by uniqueness of the minimizer of E, $U_0 = U_*$ and $\mu_0 = \mu_*$. Furthermore, these are the only possible limits for sequences extracted from U_ε and μ_ε, hence the whole families U_ε and μ_ε converge to U_* and μ_*. We have thus proved Theorem 3 in the case $\lambda > 0$. In the case $\lambda = 0$, returning to (IV.20), we have

$$\begin{aligned} J(u) &\geq J(\rho) - \frac{1}{2}\int_\Omega \omega^2 r^2 + \frac{1}{2}\int_\Omega \rho^2 |v|^2 \frac{|\nabla U|^2}{\eta^4} \\ &\geq F(\rho) + \frac{\omega^2}{2}\int_\Omega r^2(\rho^2 - 1) + \frac{1}{2}\int_\Omega |\nabla U|^2 \\ &\geq F(\rho) + \frac{1}{2}\int_\Omega |\nabla U|^2 + o(\omega^2), \end{aligned}$$

since $\int_\Omega |1-\rho^2| = o(1)$. On the other hand, we saw in Proposition IV.1 that for $\omega \ll \frac{1}{\varepsilon^2}$ we have the upper bound $\min J \leq F(\rho) + o(\omega^2)$, hence if u minimizes J, $\frac{1}{2}\int_\Omega |\nabla U|^2 \leq o(\omega^2)$. We conclude that $\frac{U}{\omega} \to 0$ in $H^1_0(\Omega)$, and that the assertions of Theorem 3 remain true in the case $\lambda = 0$, $\omega \ll \frac{1}{\varepsilon^2}$.

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