Lowest Landau level approach in superconductivity for the Abrikosov lattice close to H_{c_2}

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Abstract

We study the Ginzburg-Landau energy for a superconductor submitted to an applied magnetic field $h_{\rm ex}$ just below the "second critical field" H_{c_2}. When the Ginzburg-Landau parameter ε is small, we show that the mean energy per unit volume can be approximated by a reduced energy on a torus. Moreover, we expand this reduced energy in terms of $H_{c_2} - h_{\rm ex}$: when this quantity gets small, the problem amounts to a minimization problem on a finite-dimensional space, equivalent to the "lowest Landau level" in other approaches. This connects the Ginzburg-Landau energy to the "Abrikosov problem" of locating vortices optimally on a lattice.

1 Introduction

A superconducting material subject to an external magnetic field of intensity $h_{\rm ex}$ is described by its wave-function u, a complex-valued order parameter, and its potential-vector A so that $h := \operatorname{curl} A$ is the induced magnetic field in the sample; $|u|^2$ measures the local density of superconducting electrons in the material. The response of the material varies according to the value of the external field $h_{\rm ex}$ and the value of the Ginzburg-Landau parameter ε (inverse of the usual Ginzburg-Landau parameter κ), which is taken to be small to model extreme type II superconductors. The state of the superconductor can be studied through the minimization of the Ginzburg-Landau energy

$$J_\Omega(u, A) = \frac{1}{2}\int_\Omega |\nabla_A u|^2 + |\operatorname{curl} A - h_{\rm ex}|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2}. \tag{1.1}$$

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Here $\nabla_A = \nabla - iA$ is the covariant gradient and Ω is a two-dimensional bounded and simply-connected domain, representing the cross-section of the material. We refer to the book [SS1] for mathematical results that have been obtained, and to [SST, T] for physics reference. See also [AD].

In this paper, we are interested in the regime where the intensity of the applied magnetic field $h_{\rm ex}$ is of order $1/\varepsilon^2$, thus we set

$$h_{\rm ex} = \frac{b}{\varepsilon^2}. \tag{1.2}$$

For fixed b, if ε is sufficiently small, it is known that A is such that $\operatorname{curl} A$ is very close to $h_{\rm ex}$ (see e.g. Fournais-Helffer [FH2]). The behaviour of u, on the contrary, is strongly dependent on b: when b is too large ($b > b_0 \simeq 1.695$), the only critical point of the energy is the normal solution $u = 0$, and A is such that $\operatorname{curl} A = h_{\rm ex}$ (see [GP]). The behaviour of u is analyzed by considering the linearized problem around the normal solution $u = 0$, $\operatorname{curl} A = h_{\rm ex}$. Two eigenvalues emerge: the value of b_0 corresponds to an eigenvalue problem for a half-space, studied by many authors, see e.g. [LP1, LP2, HM], while $b = 1$ is the eigenvalue for the full space $\mathbb{R}^2$. The value $b = 1$, that is $h_{\rm ex} = 1/\varepsilon^2$ is also called the second critical field H_{c_2}, while b_0/ε^2 is the third critical field H_{c_3}.

For $b \in (1, b_0)$, the minimizer is such that $|u|$ is very small, except on a thin boundary layer close to the surface of the sample, referred to as surface superconductivity, see [P1, Al2, HP, FH1]. It is this boundary layer which is blown up and approximated by a half-space problem. The formal computations of Abrikosov [Abr] indicate a bifurcation at $b = 1$. As soon as $b < 1$, superconductivity is no longer present only on the surface of the sample but also in the bulk. For b close to 1 (but smaller), it is expected that the modulus of the minimizer is small but nonzero and vanishes at isolated points, called vortices, located on a triangular lattice, referred to as the Abrikosov lattice. The rigorous proof of the optimal location of the vortices is an open problem, related to the minimization of a reduced energy. This is also called the Abrikosov problem. It is our aim in this paper to derive properties on the minimal energy and relate it, for b close to 1, to the Abrikosov problem of minimizing an energy on a torus.

The situation for vortices is quite different from the smaller field case $h_{\rm ex} \ll 1/\varepsilon^2$, or $b \to 0$ (refer to [SS1] for this regime) where vortices are local perturbations of the wave function, while in this paper, their size ε is comparable to the interdistance between them. In particular the term in $\int (1 - |u|^2)^2$ is no longer negligible.

Let us first be more precise about the space of minimization. The energy J_Ω admits the gauge invariance $J_\Omega(u, A) = J_\Omega(ue^{i\Phi}, A + \nabla\Phi)$ for any smooth function Φ, and the physically relevant quantities, such as the modulus of the wave function or the current are gauge-invariant, see e.g. [SS1]. It is possible by a gauge change to restrict the minimization of the energy to the space

$$\mathcal{H} = \{(u, A) \text{ s.t. } u \in H^1(\Omega),\ A \in H^1(\Omega)\}.$$

The problem of minimizing J_Ω for b close to 1 has already been studied [Al1, SS2]. In

[SS2], it is proved that the energy is uniformly distributed in the sample, leading to the definition of a function $f(b)$:

Theorem 1 ([SS2]). *Let $0 \le b \le 1$. There exists a continuous increasing function f from $[0,1]$ to $[0,\frac{1}{4}]$, such that, as $\varepsilon \to 0$, for (u,A) any minimizer of J_Ω in $\mathcal{H}$, for all $1 \ge R_\varepsilon \gg \varepsilon$, and all balls B_{R_ε} of radius R_ε included in Ω, we have*

$$\lim_{\varepsilon\to 0} \frac{\varepsilon^2 J_{B_{R_\varepsilon}}(u,A)}{|B_{R_\varepsilon}|} = \lim_{\varepsilon\to 0} \frac{\varepsilon^2 \min_{\mathcal{H}} J_{B_{R_\varepsilon}}}{|B_{R_\varepsilon}|} = f(b) \tag{1.3}$$

where $|.|$ denotes the two-dimensional Lebesgue measure of a set. Moreover, there exists a positive constant α such that

$$\alpha(1-b)^2 \le 1 - 4f(b) \le (1-b)^2. \tag{1.4}$$

Thus, in (1.3), the mean energy of the minimizer in each ball B_{R_ε} is very close to the minimal energy on the ball, provided that R_ε is much larger than ε, that is there should be a large number of vortices in the ball B_{R_ε}. The value of this minimal energy is independent of the ball, hence the equidistribution of the energy. The lower bound for $1-4f(b)$ is obtained by constructing a test-function which is periodic with respect to some square lattice. Estimates on the convergence of the average of the L^2 and L^4 norms of u, establishing that they decrease like $\sqrt{1-b}$ as $b \to 1$, thus showing an average decrease of bulk-superconductivity, are also proved in [SS2].

In this paper, we want to better characterize $f(b)$ in the limit when b tends to 1, and relate it to a minimization problem on a torus. This relies on the study of the energy on a reduced space, which is the first eigenspace for a magnetic operator, also called lowest Landau level (LLL). This functional space is comprised of holomorphic functions multiplied by a particular Gaussian. The lowest Landau level was used in [AB, ABN] to analyze the vortex lattice in Bose-Einstein condensates. Our goal here is to show that a similar framework applies to the Ginzburg-Landau energy. Because our problem is posed on a torus, the space is finite dimensional, and the dimension is related to the number of zeroes, while in the case of BEC, the LLL is of infinite dimension. The results that we show are related to those obtained by Almog in [Al1] for rectangles. We hope to provide here a short and simple presentation of these results and to bridge between the works of [Al1] and [ABN].

In order to present our result, we rescale the functions u and A, around some origin in Ω, as follows:

$$u \to u\left(\frac{\varepsilon x}{\sqrt{b}}\right) \tag{1.5}$$

$$A \to \frac{\varepsilon}{\sqrt{b}} A\left(\frac{\varepsilon x}{\sqrt{b}}\right) \tag{1.6}$$

and the energy J_Ω turns into $G_{\Omega\sqrt{b}/\varepsilon}$ where, for any domain $\mathcal{D}$, we denote

$$G_{\mathcal{D}}(u,A) = \frac{1}{2}\int_{\mathcal{D}} |\nabla_A u|^2 + \frac{b}{\varepsilon^2}|\operatorname{curl} A - 1|^2 + \frac{(1-|u|^2)^2}{2b}. \tag{1.7}$$

Theorem 1 from [SS2] implies that if (u, A) is a minimizer of $G_{\Omega\sqrt{b}/\varepsilon}$, then for any R_ε with $1/\varepsilon \geq R_\varepsilon \gg 1$, as ε tends to 0, we have

$$\frac{f(b)}{b} = \lim_{\varepsilon\to 0,\ R_\varepsilon\to\infty} \frac{G_{B_{R_\varepsilon}}(u, A)}{|B_{R_\varepsilon}|} = \lim_{\varepsilon\to 0,\ R_\varepsilon\to\infty} \frac{\min_{\mathcal{H}} G_{B_{R_\varepsilon}}}{|B_{R_\varepsilon}|}. \tag{1.8}$$

We now want to reduce the Ginzburg-Landau energy to a periodic setting, in a sense that we will make precise below. As pointed out in [SS2], in the relation (1.8), balls can be replaced without loss of generality by squares, or other shapes like parallelograms. In what follows, instead of considering balls, we will consider parallelograms, that is unit cells for periodic problems: for $R \in \mathbb{R}_+^*$, and $\tau = \tau_1 + i\tau_2 \in \mathbb{C}\backslash\mathbb{R}$, we set $K_{\tau,R}$ to denote the unit parallelogram of the lattice

$$\mathcal{L}_{\tau,R} = R(\mathbb{Z} \oplus \tau\mathbb{Z}).$$

We will consider τ as fixed, and let $R \to \infty$ in order to study large tori with fixed "shape". Our main result is going to reduce the calculation of $f(b)$, as b tends to 1, given by (1.8), to a minimization problem for (u, A) where u lies in a finite-dimensional subspace of "periodic" functions and $A = A_0$ is given by

$$A_0 = \frac{1}{2}(-y, x) \text{ in } K_{\tau,R}. \tag{1.9}$$

For that purpose, we define the energy $\mathcal{G}_{\tau,R}(u) = G_{K_{\tau,R}}(u, A_0)$, that is

$$\mathcal{G}_{\tau,R}(u) = \frac{1}{2}\int_{K_{\tau,R}} |\nabla_{A_0} u|^2 + \frac{(1-|u|^2)^2}{2b}. \tag{1.10}$$

Let us point out that $\mathcal{G}_{\tau,R}(u)$ no longer depends on ε, except through the fact that R needs to be less than $1/\varepsilon$. The next step is to set a framework for "periodic" boundary conditions, in the sense that the gauge-invariant quantities are periodic, namely $|u|$ and $|\nabla_{A_0} u|$ should be periodic, but not u itself. More precisely, u and $\nabla_{A_0} u$ taken at the point $(z + nR + mR\tau)$ for two integers n and m, have to be gauge equivalent to u and $\nabla_{A_0} u$ [BGT]. This yields the definition of the following space for u (we will alternatively use complex coordinates and coordinates in $\mathbb{R}^2$) :

$$E_{\tau,R} = \{u \in H^1(K_{\tau,R}, \mathbb{C}),\ \text{s.t.}\ u(z+R) = e^{\frac{i\pi N y}{R\tau_2}} u(z),\ u(z+R\tau) = e^{\frac{i\pi N}{R\tau_2}(\tau_1 y - \tau_2 x)} u(z)\} \tag{1.11}$$

together with the quantization condition: $|K_{\tau,R}|/(2\pi)$ is an integer, that is there exists $N \in \mathbb{N}^*$, such that

$$R^2\tau_2 = 2\pi N. \tag{1.12}$$

This procedure is performed on the full Ginzburg-Landau energy in [BGT, Du, Ay] and we refer to [NV] and the references therein for the quantification of the torus. In fact, A_0 itself can be extended by periodicity (see [BGT]), but we do not enter into the details here. One may also check that $u \in E_{\tau,R}$ implies that the total degree of zeroes of u in the torus is equal to N, also proportional to the flux of the magnetic field.

Our main result is the following.

Theorem 2. *For every $\tau \in \mathbb{C} \setminus \mathbb{R}$, let $K_{\tau,R}$ denote the unit parallelogram of the lattice $\mathcal{L}_{\tau,R} = R(\mathbb{Z} \oplus \tau\mathbb{Z})$ and $G_{K_{\tau,R}}$ the Ginzburg-Landau energy (1.7) in this domain. Let $f(b)$ be the function defined in Theorem 1. Then, for all $b \in (0,1)$, for all $\tau \in \mathbb{C} \setminus \mathbb{R}$, we have*

$$f(b) = \lim_{\varepsilon\to 0,\ R\to\infty,\ R<1/\varepsilon} b\frac{\min_{\mathcal{H}} G_{K_{\tau,R}}}{|K_{\tau,R}|} = \lim_{R\to\infty,\ |K_{\tau,R}|/(2\pi)\in\mathbb{N}} b\frac{\min_{E_{\tau,R}} \mathcal{G}_{\tau,R}}{|K_{\tau,R}|} \tag{1.13}$$

where $\mathcal{G}_{\tau,R}$ is defined in (1.10) and $E_{\tau,R}$ in (1.11). Moreover, we have the following expansion for $f(b)$ as b approaches 1:

$$\lim_{b\to 1} \frac{4f(b)-1}{(1-b)^2} = \lim_{R\to\infty} \min_{v\in L_{\tau,R}} F_{\tau,R}(v) \quad \forall \tau \in \mathbb{C}\setminus\mathbb{R} \tag{1.14}$$

where

$$F_{\tau,R}(v) = \frac{1}{|K_{\tau,R}|}\int_{K_{\tau,R}} |v|^4 - 2|v|^2 \tag{1.15}$$

and $L_{\tau,R}$ is a finite-dimensional subspace of $E_{\tau,R}$ defined by functions such that $\mathcal{D}_{A_0}v := (\partial_1 + i\partial_2 + \frac{1}{2}(x+iy))v = 0$.

In (1.13), the first equality is a consequence of the definition of $f(b)$ and follows from (1.8), while the second equality is a reduction to an energy which no longer depends on ε. The information provided by (1.13) allows us to reduce the minimization to a subspace of $E_{\tau,R}$ and thus obtain in (1.14), a more precise expansion of $f(b)$ as $b \to 1$ than the one of (1.4). Let us emphasize that the limit on the r.h.s of (1.14) is thus independent of τ.

The space $L_{\tau,R}$ can be viewed as the finite dimensional analogue of the "lowest Landau level" in [ABN]. It corresponds to the eigenspace for the smallest eigenvalue of the Schrödinger operator with magnetic field $-\nabla^2_{A_0}$, as we will see later. The functions in this space are explicit: they can be described using the Jacobi Theta function and are completely determined by their N zeroes on each lattice cell, N being defined by the quantization condition (1.12). The dimension of $L_{\tau,R}$ is also the integer N.

Let us give some ideas about the proof of our theorem. The first step consists in reducing the minimization of $G_{K_{\tau,R}}$ over $\mathcal{H}$, to a minimization for $u \in E_{\tau,R}$, and $A = A_0$. This reduction uses estimates on A for a minimizer and tricks introduced in [SS2]. This proves (1.13). The error is unfortunately like $O(R)$, which does not allow to carry out estimates on how close the minimizers of these two problems are.

The second step consists in studying the minimization of $\mathcal{G}_{\tau,R}$ when $u \in E_{\tau,R}$: we will show that if $u \in E_{\tau,R}$, we have the crucial relation (see Corollary 3.1)

$$b\frac{\mathcal{G}_{\tau,R}(u)}{|K_{\tau,R}|} = \frac{1}{4} + \frac{b}{2|K_{\tau,R}|}\int_{K_{\tau,R}} |\mathcal{D}_{A_0}u|^2 + \frac{(1-b)^2}{4} F_{\tau,R}\left(\frac{u}{\sqrt{1-b}}\right), \tag{1.16}$$

where $F_{\tau,R}$ is defined in (1.15) and $\mathcal{D}_{A_0}u$ is defined at the end of Theorem 2. When b tends to 1, the last two terms are of different order and this forces, for an energy minimizer, $\mathcal{D}_{A_0}u$ to tend to 0. This leads us to analyze in details the space of functions u such that

$\mathcal{D}_{A_0}u = 0$. It is the first eigenspace for $-\nabla^2_{A_0}$ in $E_{\tau,R}$, which we called $L_{\tau,R}$. Thus, we may project orthogonally a minimizer u of $\mathcal{G}_{\tau,R}$ onto this first eigenspace $L_{\tau,R}$ and prove that its projection u_Π is an almost minimizer of the energy $F_{\tau,R}$ restricted to $L_{\tau,R}$. We also get that $u - u_\Pi$ is small in the following sense:

Proposition 1.1. *Let u be a minimizer of $\mathcal{G}_{\tau,R}$ in $E_{\tau,R}$ and let $v = u/\sqrt{1-b}$. Let v_Π be the L^2 orthogonal projection of v onto $L_{\tau,R}$ and $w = v - v_\Pi$. Then for $\beta < 1$, there exists a constant $C_{\beta,\tau}$ such that, for R large,*

$$\|v\|_{C^{0,\beta}(K_{\tau,R})} \leq C_{\beta,\tau}R \tag{1.17}$$

$$\|w\|_{C^{0,\beta}(K_{\tau,R})} \leq C_{\beta,\tau}R\sqrt{1-b}. \tag{1.18}$$

Moreover,

$$F_{\tau,R}(v_\Pi) = \min_{L_{\tau,R}} F_{\tau,R} + O(|K_{\tau,R}|\sqrt{1-b}). \tag{1.19}$$

If R is large, but b is sufficiently close to 1, the last term in the estimate is a small error. The proof of this proposition relies on estimates of Lu and Pan [LP2], also proved in [AB], which are analogues of elliptic estimates applied to the operator $-\nabla^2_{A_0}$.

Let us finish with a discussion on related works in the literature.

Almog [Al1] studies a similar problem, but instead of considering general tori, he works on rectangles only (that is $\tau = li$ for some l and the domain is $K_{li,R}$), with some conditions on their size related to the way $b \to 1$. Here, we work on all tori shapes, we do not have restrictions on the rate of convergence of $b \to 1$. His proof is two-fold: on the one hand, he proves that if u is a minimizer of $G_{\mathcal{D}}$, for an arbitrary domain $\mathcal{D}$, then its L^∞ norm is small in terms of $b - 1$. It is this estimate which allows him to truncate u on the boundary of subrectangles to 0, so that the truncated function becomes an element of $E_{li,R}$ and the energy he considers is $G_{K_{li,R}}$. His constraints on the size of the rectangles provide an estimate on how close the initial minimizer is to that on $E_{li,R}$. The core of his results is an equivalent of Proposition 1.1, that is the projections onto the lowest Landau level, that relies on rather technical computations instead of elliptic estimates as we do. But essentially the results are similar in nature. The price that we pay for our lack of constraints is that we do not have an estimate on the closeness of u to the minimizer of $F_{\tau,R}$.

The minimization of $F_{\tau,R}$ over $L_{\tau,R}$ is another formulation of the Abrikosov problem: it is expected that, as R tends to infinity, the zeroes of the minimizer are almost located on a triangular lattice whose unit cell has volume 2π. The main difficult open question is to prove this statement. It would be already satisfactory (but difficult) to prove that if $\tau = j = e^{2i\pi/3}$ (that is the large torus is already triangular), then the minimum of $F_{j,R}$ on $L_{j,R}$ is such that its zeroes are exactly on a triangular lattice of volume 2π. In other words, we expect the modulus of the solution to be periodic with respect to the small lattice $\mathcal{L}_{j,r_0}$ with $r_0^2\sqrt{3} = 4\pi$, additionally to the initial one. F. Nier [Ni] was able to prove recently that this solution with zeroes on a triangular lattice provides a local minimum of the energy $F_{j,R}$. The other fact which was previously known is that if the zeroes of u are already assumed to be on a lattice of shape τ then the one minimizing $F_{\tau,R}$ is the triangular one, as proved in [NV, ABN].

The work in [ABN] deals with rotating Bose-Einstein condensates, described by the Gross-Pitaevskii energy, when the rotational velocity Ω tends to the transverse trapping velocity that is set to 1. This is the equivalent of the limit $b \to 1$, with the role of the applied magnetic field played by the rotational velocity. When Ω tends to 1, the minimizer of the Gross-Pitaevskii energy can be restricted to the lowest Landau level, that is the first eigenspace for $-\nabla^2_{A_0}$ in $\mathbb{R}^2$ (instead of a bounded domain) as done in [AB]. The proof relies on the projection onto the LLL and elliptic estimates, as we do here for Proposition 1.1. The difference between our energy and the one for condensates is that the latter is posed in the whole space $\mathbb{R}^2$ but a trapping potential makes the problem compact: the mean value of $|u|^2$ on several cells of vortices, instead of being almost constant, is close, on a large scale, to an inverted parabola. In particular, there is no invariance of the mean energy per unit volume in the domain, but on the contrary dependence on it. Nevertheless, at the limit $b = 1$ or $\Omega = 1$, the two limiting problems, for condensates and superconductors, given by the minimization of $F_{\tau,R}$ should be the same, though the proof is still open in the case of condensates [Aft, ABN].

The paper is organized as follows: first we prove (1.13) in Section 2, then we study the operator $\mathcal{D}_{A_0}$, its eigenspace and its spectrum in Section 3, and finally, in Section 4, we prove Proposition 1.1 and the rest of Theorem 2.

Open problems

- The first main open problem is the one mentioned above, which consists in showing that, as R tends infinity, the minimizer of $F_{\tau,R}$ in $L_{\tau,R}$ has zeroes that form an almost triangular lattice, and an exact triangular lattice when the torus is already triangular, i.e. $\tau = j$.

- Does the fact that $F_{\tau,R}(v_\Pi) \sim \min F_{\tau,R}$ imply that v_Π is close in some sense to the set of minimizers of $F_{\tau,R}$ in $L_{\tau,R}$? In what norm is there convergence? In particular do the zeroes of v_Π converge?

- Link more precisely the minimizers of $G_{K_{\tau,R}}$ to the the minimizers restricted to $E_{\tau,R}$, as done in [Al1] but with sharper estimates valid in a wider regime.

- This would probably require some better L^∞ estimates on minimizers u of G_Ω: do estimates of the sort $|u(x)| \leq C_\delta \sqrt{1-b}$ when dist $(x, \partial G_\Omega) \geq \delta$ hold?

- Can anything else be said when b is fixed and does not tend to 1 and can $f(b)$ then be better characterized? In particular, can we estimate the number of Landau levels filled according to the distance of b to 1?

2 Reduction to a torus

In this section b is fixed in $(0, 1)$. We are going to prove the following:

Proposition 2.1. *Let us recall that* A_0 *is given by (1.9), i.e. a vector field such that* div $A_0 \equiv 0$ *and* curl $A_0 \equiv 1$. *The function* $f(b)$ *defined in Theorem 1 has the property (1.13).*

This way, studying $f(b)$ reduces to minimizing this simpler energy $\mathcal{G}$ on large tori (of any shape). We first show intermediate lemmas. The first one recalls some useful a priori estimates (see [FH2] Proposition 4.4):

Lemma 2.1. *Let* $\mathcal{D}$ *be a bounded domain,* (u, A) *be a minimizer of* $G_{\mathcal{D}}$, *then* $|u| \le 1$ *and there exists a constant* C *independent of* ε *and* b *such that*

$$\|\mathrm{curl}\, A - 1\|_{C^1(\mathcal{D})} \le \frac{C\varepsilon}{\sqrt{b}}, \quad \|\nabla_A u\|_{L^\infty(\mathcal{D})} \le \frac{C}{\sqrt{b}}.$$

Following exactly the method of [SS2], Lemma 3.1, we can show the following result:

Lemma 2.2. *Let* $f(b)$ *be given by Theorem 1, and* (u, A) *a minimizer of* $G_{\Omega\frac{\sqrt{b}}{\varepsilon}}$. *Then for all* R *such that* $1 \ll R < 1/\varepsilon$, *and* $K_{\tau,R} \subset \Omega\frac{\sqrt{b}}{\varepsilon}$,

$$\frac{G_{K_{\tau,R}}(u,A)}{|K_{\tau,R}|} = \frac{\min_{\mathcal{H}} G_{K_{\tau,R}}(u,A)}{|K_{\tau,R}|} + o(1) = \frac{f(b)}{b} + o(1) \quad \text{as } \varepsilon \to 0,$$

so that (u, A) *is an "almost minimizer" of* $G_{K_{\tau,R}}$. *Moreover, as* $\varepsilon \to 0$,

$$\frac{1}{|K_{\tau,R}|}\int_{K_{\tau,R}} |u|^4 \longrightarrow 1 - 4f(b) \tag{2.1}$$

$$\frac{1-4f(b)}{1-b} - o(1) \le \frac{1}{|K_{\tau,R}|}\int_{K_{\tau,R}} |u|^2 \le \sqrt{1-4f(b)} + o(1). \tag{2.2}$$

This amounts to writing the result of Theorem 1, but in parallelograms instead of balls. The results corresponding to (2.1) and (2.2) are also proved in [SS2].

In the proof of Proposition 2.1, we can thus restrict, in a first step, to a minimizer of $G_{K_{\tau,R}}$. We next reduce to the case $A = A_0$.

Lemma 2.3. *Let* (v, B) *be a minimizer of* $G_{K_{\tau,R}}$, *then* (v, B) *is gauge-equivalent to some* (u, A) *such that*

$$\|A - A_0\|_{L^\infty(K_{\tau,R})} \le C_\tau \frac{\varepsilon R}{\sqrt{b}}$$

where C_τ *is a constant depending only on* τ.

Proof. Let us solve

$$\begin{cases} \Delta\phi = -\mathrm{div}\, B & \text{in } K_{\tau,R} \\ \frac{\partial\phi}{\partial\nu} = A_0\cdot\nu - B\cdot\nu & \text{on } \partial K_{\tau,R}. \end{cases}$$

Then we take $A = B + \nabla\phi$. This way, A satisfies div $A = 0 =$ div A_0 and $(A - A_0)\cdot\nu = 0$ on $\partial\Omega$. Thus we may write $A - A_0 = \nabla^\perp \xi$ with $\xi = 0$ on $\partial K_{\tau,R}$. Moreover,

$$\Delta\xi = \operatorname{curl} A - \operatorname{curl} A_0 = \operatorname{curl} B - 1$$

In view of Lemma 2.1, we have $\|\Delta\xi\|_{L^\infty(K_{\tau,R})} \le \frac{C\varepsilon}{\sqrt{b}}$. By elliptic regularity, we deduce

$$\|\nabla\xi\|_{L^\infty(K_{\tau,R})} \le \frac{C_\tau R\varepsilon}{\sqrt{b}}$$

and the result follows. □

Thus, by a gauge-transformation, we may reduce to such a (u, A) and are going to prove that its energy is close to $G_{K_{\tau,R}}(u, A_0) = \mathcal{G}_{\tau,R}(u)$, which no longer depends on ε.

Lemma 2.4. *We have*

$$f(b) = \lim_{R\to\infty} b\frac{\min_{H^1}\mathcal{G}_{\tau,R}}{|K_{\tau,R}|}. \tag{2.3}$$

Proof. In view of Lemma 2.3, we have

$$\int_{K_{\tau,R}} |\nabla_A u|^2 = \int_{K_{\tau,R}} |\nabla_{A_0} u|^2 + O(\varepsilon^2 R^4/b)$$

If follows that

$$G_{K_{\tau,R}}(u, A) \ge \frac{1}{2}\int_{K_{\tau,R}} |\nabla_A u|^2 + \frac{(1-|u|^2)^2}{2b} = G_{K_{\tau,R}}(u, A_0) + O(\varepsilon^2 R^4/b).$$

Therefore, by minimality of (u, A), $\min_{\mathcal{H}} G_{K_{\tau,R}} \ge \min_{H^1} G_{K_{\tau,R}}(., A_0) + O(\varepsilon^2 R^4/b)$. Also $\min G_{K_{\tau,R}} \le \min G_{K_{\tau,R}}(., A_0)$. We deduce that

$$\min_{\mathcal{H}} G_{K_{\tau,R}} = \min_{H^1} \mathcal{G}_{\tau,R} + O(\varepsilon^2 R^4/b)$$

and that (u, A) is an almost minimizer of $\mathcal{G}_{\tau,R}$ in that sense. In view of Lemma 2.2, we deduce that

$$f(b) = \lim_{\varepsilon\to 0, 1\ll R\ll 1/\varepsilon} b\frac{\min_{H^1}\mathcal{G}_{\tau,R}}{|K_{\tau,R}|}.$$

But since $\mathcal{G}_{\tau,R}$ does not depend on ε, we may suppress the constraints on R and just replace them with $R\to\infty$. □

Proof of Proposition 2.1. Let us consider χ a cut-off function which is identically equal to 1 in $K_{\tau,R-2}$, to 0 in $K_{\tau,R}\backslash K_{\tau,R-1}$ and such that $|\nabla\chi| \le C$. A similar proof to that of [SS2], Lemma 3.1, allows to check that

$$\mathcal{G}_{\tau,R}(u) = G_{K_{\tau,R}}(u, A_0) = G_{K_{\tau,R}}(\chi u, A_0) + O(R) = \mathcal{G}_{\tau,R}(\chi u) + O(R).$$

It follows that

$$\frac{\min_{H^1(K_{\tau,R})} \mathcal{G}_{\tau,R}}{|K_{\tau,R}|} \sim \frac{\min_{H^1_0(K_{\tau,R})} \mathcal{G}_{\tau,R}}{|K_{\tau,R}|} \quad \text{as } R \to \infty. \tag{2.4}$$

But H^1_0 can be viewed as a subspace of $E_{\tau,R}$ (conditions (1.11) are satisfied) thus

$$\frac{\min_{H^1_0(K_{\tau,R})} \mathcal{G}_{\tau,R}}{|K_{\tau,R}|} \geq \frac{\min_{E_{\tau,R}} \mathcal{G}_{\tau,R}}{|K_{\tau,R}|} \geq \frac{\min_{H^1(K_{\tau,R})} \mathcal{G}_{\tau,R}}{|K_{\tau,R}|}$$

and from (2.4) and (2.3), it follows that Proposition 2.1 holds. □

3 The operator $\mathcal{D}_{A_0}$ and its first eigenspace

We now study the operator $-\nabla^2_{A_0}$ over the space $E_{\tau,R}$. For the sake of completeness, we include the proof of results which are known in other contexts. We will need the Theta function, naturally associated with the lattice $\mathbb{Z} \oplus \tau\mathbb{Z}$, defined by

$$\Theta_\tau(v) = \frac{1}{i} \sum_{n=-\infty}^{+\infty} (-1)^n e^{i\pi\tau(n+1/2)^2} e^{(2n+1)\pi i v}, \quad v \in \mathbb{C}. \tag{3.1}$$

We refer the reader to [Cha] for details. The Theta function vanishes exactly on the lattice $\mathbb{Z} \oplus \tau\mathbb{Z}$ and satisfies

$$\Theta_\tau(v) = -\Theta_\tau(-v), \quad \Theta_\tau(v+1) = -\Theta_\tau(v), \quad \Theta_\tau(v+\tau) = -e^{-i\pi\tau} e^{-2\pi i v} \Theta_\tau(v). \tag{3.2}$$

Proposition 3.1. *The operator $-\nabla^2_{A_0}$ is self-adjoint positive over the subspace $E_{\tau,R}$. Its lowest eigenvalue is equal to 1, and the associated eigenspace, called $L_{\tau,R}$, has complex dimension $N = \frac{|K_{\tau,R}|}{2\pi}$ and is described by the functions*

$$u(z) = \lambda e^{-|z|^2/4} e^{z^2/4 - i\pi N z/R} \prod_{k=1}^{N} \Theta_\tau\left(\frac{z - z_k}{R}\right) \tag{3.3}$$

where λ is any complex number, z_k are N points in $K_{\tau,R}$, satisfying the constraint

$$\sum_{k=1}^{N} z_k = RN\frac{1+\tau}{2} \quad mod\ \mathcal{L}_{\tau,R}.$$

The second eigenvalue of $-\nabla^2_{A_0}$ is greater than 3.

Let us point out that this space is the finite dimensional equivalent of the lowest Landau level as used e.g. in [GJ, ABN].

Corollary 3.1. *Let* $u \in E_{\tau,R}$ *and let* $\mathcal{D}_{A_0} = \partial_1 + i\partial_2 + \frac{1}{2}(x+iy)$*, then*

$$\begin{aligned}\mathcal{G}_{\tau,R}(u) = G_{K_{\tau,R}}(u, A_0) &= \frac{1}{2}\int_{K_{\tau,R}} |u|^2 + |\mathcal{D}_{A_0}u|^2 + \frac{(1-|u|^2)^2}{2b} \\ &= \frac{1}{2}\int_{K_{\tau,R}} |\mathcal{D}_{A_0}u|^2 + \frac{|K_{\tau,R}|}{b}\left(\frac{1}{4} + \frac{(1-b)^2}{4}F_{\tau,R}(\frac{u}{\sqrt{1-b}})\right) \quad (3.4)\end{aligned}$$

If additionally $u \in L_{\tau,R}$*, then* $\mathcal{D}_{A_0}u = 0$ *and*

$$\begin{aligned}\mathcal{G}_{\tau,R}(u) = \frac{1}{2}\int_{K_{\tau,R}} |u|^2 + \frac{(1-|u|^2)^2}{2b} &= \frac{|K_{\tau,R}|}{4b} + \frac{1}{4b}\int_{K_{\tau,R}} |u|^4 - 2(1-b)|u|^2. \\ &= \frac{|K_{\tau,R}|}{4b}\left(1 + (1-b)^2 F_{\tau,R}(\frac{u}{\sqrt{1-b}})\right). \quad (3.5)\end{aligned}$$

Proof of Proposition 3.1. We introduce the operator

$$\mathcal{D}_{A_0} = \partial_1 + i\partial_2 + \frac{1}{2}(x+iy) = 2\partial_{\overline{z}} + \frac{z}{2} \quad (3.6)$$

in complex coordinates notation, where $\partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y)$. This operator corresponds to the creation operator in second quantification and is naturally associated with $-\nabla^2_{A_0}$ as we will see below, cf. [GJ]. In [SS2], it was introduced for a general field A following the Bogomolnyi trick for self-duality.

The adjoint of $\mathcal{D}_{A_0}$ with respect to the scalar product in L^2 is $\mathcal{D}^*_{A_0} = -2\partial_z + \frac{1}{2}\overline{z}$. Straightforward computations yield

$$\begin{aligned}\mathcal{D}_{A_0}\mathcal{D}^*_{A_0} &= -4\partial_{z\overline{z}} - z\partial_z + \overline{z}\partial_{\overline{z}} + \frac{1}{4}|z|^2 + 1, \\ \mathcal{D}^*_{A_0}\mathcal{D}_{A_0} &= -4\partial_{z\overline{z}} - z\partial_z + \overline{z}\partial_{\overline{z}} + \frac{1}{4}|z|^2 - 1,\end{aligned}$$

hence we deduce the canonical relations:

$$\mathcal{D}_{A_0}\mathcal{D}^*_{A_0} + \mathcal{D}^*_{A_0}\mathcal{D}_{A_0} = -2\Delta + \tfrac{1}{2}|z|^2 - 2(iy\partial_y + ix\partial_y) = -2\nabla^2_{A_0}, \quad (3.7)$$

$$\mathcal{D}_{A_0}\mathcal{D}^*_{A_0} - \mathcal{D}^*_{A_0}\mathcal{D}_{A_0} = 2I. \quad (3.8)$$

In particular, $-\nabla^2_{A_0} = \mathcal{D}^*_{A_0}\mathcal{D}_{A_0} + I$. One may check that the periodicity conditions (1.11) ensure that $\nabla_{A_0}u$ is really periodic with respect to $\mathcal{L}_{\tau,R}$. Thus the previous relation implies (after an integration by parts which yields no boundary term)

$$\int_{K_{\tau,R}} |\nabla_{A_0}u|^2 = \int_{K_{\tau,R}} |\mathcal{D}_{A_0}u|^2 + |u|^2. \quad (3.9)$$

Therefore, the lowest eigenvalue of $-\nabla^2_{A_0}$ in $E_{\tau,R}$, characterized via the Rayleigh quotient

$$\min_{u \in E_{\tau,R}} \frac{\int_{K_{\tau,R}} |\nabla_{A_0}u|^2}{\int_{K_{\tau,R}} |u|^2}$$

is equal to 1 and the eigenspace $L_{\tau,R}$ identifies with the solutions of $\mathcal{D}_{A_0}u = 0$ in $E_{\tau,R}$. In particular, returning to the form (3.6), we see that it is made up of functions $u = f(z)e^{-|z|^2/4}$ with f holomorphic. We can be more specific about the structure of f as we will see below.

Let us now consider $u \in L^{\perp}_{\tau,R} = (Ker\mathcal{D}_{A_0})^{\perp}$ (where the orthogonal is taken for the L^2 scalar product on $E_{\tau,R}$). Clearly u belongs to the closure of the range of $\mathcal{D}^*_{A_0}$. We may thus assume $u = \mathcal{D}^*_{A_0}v$. In (3.9), we replace $\mathcal{D}_{A_0}u$ by $\mathcal{D}_{A_0}\mathcal{D}^*_{A_0}v$ and use (3.8) to find

$$\begin{aligned}\langle -\nabla^2_{A_0}u, u\rangle = \langle (I + \mathcal{D}^*_{A_0}\mathcal{D}_{A_0})u, u\rangle &= \|u\|^2 + \langle 2v + \mathcal{D}^*_{A_0}\mathcal{D}_{A_0}v, \mathcal{D}_{A_0}u\rangle \\ &= \|u\|^2 + 2\langle \mathcal{D}^*_{A_0}v, u\rangle + \langle \mathcal{D}^*_{A_0}\mathcal{D}_{A_0}v, \mathcal{D}_{A_0}\mathcal{D}^*_{A_0}v\rangle = 3\|u\|^2 + \|\mathcal{D}^*_{A_0}\mathcal{D}_{A_0}v\|^2.\end{aligned}$$

We conclude that, for $u \in L^{\perp}_{\tau,R}$,

$$\langle -\nabla^2_{A_0}u, u\rangle \geq 3\|u\|^2,$$

and this proves the last assertion of the proposition.

Let us now characterize better the eigenspace $L_{\tau,R}$. We have seen that it is made up of functions $u = f(z)e^{-|z|^2/4}$ with f holomorphic, satisfying the periodicity condition (1.11), which implies in particular that $|u|$ is periodic. Since f is holomorphic, it has a finite number of zeroes in $K_{\tau,R}$. Let us call N_0 this number and call z_k the zeroes. We are going to prove that (3.3) holds. This is a consequence of Hadamard's factorization theorem. The proof is similar to that in [ABN]. Since f and the function $\prod_{k=1}^{N_0}\Theta_\tau(\frac{z-z_k}{R})$ have the same zeroes, their quotient is an analytic function which does not vanish in $K_{\tau,R}$. Hence one can find an analytic function ϕ such that

$$f(z) = e^{\phi(z)}\prod_{k=1}^{N_0}\Theta_\tau(\frac{z-z_k}{R}).$$

The $\mathcal{L}_{\tau,R}$-periodicity of $|u(z)| = e^{-\frac{|z|^2}{4}}|f(z)|$ implies the upper bound

$$\forall z \in \mathbb{C},\ e^{\mathbb{R}\mathrm{e}(\phi(z)) - \frac{|z|^2}{4}}\left|\prod_{k=1}^{N_0}\Theta_\tau(\frac{z-z_k}{R})\right| \leq C_1\,.$$

Therefore, when the periodicity cell Q is chosen such that $\mathcal{L}_{\tau,R}\cap\partial Q = \varnothing$, there exists a constant $C > 0$ such that $\forall z \in \partial Q + \mathcal{L}_{\tau,R},\quad \mathbb{R}\mathrm{e}\,(\phi(z)) \leq C\,|z|^2 + \ln(CC_1)\,$. Since $\mathbb{R}\mathrm{e}\,(\phi(z))$ is a harmonic function, the maximum principle yields $\forall z \in \mathbb{C},\quad \mathbb{R}\mathrm{e}\,(\phi(z)) \leq C'(|z|^2 + 1)$ for some constant $C' > 0$. The Hadamard factorization theorem, see [Boa], then implies that ϕ is a harmonic polynomial of degree 2, that is there exists $(\delta, \eta, \beta) \in \mathbb{C}^3$ such that

$$f(z) = e^{\delta+\eta z+\beta z^2}\prod_{k=1}^{N_0}\Theta_\tau(\frac{z-z_k}{R}).$$

Conditions (1.11) and the properties on Theta (3.2) imply that

$$\begin{aligned}&e^{\eta R+\beta R^2-R^2/4}e^{2\beta Rz-Rx/2}(-1)^{N_0} = e^{\frac{i\pi Ny}{R\tau_2}}\\ &(-1)^{N_0}e^{\eta R\tau+R^2\tau^2/4-R^2|\tau|^2/4+R\tau z/2-Rx\tau_1/2-Ry\tau_2-i\pi\tau N_0-2i\pi N_0 z/R+2i\pi\sum_k z_k/R} = e^{\frac{i\pi N}{R\tau_2}(\tau_1 y-\tau_2 x)}\end{aligned}$$

The first equality yields

$$\beta = \frac{1}{4}, \quad R^2\tau_2 = 2\pi N, \quad \eta = -i(\frac{\pi N}{R} + \frac{2k\pi}{R}) \tag{3.10}$$

for some integer k, while it follows from the second one that

$$N_0 = N, \quad \eta = -i\frac{\pi N}{R}, \tag{3.11}$$

and

$$\sum_{k=1}^{N} z_k = RN\frac{1+\tau}{2} \text{ mod } \mathcal{L}_{\tau,R}. \tag{3.12}$$

This yields (3.3). □

Proof of Corollary 3.1. Equality (3.4) is a direct consequence of (3.9) while (3.5) follows from the definition of $L_{\tau,R}$. □

4 The behavior as $b \to 1$

The aim of this section is to prove Proposition 1.1.

4.1 Upper bound for the energy

Lemma 4.1. *We have*

$$\min_{L_{\tau,R}} F_{\tau,R} \leq \frac{-1}{2\pi\gamma(\tau)}$$

where $\gamma(\tau)$ denotes the Abrikosov parameter of the lattice $\mathcal{L}_{\tau,\nu}$ defined by $\nu^2 = 2\pi/\tau_2$ and

$$\gamma(\tau) = \min_{L_{\tau,\nu}, \nu^2\tau_2 = 2\pi} \frac{\int_{K_{\tau,\nu}} |v|^4}{\left(\int_{K_{\tau,\nu}} |v|^2\right)^2}. \tag{4.1}$$

The computation of $\gamma(\tau)$ has been made in [ABN]: it is equal to the Abrikosov parameter and related to a series for Husimi functions for which Voros-Nonenmacher [NV] studied the minimizer in terms of τ. It follows from these two papers that $\min_\tau \gamma$ is achieved when $\tau = j$, that is for the triangular lattice. An approximate value for $1/\gamma(j)$ is 0.86.

Proof. One specific choice of test function is to fix a lattice with a unit cell having a volume 2π, that is $\mathcal{L}_{\tau,\nu} = \nu(\mathbb{Z} \oplus \tau\mathbb{Z})$ with $\nu^2\tau_2 = 2\pi$. Then as seen in Proposition 3.1, the corresponding first eigenspace $L_{\tau,\nu}$ is one-dimensional and spanned by u_τ, which is multiple of $\Theta_\tau((z - (1+\tau)/2)/\nu)$, as given by (3.3).

Now if v is any function in $L_{\tau,\nu}$, we have $v = \lambda u_\tau$ and $F_{\tau,\nu}(\lambda u_\tau)$ is minimal for $\lambda^2 = (\int_{K_{\tau,\nu}} |u_\tau|^2)/(\int_{K_{\tau,\nu}} |u_\tau|^4)$. This implies in particular that

$$F_{\tau,\nu}(\lambda u_\tau) = -\frac{1}{|K_{\tau,\nu}|} \frac{\left(\int_{K_{\tau,\nu}} |u_\tau|^2\right)^2}{\int_{K_{\tau,\nu}} |u_\tau|^4}.$$

Since $\gamma(\tau)$ is invariant when the function v varies in $L_{\tau,\nu}$, we have

$$\min_{L_{\tau,\nu}} F_{\tau,\nu} = \frac{-1}{2\pi\gamma(\tau)}.$$

Now if $R^2\tau_2/2\pi$ is an integer, (1.12) is satisfied and a specific test function in $E_{\tau,R}$ can be constructed by simply extending the previous λu_τ by "periodicity" i.e. taking $\lambda u_\tau(z + \nu(n + m\tau))$, for n, m integers. The periodicity of $|u_\tau|$ implies that

$$F_{\tau,R}(\lambda u_\tau) = F_{\tau,\nu}(\lambda u_\tau) = -\frac{1}{2\pi\gamma(\tau)}, \tag{4.2}$$

hence the result. □

Remark 4.1. *In order to try to minimize $F_{\tau,R}$, we have taken test-configurations which are periodic with respect to smaller lattices (of the same shape). However, we do not believe that the estimate above is optimal, unless $\tau = j$. The natural conjecture is rather that*

$$\liminf_{R\to\infty} \min_{L_{\tau,R}} F_{\tau,R} = -\frac{1}{2\pi\gamma(j)}.$$

This corresponds to showing that in large tori, the optimal location of vortices is still on a triangular lattice, except for negligible boundary effects. Observe that the inequality

$$\liminf_{R\to\infty} \min_{L_{\tau,R}} F_{\tau,R} \le -\frac{1}{2\pi\gamma(j)}$$

is already not so easy to obtain.

We now easily deduce

Proposition 4.1. *For any τ and any $b \in (0,1)$, we have*

$$\min_{E_{\tau,R}} \frac{\mathcal{G}_{\tau,R}}{|K_{\tau,R}|} \le \frac{1}{4b}\left(1 + (1-b)^2 \min_{L_{\tau,R}} F_{\tau,R}\right) \le \frac{1}{4b}\left(1 - (1-b)^2 \frac{1}{2\pi\gamma(\tau)}\right). \tag{4.3}$$

Corollary 4.1. *For any $b \in (0,1)$, we have*

$$f(b) \le \frac{1}{4} + \frac{(1-b)^2}{4} \inf_{\tau\in\mathbb{C}\setminus\mathbb{R}} \liminf_{R\to\infty} \min_{v\in L_{\tau,R}} F_{\tau,R}(v) \le \frac{1}{4} - \frac{(1-b)^2}{4}\frac{1}{2\pi\gamma(j)}. \tag{4.4}$$

Proof of the proposition. It suffices to build a test-configuration: let us take v_0 a minimizer of $F_{\tau,R}$ in $L_{\tau,R}$, and take $u_0 = v_0\sqrt{1-b}$. Since $u_0 \in L_{\tau,R}$, we use (3.5) and find

$$\begin{aligned}\mathcal{G}_{\tau,R}(u_0) &= \frac{|K_{\tau,R}|}{4b}\left(1+(1-b)^2 F_{\tau,R}(v_0)\right)\\ &= \frac{|K_{\tau,R}|}{4b}\left(1+(1-b)^2 \min_{L_{\tau,R}} F_{\tau,R}\right).\end{aligned}$$

The second inequality follows from Lemma 4.1. □

Proof of the corollary. In view of the characterisation of f by (1.13), we deduce

$$f(b) \le \frac{1}{4} + \frac{(1-b)^2}{4} \liminf_{R\to\infty} \min_{L_{\tau,R}} F_{\tau,R}. \tag{4.5}$$

Since this is true for every τ, minimizing the right-hand side over τ and using Lemma 4.1 gives the result (since we know that γ is minimized for $\tau = j$). □

Remark 4.2. *In [SS2] an upper bound for f of this sort was given by using configurations periodic with respect to square lattices. We see here that this upper bound can be improved by taking triangular lattices, but it is still open whether there is equality above, i.e. whether triangular-periodic configurations are energetically optimal.*

4.2 Lower bound as $b \to 1$

Proposition 4.2. *Let u be a minimizer of $\mathcal{G}_{\tau,R}$ in $E_{\tau,R}$. Then as $R \to \infty$,*

$$\frac{\mathcal{G}_{\tau,R}(u)}{|K_{\tau,R}|} \ge \frac{1}{4b} + \frac{(1-b)^2}{4b}\left(\min_{L_{\tau,R}} F_{\tau,R} + O(|K_{\tau,R}|(1-b)^{\frac{1}{2}})\right). \tag{4.6}$$

The proof consists in projecting the minimizer u onto the space $L_{\tau,R}$ and checking, using elliptic estimates, that u and its projection u_Π are close. Finally, we prove that $F_{\tau,R}(u_\Pi)$ is close to $\min_{L_{\tau,R}} F_{\tau,R}$. Unfortunately, we are not able to prove that u_Π is close to a minimizer of $F_{\tau,R}$ (see open problems). The proof is similar to that of [AB] for the reduction of the minimization of the Gross-Pitaevskii energy to the lowest Landau level. In [Al1], the projection onto the lowest Landau level is also used, but with different elliptic estimates.

Proof. - Step 1: upper bounds. From Proposition 4.1 we have

$$\frac{1}{|K_{\tau,R}|}\mathcal{G}_{\tau,R}(u) \le \frac{1-(1-b)^2/(2\pi\gamma(\tau))}{4b}$$

for any R such that $R^2\tau_2/(2\pi) \in \mathbb{N}$. Using (3.4), writing $\alpha = \frac{1}{2\pi\gamma(\tau)}$, and rearranging the terms, we find

$$\frac{1}{|K_{\tau,R}|}\int_{K_{\tau,R}} |\mathcal{D}_{A_0}u|^2 + \frac{(|u|^2-(1-b))^2}{2b} \le \frac{(1-\alpha)(1-b)^2}{2b}. \tag{4.7}$$

We let $v = u/\sqrt{1-b}$ and deduce that

$$\frac{1}{|K_{\tau,R}|}\int_{K_{\tau,R}} (1-|v|^2)^2 \leq (1-\alpha) \tag{4.8}$$

$$\frac{1}{|K_{\tau,R}|}\int_{K_{\tau,R}} |\mathcal{D}_{A_0} v|^2 \leq \frac{(1-\alpha)(1-b)}{2b}. \tag{4.9}$$

It follows that

$$\left(\int_{K_{\tau,R}} |v|^4\right)^{1/2} \leq \||v|^2 - 1\|_{L^2} + \|1\|_{L^2} \leq 2|K_{\tau,R}|^{1/2} \tag{4.10}$$

and also using the Cauchy-Schwarz inequality, that

$$\int_{K_{\tau,R}} |v|^2 \leq 2|K_{\tau,R}|. \tag{4.11}$$

- *Step 2.* The function v belongs to $E_{\tau,R}$, we may project it orthogonally (for $L^2(K_{\tau,R})$) onto $L_{\tau,R}$. Let v_Π denote its projection, and let us write

$$v = v_\Pi + w \tag{4.12}$$

Since $w \in L_{\tau,R}^\perp$, we have from Proposition 3.1 and (3.9) that

$$3\int_{K_{\tau,R}} |w|^2 \leq \int_{K_{\tau,R}} |\nabla_{A_0} w|^2 = \int_{K_{\tau,R}} |\mathcal{D}_{A_0} w|^2 + |w|^2$$

and thus

$$2\int_{K_{\tau,R}} |w|^2 \leq \int_{K_{\tau,R}} |\mathcal{D}_{A_0} w|^2$$

But since $v_\Pi \in L_{\tau,R}$ and $w \in L_{\tau,R}^\perp$ we have $\mathcal{D}_{A_0} v = \mathcal{D}_{A_0} w$ and thus, in view of (4.9), we get

$$\frac{1}{|K_{\tau,R}|}\int_{K_{\tau,R}} |w|^2 \leq \frac{(1-\alpha)(1-b)}{4b}. \tag{4.13}$$

- *Step 3:* We use the estimates of Lu-Pan [LP2], or rather the adaptation proved in [AB]: for any (f, A), there exists a constant C such that

$$\|\nabla_A f\|_{H^1(B_r)} \leq C(\|\Delta A\|_\infty, \|\mathrm{curl}\, A\|_\infty)\left(\|\nabla_A^2 f\|_{L^2(B_{2r})} + \|f\|_{L^2(B_{2r})}\right) \tag{4.14}$$

where C is independent of r.

We deduce the following estimates

Lemma 4.2. *Let u be a minimizer of $\mathcal{G}_{\tau,R}$ in $E_{\tau,R}$, $v = u/\sqrt{1-b}$, and let v_Π its orthogonal projection onto $L_{\tau,R}$ and $w = v - v_\Pi$. Then for $\beta < 1$ there exists a constant $C_{\beta,\tau}$ such that, for R large enough,*

$$\|v\|_{C^{0,\beta}(K_{\tau,R})} \le C_{\beta,\tau} R \tag{4.15}$$

$$\|w\|_{C^{0,\beta}(K_{\tau,R})} \le C_{\beta,\tau} R\sqrt{1-b}. \tag{4.16}$$

Proof. For R sufficiently large, $R > 2$ and $R\tau_2 > 2$, so that for any $x \in K_{\tau,R}$, $B(x,2) \subset M_R$, defined by $M_R = K_{\tau,R}(1 + mR + nR\tau)$, with $m, n \in \{-1, 0, 1\}$. We use the fact that since u is a minimizer of $G_{K_{\tau,R}}(., A_0)$, v solves the Euler-Lagrange equation

$$-\nabla^2_{A_0} v = \frac{v}{b}(1 - (1-b)|v|^2) \tag{4.17}$$

hence estimate (4.14) yields that for any $x \in K_{\tau,R}$,

$$\|\nabla_{A_0} v\|_{H^1(B(x,1))} \le C\|v\|_{L^2(B(x,2))} + C(1-b)\|v|v|^2\|_{L^2(B(x,2))}. \tag{4.18}$$

From (4.17) it is also easily shown (using the maximum principle) that $|u| \le 1$ hence $|v| \le 1/\sqrt{1-b}$. Moreover, since $v \in E_{\tau,R}$, $|v|$ is periodic, hence $\|v\|_{L^2(M_R)} \le 9\|v\|_{L^2(K_{\tau,R})}$. Using (4.11) we find $\|v\|_{L^2(B(x,2))} \le CR$. Similarly

$$(1-b)\|v|v|^2\|_{L^2(B(x,2))} \le (1-b)\|v\|_{L^\infty}\||v|^2\|_{L^2(B(x,2)} \le C\sqrt{1-b}R$$

by (4.10). Inserting into (4.18), we conclude that $\|\nabla_{A_0} v\|_{H^1(B_1(x))} \le CR$, where C does not depend on x in $K_{\tau,R}$. The Sobolev embedding yields (4.15).

For the estimate on w, we use (4.17) and recall that $-\nabla^2_{A_0} v_\Pi = v_\Pi$, hence

$$-\nabla^2_{A_0} w = \frac{v}{b}(1-(1-b)|v|^2) - v_\Pi = \frac{1}{b}w + \frac{1}{b}v_\Pi(1-b) - \frac{1}{b}(1-b)v|v|^2.$$

Using (4.14), we find as above for all $x \in K_{\tau,R}$,

$$\|\nabla_{A_0} w\|_{H^1(B_1(x))} \le C(\|w\|_{L^2(K_{\tau,R})} + (1-b)\|v_\Pi\|_{L^2(K_{\tau,R})} + \sqrt{1-b}\||v|^2\|_{L^2(K_{\tau,R})}).$$

Estimates (4.11)-(4.10)-(4.13) yield $\|\nabla_{A_0} w\|_{H^1(B_1(x))} \le CR\sqrt{1-b}$, where C does not depend on x in $K_{\tau,R}$. The Sobolev embedding yields (4.16). □

It follows that $\|w\|_{L^\infty} \le C|K_{\tau,R}|^{\frac{1}{2}}\sqrt{1-b}$ and $\|v_\Pi\|_{L^\infty} \le C|K_{\tau,R}|^{\frac{1}{2}}$.

- *Step 4:* We estimate $\int_{K_{\tau,R}} |v|^4 - 2|v|^2$. By definition of v_Π and w, we have

$$-2\int_{K_{\tau,R}} |v|^2 = -2\int_{K_{\tau,R}} |v_\Pi|^2 + |w|^2 \ge -2\int_{K_{\tau,R}} |v_\Pi|^2 - C|K_{\tau,R}|(1-b)$$

where the last inequality comes from (4.13). Moreover,

$$\int |v|^4 \ge \int |v_\Pi|^4 + 4(v_\Pi \cdot w)(|w|^2 + |v_\Pi|^2) \ge \int |v_\Pi|^4 - 2\left(\|v_\Pi\|^2_{L^\infty} + \|w\|^2_{L^\infty}\right)\|v_\Pi\|_{L^2}\|w\|_{L^2}$$

$$\ge \int |v_\Pi|^4 - C\sqrt{1-b}|K_{\tau,R}|^2.$$

We conclude that

$$\int_{K_{\tau,R}} |v|^4 - 2|v|^2 \geq \int_{K_{\tau,R}} |v_\Pi|^4 - 2|v_\Pi|^2 - C|K_{\tau,R}|^2\sqrt{1-b}.$$

Inserting into (3.4), we find

$$\frac{b}{|K_{\tau,R}|}\mathcal{G}_{\tau,R}(u) - \frac{1}{4} \geq \frac{(1-b)^2}{4}F_{\tau,R}(v) \geq \frac{(1-b)^2}{4}F_{\tau,R}(v_\Pi) + O(|K_{\tau,R}|(1-b)^{5/2})) \quad (4.19)$$

and (4.6) follows.

□

We deduce immediately that if b tends to 1 and $R \to \infty$ in such a way that $\sqrt{1-b}|K_{\tau,R}| \to 0$, then

$$\limsup_{R\to\infty,\sqrt{1-b}|K_{\tau,R}|\to 0} b\frac{\mathcal{G}_{\tau,R}(u)}{|K_{\tau,R}|} \geq \frac{1}{4} + \frac{(1-b)^2}{4}\min_{L_{\tau,R}} F_{\tau,R}$$

But by the characterisation (1.13), the left-hand side is equivalent as $b \to 1$ to $f(b)$ hence we deduce

$$\liminf_{b\to 1} \frac{4f(b)-1}{(1-b)^2} \geq \limsup_{R\to\infty} \min_{L_{\tau,R}} F_{\tau,R}.$$

Comparing with (4.5), we deduce that the liminf and limsup are equal and the second assertion of Theorem 2 follows. The first assertion is also easily true.

Now if u minimizes $\mathcal{G}_{\tau,R}$, and v_Π is the projection on $L_{\tau,R}$ as above, comparing (4.3) to (4.19) we deduce also that

$$F_{\tau,R}(v_\Pi) = \min_{L_{\tau,R}} F_{\tau,R} + O(|K_{\tau,R}|\sqrt{1-b}).$$

With the content of Lemma 4.2, Proposition 1.1 is proved.

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