Gamma-convergence of gradient flows with applications to Ginzburg-Landau

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Abstract

We present a method to prove convergence of gradient-flows of families of energies which Gamma-converge to a limiting energy. It provides lower bound criteria to obtain the convergence, which correspond to a sort of C^1-order Gamma-convergence of functionals. We then apply this method to establish the limiting dynamical law of a finite number of vortices for the heat-flow of the Ginzburg-Landau energy in dimension 2, retrieving in a different way the existing results for the case without magnetic field, and obtaining new results for the case with magnetic field.

I Introduction

The notion of Gamma-convergence was introduced by Ennio De Giorgi in the 70's. It provided a useful notion of convergence of a family of energy-functionals E_ε to a limiting functional F, i.e. criteria which allow to conclude that global minimizers of E_ε converge to global minimizers of F. It also unified various notions of variational convergence considered

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earlier. Since then, a large number of variational problems have successfully been put in the Gamma-convergence framework and this notion has become standard. For a presentation of Gamma-convergence (from now on denoted Γ-convergence) and many examples, we refer to the very nice book of Braides [Bra]. An early, celebrated example, conjectured by De Giorgi and proved by Modica-Mortola, was the Γ-convergence of the real-valued phase transition model i.e. the family of energies

$$E_\varepsilon(u) = \int_\Omega \varepsilon |\nabla u|^2 + \frac{1}{\varepsilon}(1-|u|^2)^2$$

with $u : \Omega \mapsto \mathbb{R}$ to a perimeter functional. Another example that we will consider in this paper is the Ginzburg-Landau functional, the complex-valued analogue of the previous energy, whose Γ-convergence properties were first studied by Bethuel, Brezis and Hélein [BBH].

When it comes to studying time-dependent versions of the problem, i.e. proving the convergence of certain flows (for example the gradient flow) of Γ-converging energies towards the flow of the limiting energy, no general criterion or result seems to be available, even though it is expected that with proper scalings, there is convergence of solutions of the gradient-flow (for a chosen structure) of Γ-converging energies to the solution of the limiting flow, for a certain structure to be determined. However this does not follow from any easy abstract argument, since it would involve commuting the limit as the parameter $\varepsilon \to 0$ and time-derivatives, which is wrong without further assumptions. Available results in this direction are proved for specific problems, with PDE rather than energy methods, i.e. without specifically using the Γ-convergence structure.

For example in the aforementioned case of real-valued phase-transitions, the convergence of the solutions of the gradient flow (called the Allen-Cahn equation) to the mean curvature flow (which is the gradient flow for the perimeter functional) in stronger or weaker senses was established by De Mottoni-Schatzman [dMS] and X. Chen [Ch], then Evans-Soner-Souganidis [ESS] via viscosity solutions and the level-set approach to mean-curvature flow treated the situation even after the appearance of singularities, so did later Ilmanen [I] connecting it to the notion of Brakke flow. It also requires some proper time-rescaling, and dimension 1 is special because interfaces move exponentially slowly. For the (complex-valued) Ginzburg-Landau functional without magnetic field, the expected convergence (for various types of flows) in 2D has also been proved by PDE methods, by Lin [Li1] and Jerrard-Soner [JS1] for heat flow, Lin-Xin and Jerrard-Colliander [LX, CJ1, CJ2] for Schrödinger flows, Lin [Li2] and Jerrard [J2] for wave flow, all up to "collision-time"; the advantage of these PDE methods over the one we present here being that they also work for dispersive equations. In dimension 3, the limiting heat-flow of Ginzburg-Landau is again the mean-curvature (Brakke) flow (see [AS, LR, BOS]).

In this paper, we focus on heat flow or gradient flows, and make an attempt towards a more systematic treatment of the convergence relying on the Γ-convergence structure (for another formal attempt of this sort, see [F]). This convergence cannot follow from Γ-convergence only: slightly perturbing the energy landscape of E_ε may add local minima which disappear in the limit. Thus extra conditions are required to guarantee that the C^1-structure of the energy landscape also converges. Our scheme is a time-dependent analogue of the Γ-convergence scheme, or a C^1 form of Γ-convergence. Like the basic theorems of Γ-convergence, the abstract result is extremely simple to state and prove. We then prove that this scheme can actually be applied, by using it first to recover the convergence result for the heat flow of the Ginzburg-Landau equations (the result of [Li1, JS1]) and second to obtain a convergence result for the magnetic Ginzburg-Landau energy, which is a new result. We hope this scheme is sufficiently robust (or can be made so) to apply to other interesting examples.

The criteria we state also contain information on the limiting gradient-flow. Indeed, we mentioned above that the solutions of the gradient-flows (for a certain gradient structure) are expected to converge to the solution of the gradient-flow of the limiting energy, but for what limiting structure and what time scaling? This is not obvious, and is usually determined during the case by case convergence study. In our scheme the limiting gradient-flow is determined by certain criteria which have to be verified, and this may appear more naturally.

Finally, let us point out that the idea of this method extends to second order and allows to find criteria (basically that the C^2-structure is preserved under the convergence) to prove that stability/instability of critical points of E_ε carries through to critical points of F. This is the object of the paper [S3], in which it is again applied to Ginzburg-Landau.

I.1 The abstract result

The abstract framework that we wish to consider is the following. Let E_ε be a family of C^1 functionals defined over $\mathcal{M}$, an open subset of an affine space associated to a Banach space $\mathcal{B}$. We assume that E_ε Γ-converges to the C^1 functional F defined over $\mathcal{N}$, which we assume here for simplicity to be an open set of a finite-dimensional vector space $\mathcal{B}'$. This is the definition that we will use :

Definition 1 *E_ε Γ-converges along the trajectory $u_\varepsilon(t)$ ($t \in [0,T)$) in the sense S to F if there exists $u(t) \in \mathcal{N}$ and a subsequence (still denoted u_ε) such that $\forall t \in [0,T)$, $u_\varepsilon(t) \stackrel{S}{\rightharpoonup} u(t)$ and*

$$\forall t \in [0,T), \quad \liminf_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon(t)) \geq F(u(t)).$$

The sense S is to be specified in each case, it can be a weak convergence of u_ε in a certain norm or distance, it can be a convergence of some function of u_ε... Notice that u_ε and u do not generally belong to the same space. Usually Γ-convergence requires a limsup condition that will not be needed here but rather implied by the extra conditions we add.

We assume that $\mathcal{B}$ embeds continuously into a Hilbert space X_ε, resp. $\mathcal{B}'$ into Y.

Definition 2 *If the differential $dE_\varepsilon(u)$ of E_ε at u, is also linear continuous on X_ε, we call the vector of X_ε which represents it the gradient of E_ε at $u \in \mathcal{M}$ for the structure X_ε, and* *we denote it by $\nabla E_\varepsilon(u)$. We have for any $\phi \in X_\varepsilon$*

$$\frac{d}{dt}_{|t=0} E_\varepsilon(u+t\phi) = dE_\varepsilon(u)\cdot\phi = \langle \nabla E_\varepsilon(u), \phi\rangle_{X_\varepsilon}.$$

If this gradient does not exist, we use the convention $\|\nabla E_\varepsilon(u)\|_{X_\varepsilon} = +\infty$.

The same is done for F, and since $\mathcal{B}'$ is assumed to be finite-dimensional, the C^1 character of F implies the existence of its gradient for the structure Y.

Definition 3 *A solution of the gradient-flow for E_ε with respect to the structure X_ε on $[0,T)$ is a map $u_\varepsilon \in H^1((0,T), X_\varepsilon)$ such that*

$$\partial_t u_\varepsilon = -\nabla E_\varepsilon(u_\varepsilon) \in X_\varepsilon \tag{I.1}$$

for a.e. $t\in[0,T)$ (where the gradient with respect to the structure X_ε is taken in the sense of Definition 2).

Such a solution is conservative *if for all $t \in [0,T)$,*

$$E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(t)) = \int_0^t \|\partial_t u_\varepsilon(s)\|^2_{X_\varepsilon}\, ds.$$

If u_ε is a family of solutions on $[0,T)$ of the gradient-flow for E_ε along which E_ε Γ-converges to F (in the sense of Definition 1), and $u_\varepsilon \stackrel{S}{\rightharpoonup} u$, we define the energy-excess *$D(t)$ by*

$$D_\varepsilon(t) = E_\varepsilon(u_\varepsilon(t)) - F(u(t)), \qquad D(t) = \limsup_{\varepsilon\to 0} D_\varepsilon(t) \geq 0.$$

A family of solutions of the gradient-flow is said to be well-prepared initially if $D(0)=0$.

The energy excess D should be considered as a small perturbation, and can be taken to be 0 in a first reading of what follows.

We define similarly the gradient flow of F. Notice that in practice, a sufficiently smooth solution is conservative, so if smoothness can be deduced from (I.1), all solutions are conservative. Our first main result is

Theorem 1 *Let E_ε and F be C^1 functionals over $\mathcal{M}$ and $\mathcal{N}$ respectively, and let u_ε be a family of conservative solutions of the flow for E_ε ($\partial_t u_\varepsilon = -\nabla E_\varepsilon(u_\varepsilon)$) on $[0,T)$, with $u_\varepsilon(0) \stackrel{S}{\rightharpoonup} u_0$, along which E_ε Γ-converges to F in the sense of Definition 1. Assume moreover that 1) and either 2) or 2') below are satisfied:*

1) *(lower bound) For a subsequence such that $u_\varepsilon(t) \stackrel{S}{\rightharpoonup} u(t)$, we have $u \in H^1((0,T), Y)$ and there exists $f \in L^1((0,T))$ such that for every $s\in[0,T)$,*

$$\liminf_{\varepsilon\to 0} \int_0^s \|\partial_t u_\varepsilon(t)\|^2_{X_\varepsilon}\, dt \geq \int_0^s \left(\|\partial_t u\|^2_Y - f(t)D(t)\right)\, dt. \tag{I.2}$$

2) (construction) If $u_\varepsilon(t) \overset{S}{\rightharpoonup} u(t)$, there exists a locally bounded function g on $[0,T)$ such that for any $t_0 \in [0,T)$ and any v defined in a neighborhood of t_0 satisfying

$$v(t_0) = u(t_0), \quad \partial_t v(t_0) = -\nabla F(u(t_0))$$

there exists $v_\varepsilon(t)$ such that $v_\varepsilon(t_0) = u_\varepsilon(t_0)$ and, letting D be the energy excess of u_ε,

$$\text{(I.3)} \qquad \begin{aligned} &\limsup_{\varepsilon\to 0} \|\partial_t v_\varepsilon(t_0)\|^2_{X_\varepsilon} \le \|\partial_t v(t_0)\|^2_Y + g(t_0)D(t_0) \\ &\liminf_{\varepsilon\to 0} -\frac{d}{dt}_{|t=t_0} E_\varepsilon(v_\varepsilon) \ge -\frac{d}{dt}_{|t=t_0} F(v) - g(t_0)D(t_0). \end{aligned}$$

2') There exists a locally bounded function g on $[0,T)$ such that for any $t \in [0,T)$

$$\text{(I.4)} \qquad \liminf_{\varepsilon\to 0} \|\nabla E_\varepsilon(u_\varepsilon(t))\|^2_{X_\varepsilon} \ge \|\nabla F(u(t))\|^2_Y - g(t)D(t).$$

Then if $D(0) = 0$, i.e. if u_ε is well prepared initially, then $D(t) = 0$ on $[0,T)$, all the inequalities above are equalities, and $\forall t \in [0,T), u_\varepsilon(t) \overset{S}{\rightharpoonup} u(t)$ is the solution of the gradient-flow for F with respect to the structure Y on $[0,T)$ with initial data u_0, i.e.

$$\begin{cases} \partial_t u = -\nabla F(u) \\ u(0) = u_0. \end{cases}$$

Moreover, if 2) is satisfied, then it yields for every t_0 a family $v_\varepsilon^{t_0}(t)$ defined in a neighborhood of t_0 and, letting $v'_\varepsilon(t_0) = \partial_t v_\varepsilon^{t_0}(t_0)$,

$$\text{(I.5)} \qquad \lim_{\varepsilon\to 0} \int_0^T \|\partial_t u_\varepsilon - v'_\varepsilon\|^2_{X_\varepsilon}\, dt = 0.$$

We will prove in Lemma II.1 that 2) implies 2'). The idea of the construction 2) is to perturb u and "push it" along the direction of the expected motion i.e. along $\nabla F(u)$. If this can be done without "paying too much" in $\|\partial_t u\|^2_{X_\varepsilon}$ while decreasing the energy at least of the expected amount (that is (I.3)), then it implies that the slope was at least the expected limiting one $\|\nabla F(u)\|_Y$ or (I.4) holds.

We will also prove

Proposition I.1 *Under the same hypotheses, if u_ε is any family of solutions on $[0,T)$ of the time-rescaled equation $\partial_t u_\varepsilon = -\lambda_\varepsilon \nabla E_\varepsilon(u_\varepsilon)$ with $D(0) = 0$, then*
- if $\lambda_\varepsilon \ll 1$, then, for a.e. $t \in [0,T)$, $u_\varepsilon(t) \overset{S}{\rightharpoonup} u_0 \in \mathcal{N}$ (i.e. no motion)
- if $\lambda_\varepsilon \gg 1$ then, for a.e. $t \in [0,T)$, $u_\varepsilon(t) \overset{S}{\rightharpoonup} u \in \mathcal{N}$ with $\nabla F(u) = 0$ (instantaneous motion to a critical point).

So this method is akin to a C^1 version of Γ-convergence, obtained through the control of convergence of the norms of tangents to curves. Criteria 1) and 2), 2') are expected to hold for a general class of $u_\varepsilon(t)$ (not only solutions to the gradient flow).

The difficulty in applying this theorem to concrete situations is in proving that these criteria are satisfied, in particular perform the perturbation construction of 2). When it can be done, it should allow to use the PDE less, relying more on energy comparison techniques (even though we will see that the PDE is used to check some hypotheses of the theorem).

As mentioned before, once structures X_ε and Y such that 1) and 2) are verified are identified, it provides the suitable time-scaling for the ε problems (see Proposition I.1) and the limiting gradient-structure. For example, in the case of the Ginzburg-Landau equation, we will see that we need to take $\|.\|_{X_\varepsilon} = \frac{1}{\sqrt{|\log \varepsilon|}}\|.\|_{L^2(\Omega)}$ and $\|.\|_Y = \frac{1}{\sqrt{\pi}}\|.\|_{(\mathbb{R}^2)^n}$ (the canonical Euclidean scalar product on $(\mathbb{R}^2)^n$).

I.2 Generalizations

The procedure above can be generalized in several ways. First it can be extended to the case where the limiting space $\mathcal{B}'$ is not finite-dimensional, and embeds into a Hilbert space Y. Inspecting the proof (see Section II) shows that it suffices to require that for $u(t)$ limit of solutions u_ε, the gradient $\nabla F(u(t))$ with respect to the structure Y exists and that $\int_0^t \langle \nabla F(u(s)), \partial_t u(s)\rangle_Y \, ds = F(u(t)) - F(u(0))$ holds.

Second of all, a natural and more general framework to work on would be that of manifold-type spaces, i.e. that $\mathcal{M}$, and more commonly the limiting space $\mathcal{N}$, has a tangent space $T_u\mathcal{N}$ at each point, which embeds into a Hilbert space Y_u depending on the point. Then $\partial_t u = -\nabla F(u)$ would be an equation holding in $Y_{u(t)}$, with the gradient taken with respect to the structure $Y_{u(t)}$. The criteria replacing 1), 2) and 2') in Theorem 1 should then be exactly the same with Y replaced by Y_u everywhere, for example $\liminf_{\varepsilon\to 0} \int_0^s \|\partial_t u_\varepsilon(t)\|^2_{X_\varepsilon} \, dt \geq \int_0^s \left(\|\partial_t u\|^2_{Y_{u(t)}} - f(t)D(t) \right) dt$.

The proof can also be reproduced step by step with this formal replacement. The difficulty is rather to make sense of these manifold-type structures which do not in general have atlas structures. Such a framework is appropriate for a number of problems. One example is the convergence of the Allen-Cahn equation to the mean curvature flow, where the expected structure is the weighted Lebesgue space L^2_μ where μ is the limiting measure carried by the surface or varifold. Another example is the case of Ginzburg-Landau with large number of vortices, where the limiting space should be (up to rescaling) the space of probability measures μ endowed with the 2-Wasserstein distance, the appropriate structure on the tangent bundle being again L^2_μ. In this particular case, the "manifold structure" and the notion of trajectories and gradient flows for convex functionals for this structure have been given a rigorous meaning by Ambrosio et al. in [AGS].

Thus, we see how some curvature may appear through the limiting process, when the structure on the tangent space Y_u can indeed depend on the point u even when the original space X_ε does not. The work by F. Otto [O] on the porous medium equation seems to be the first that exploited the interpretation of the evolution as a gradient flow for such a curved structure, in that case in order to derive information on the long-time behavior.

However, the scheme in its present form would allow to prove convergence only as long as the limiting flow is a classical flow, i.e. before the apparition of singularities.

A third possible generalization would be treating the case of a possible limiting functional F which is not C^1 with respect to the structure Y (or Y_u) but which is convex so that the gradient of F can be replaced by its subdifferential ∂F. Then, one would need to replace $\|\nabla F(u)\|_{Y_u}$ by a norm of $\partial F(u)$ defined as the maximum of the norms of the possible slopes in $\partial F(u)$. Again, the proof carries through formally and one is led to gradient flows defined through subdifferentials for convex functionals (see [Bre] and [AGS] again).

I.3 Statement of the results on Ginzburg-Landau

Ginzburg-Landau functionals arise in condensed matter physics, they serve to model superconductivity, superfluidity, Bose-Einstein condensates. They involve a complex-valued order parameter, that we denote u, which describes the local state of the material, $|u|^2 \le 1$ being a local density. We will only consider the 2-dimensional case of a bounded simply connected domain $\Omega \subset \mathbb{R}^2$. The Ginzburg-Landau energy without magnetic field is

$$F_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2, \tag{I.6}$$

defined over $H^1(\Omega, \mathbb{C})$. For superconductivity, one considers the gauge-invariant functional

$$J(u,A) = \frac{1}{2}\int_\Omega |\nabla u - iAu|^2 + |\mathrm{curl}\, A - h_{\mathrm{ex}}|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2, \tag{I.7}$$

where $A : \Omega \mapsto \mathbb{R}^2$ and h_{ex} is the intensity of the applied field. Both energies are to be studied in the limit $\varepsilon \to 0$. A key feature of these functionals is the existence of *vortices* of u, i.e. isolated zeros of $|u|$ with nonzero winding number $d \in \mathbb{Z}$ of $u/|u|$ around such a zero, or in other words topological singularities of $u/|u|$. The limit $\varepsilon \to 0$ corresponds to strongly repulsive point-like vortices. For the asymptotics of (I.6) as $\varepsilon \to 0$ we refer to [BBH] and to the subsequent vast literature. For more on (I.7), we refer to Section IV. Both energies have been proved to Γ-converge, under certain hypotheses and in a sense that we will specify below, to a function which depends only on the vortex-locations, i.e. to a function on Ω^n. Theorem 1 then allows to derive the limiting dynamics for the vortex points when studying the gradient flow for (I.6) or (I.7). We will restrict in this paper to the case where the number of vortices is initially bounded independently of ε.

The case with no magnetic field

It has been essentially shown (see [BBH], [S1, SS7] for a complete proof) that if $g : \partial\Omega \to S^1$ is a fixed map with degree $d > 0$, there exists a function $W_g(\mathbf{a}, \mathbf{d})$, defined for an arbitrary positive integer n and $\mathbf{a} \in \Omega^n$, $\mathbf{d} \in \mathbb{Z}_*^n$, and a universal constant C_0, satisfying the following. If $F_\varepsilon(u_\varepsilon) \le C|\log \varepsilon|$, with $u_\varepsilon \in H^1_g(\Omega)$ (the set of H^1 maps agreeing with g on $\partial\Omega$), then as $\varepsilon \to 0$ and modulo a subsequence,

$$\mathrm{curl}\,(iu_\varepsilon, \nabla u_\varepsilon) \rightharpoonup 2\pi \sum_{i=1}^n d_i \delta_{a_i} \tag{I.8}$$

with $\sum_i d_i = d$, where $(.,.)$ denotes the inner product in $\mathbb{C}$ identified with $\mathbb{R}^2$ ((I.8) is the sense of convergence S that we will use), and

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) - \pi\sum_{i=1}^n |d_i||\log\,\varepsilon| - C_0\sum_{i=1}^n |d_i| \geq W_g(\mathbf{a},\mathbf{d}), \tag{I.9}$$

where $\mathbf{a} = (a_1,\dots,a_n)$, $\mathbf{d} = (d_1,\dots,d_n)$. Moreover, if u_ε is a critical point (resp. minimizer) of F_ε on H^1_g, then $\mathbf{a}$ is a critical point (resp. minimizer) of W_g. An equivalent of W_g exists for Neumann boundary conditions, we denote it W_n. We write W when stating a result that applies to both cases.

Theorem 1 yields in this case

Theorem 2 *Let u_ε be a family of solutions of*

$$\frac{1}{|\log\,\varepsilon|}\partial_t u = \Delta u + \frac{u}{\varepsilon^2}(1-|u|^2), \tag{I.10}$$

with either Dirichlet ($u_\varepsilon = g$) or homogeneous Neumann boundary conditions, such that $\operatorname{curl}(iu_\varepsilon, \nabla u_\varepsilon)(0)$ *converges to* $2\pi\sum_{i=1}^n d_i\delta_{a_i^0}$*, where a_i^0 are distinct points of Ω and $d_i = \pm 1$, and that $u_\varepsilon(0)$ is well-prepared in the sense that*

$$F_\varepsilon(u_\varepsilon(0)) - \pi n|\log\,\varepsilon| - nC_0 \leq W(a_i^0, d_i) + o(1). \tag{I.11}$$

Then there exists a time $T^* > 0$ such that $\operatorname{curl}(iu_\varepsilon, \nabla u_\varepsilon) \rightharpoonup 2\pi\sum_i d_i\delta_{a_i(t)}$ *and*

$$F_\varepsilon(u_\varepsilon(t)) \leq \pi n|\log\,\varepsilon| + nC_0 + W(a_i(t), d_i) + o(1) \tag{I.12}$$

for all $t \in [0, T^*)$, with

$$\frac{da_i}{dt} = -\frac{1}{\pi}\partial_i W(a_i(t), d_i), \quad a_i(0) = a_i^0, \tag{I.13}$$

where the degrees d_i remain constant. T^* is the minimum of the collision time and of the exit time from Ω (in the Neumann case) for this law of motion.

Moreover, for all $B_i(t)$ disjoint open balls centered at $a_i(t)$, $\mathbf{1}_{B_i(t)}$ denoting the characteristic function of $B_i(t)$, we have for all $T < T^*$,

$$\frac{1}{|\log\,\varepsilon|}\int_{\Omega\times[0,T]} \left|\partial_t u_\varepsilon - \sum_i \mathbf{1}_{B_i(t)}\frac{da_i}{dt}\cdot\nabla u_\varepsilon\right|^2 dt \to 0 \quad \text{as } \varepsilon \to 0. \tag{I.14}$$

We thus recover the same type of result as in [Li1, JS1] i.e. the convergence to the flow of the limiting energy, up to collision time. In other time-scalings, one can also retrieve the results of [Li1, JS1] by applying Proposition I.1. Our initial condition (I.11) is slightly more restrictive, however we will prove in [S4] that it is satisfied after an infinitely small time. We will also deal in [S4] with vortex-collisions and extending the result after time T^*.

The estimate (I.14) is new, up to our knowledge. It expresses that u_ε is very close to being simply transported at the velocity $\frac{da_i}{dt}$ around each a_i.

The method we follow for deriving the dynamics consists in applying Theorem 1. The appropriate X_ε structure is a rescaled version of the L^2 norm, and the Y structure a rescaled version of the Euclidian norm on Ω^n. The lower bound relating the time-variation of u, to the velocity of the underlying vortices, needed to fulfil condition 1) is provided for the study of (I.6) and (I.7) by a result of [SS6], Theorem 3 and Corollary 7 (see also [J2], Proposition 3). The heart of the matter is then to perform an adequate construction to fulfill condition 2).

The case with magnetic field

In the case of the functional with magnetic field, the result we prove is new. For a physical presentation of the Ginzburg-Landau model of superconductivity model, see [T]. The energy is (I.7) with unknowns the order parameter $u : \Omega \mapsto \mathbb{C}$, and the vector potential $A : \Omega \mapsto \mathbb{R}^2$. The notation ∇_A will denote the covariant gradient $\nabla - iA$, and $h = \operatorname{curl} A = \partial_1 A_2 - \partial_2 A_1$ is the induced magnetic field, while $h_{\rm ex}$ is the intensity of the applied (uniform, constant) magnetic field. The Ginzburg-Landau energy is invariant under the gauge-transformations

$$\begin{cases} u \mapsto ue^{iw} \\ A \mapsto A + \nabla w. \end{cases}$$

The statics of (I.7) have been intensively studied recently. We will focus in particular on the regime $h_{\rm ex} = O(|\log \varepsilon|)$ which is a suitable regime to study vortices. We will assume that

$$h_{\rm ex} = \lambda |\log \varepsilon| \quad 0 < \lambda < \infty. \tag{I.15}$$

In this regime we have obtained various results about the minimizers and critical points of J, see [S1, SS1, SS2, SS3, SS4]. The (heat flow) Ginzburg-Landau equations as proposed by Gorkov-Eliashberg are

$$\begin{cases} \partial_t u + iu\Phi = \nabla_A^2 u + \dfrac{u}{\varepsilon^2}(1-|u|^2) & \text{in } \Omega \\ \partial_t A + \nabla\Phi = \nabla^\perp h + (iu, \nabla_A u) & \text{in } \Omega \\ (iu, \nabla_A u)\cdot n = 0 & \text{on } \partial\Omega \\ h = h_{\rm ex} & \text{on } \partial\Omega. \end{cases} \tag{I.16}$$

These are the gradient flow for essentially the same L^2 structure as in the case without magnetic field. Observe that here there is no need to rescale in time to see motion of vortices. The quantity Φ makes the equations invariant under the gauge-transformations :

$$\begin{cases} u \mapsto ue^{iw} \\ A \mapsto A + \nabla w \\ \Phi \mapsto \Phi - \partial_t w. \end{cases} \tag{I.17}$$

The quantity $(iu, \nabla_A u)$ is the superconducting current also denoted j, it is a gauge-invariant quantity. The equivalent of the Jacobian curl $(iu, \nabla u)$ in this case is $j + h$, which is also gauge-invariant. The quantity $-\partial_t A - \nabla\Phi = \mathcal{E}$ represents the electric current generated by the evolution of the system, and $\mathcal{F} = \partial_t u + iu\Phi$ is such that $\langle \mathcal{F}, iu\rangle$ is the charge.

The well-posedness of (I.16) (hence the existence of solutions) once a gauge has been chosen, was first established by Du [Du]. There have also been formal studies of the dynamics: Pismen-Rubinstein [PR], E [E], Chapman-Rubinstein-Schatzman [CRS]. The vortex dynamics in the limit $\varepsilon \to 0$ has been rigorously established in the case $h_{\text{ex}} = O(1)$ (for which $\lambda = 0$) by Spirn [Sp]. Rather than reproving this result with our method, we focus on the case $\lambda > 0$.

Let ξ_0 be as in [S1, S2, SS1, SS3] the solution of

$$\text{(I.18)} \qquad \begin{cases} -\Delta\xi_0 + \xi_0 + 1 = 0 & \text{in } \Omega \\ \xi_0 = 0 & \text{on } \partial\Omega \end{cases}$$

and

$$\text{(I.19)} \qquad J_0 = \frac{1}{2}\int_\Omega |\nabla\xi_0|^2 + \xi_0^2.$$

From [S1, S2, SS1, SS5], if $\text{curl}\,(iu_\varepsilon, \nabla u_\varepsilon) \rightharpoonup 2\pi\sum_i d_i\delta_{a_i}$, with fixed degrees d_i, then

$$\liminf_{\varepsilon\to 0} \frac{J(u,A) - h_{\text{ex}}^2 J_0 - \pi\sum_i |d_i||\log\,\varepsilon|}{|\log\,\varepsilon|} \geq 2\pi\lambda\sum_i d_i\xi_0(a_i),$$

hence the Γ-convergence in the sense of Definition 1. Theorem 1 yields

Theorem 3 *Let $(u_\varepsilon, A_\varepsilon, \Phi_\varepsilon)$ be a family of solutions of (I.16) with (I.15). We assume that $\text{curl}\,((iu_\varepsilon, \nabla u_\varepsilon - iA_\varepsilon u_\varepsilon) + A_\varepsilon)(0) \rightharpoonup 2\pi\sum_{i=1}^n d_i\delta_{a_i^0}$, with $d_i = \pm 1$, and that $(u_\varepsilon(0), A_\varepsilon(0))$ is well-prepared in the sense that*

$$J(u_\varepsilon(0), A_\varepsilon(0)) \leq h_{\text{ex}}^2 J_0 + \pi n|\log\,\varepsilon| + 2\pi h_{\text{ex}}\sum_i d_i\xi_0(a_i^0) + o(|\log\,\varepsilon|).$$

Then, there exists a time $T^* > 0$ such that, for all $t \in [0, T^*)$,

$$\text{curl}\,\left((iu_\varepsilon, \nabla_{A_\varepsilon} u_\varepsilon) + A_\varepsilon\right)(t) \rightharpoonup 2\pi\sum_{i=1}^n d_i\delta_{a_i(t)}$$

with

$$\forall i \quad \frac{da_i}{dt} = -d_i\lambda\nabla\xi_0(a_i(t)), \quad a_i(0) = a_i^0.$$

T^* is the minimum of the collision time and of the exit time from Ω for this law of motion. Moreover, for all $B_i(t)$ disjoint open balls centered at $a_i(t)$ and all $T < T^*$,

$$\frac{1}{|\log\,\varepsilon|}\int_{\Omega\times[0,T]} \left|\partial_t u_\varepsilon + iu\Phi_\varepsilon - \sum_i \mathbf{1}_{B_i(t)}\frac{da_i}{dt}\cdot\nabla u_\varepsilon\right|^2 + |\partial_t A_\varepsilon + \nabla\Phi_\varepsilon|^2\,dt \to 0.$$

Thus, in this case, vortices move in the potential ξ_0 without interacting. This is due to the fact that to leading order the energy has no interaction term and is consistent with results in [S1, SS4] where it is proved that in this regime, the vortices of global minimizers of J concentrate as $\varepsilon \to 0$ at the global minima of ξ_0 and those of critical points at critical points of ξ_0. If the domain is convex then ξ_0 is convex and has a single critical point $x_0 \in \Omega$ which is also a global minimum. Then Theorem 3 implies that vortices of positive degree go to x_0 while vortices of negative degree exit the domain through the boundary, unless collisions occur in the meantime. Also, the velocity is proportional to λ hence to the applied field. When $h_{\text{ex}} \ll |\log \varepsilon|$ then $\lambda = 0$ and vortices do not move in this time-scale, while $\lambda = +\infty$ when $h_{\text{ex}} \gg |\log \varepsilon|$ and vortices move infinitely fast.

This case differs from the case without magnetic field, or with magnetic field bounded independently of ε treated in [Sp], in two ways. First, the global minimizer of the energy may have a number of vortices diverging when $\varepsilon \to 0$, hence the fact that the vorticity remains bounded must be proved before implementing the strategy of Theorem 1. Second, the variation in time of the energy is expected to be of the order $|\log \varepsilon|$. Thus one must track which part remains concentrated in the vortex cores and what happens to the part which is not. Theorem 1 does this, while it seems unpractical to adapt the methods of [Li1, JS1, Sp].

We did not try to obtain the strongest possible convergence but rather focused on the limiting dynamics, working with the convergence in the sense S which can be very weak.

The paper is divided into three more sections. Section II contains the proof of the abstract result, Theorem 1. Section III is devoted to the case without magnetic field with the new proof of Theorem 2, and Section IV to the case with magnetic field and Theorem 3.

Remark on notations: C always denotes a positive constant independent of ε.

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II Proof of Theorem 1

The aim of this section is to prove Theorem 1, which links the gradient flows of E_ε and F when E_ε Γ-converges to F. The proof relies on the idea of steepest descent, i.e. that the solution u of the gradient flow maximizes $|\frac{d}{dt}_{|t=0} E(v(t))|$ for $\|\partial_t v(0)\|_{X_\varepsilon}$ given.

First we prove that condition 2) implies condition 2'). This clearly follows from

Lemma II.1 *Let the functionals E_ε, F be as in Theorem 1 and let $u_\varepsilon \overset{S}{\rightharpoonup} u$ be such that*

$\limsup_{\varepsilon\to 0} \|\nabla E_\varepsilon(u_\varepsilon)\|_{X_\varepsilon} \le C$. *If for any* v *defined in a neighborhood of* $t=0$ *satisfying*

$$\left\{\begin{array}{l} v(0)=u \\ \partial_t v(0) = -\nabla F(u) \end{array}\right. \tag{II.1}$$

there exists $v_\varepsilon \in \mathcal{M}$ *of class* C^1 *in a neighborhood of* 0*, such that* $v_\varepsilon(0)=u_\varepsilon$*, and* δ*,* δ'*, such that*

$$\limsup_{\varepsilon\to 0} \|\partial_t v_\varepsilon(0)\|^2_{X_\varepsilon} \;\le\; \|\partial_t v(0)\|^2_Y + \delta \tag{II.2}$$

$$\liminf_{\varepsilon\to 0} -\frac{d}{dt}_{|t=0} E_\varepsilon(v_\varepsilon) \;\ge\; -\frac{d}{dt}_{|t=0} F(v) - \frac{\delta'}{2}. \tag{II.3}$$

Then,

$$\liminf_{\varepsilon\to 0} \|\nabla E_\varepsilon(u_\varepsilon)\|^2_{X_\varepsilon} \ge \|\nabla F(u)\|^2_Y - (\delta+\delta').$$

Observe that in this lemma δ and δ' can be positive or negative.

Proof of the lemma: Let u_ε be as above and let v satisfy (II.1). Then

$$-\frac{d}{dt}_{|t=0} F(v) = -\langle \nabla F(u), \partial_t v(0)\rangle = \|\partial_t v(0)\|^2_Y = \|\nabla F(v(0))\|^2_Y = \|\nabla F(u)\|_Y.$$

From the hypothesis there exists $v_\varepsilon \in \mathcal{M}$ such that $v_\varepsilon(0)=u_\varepsilon(0)$ and (II.2) and (II.3) hold. Since $-\frac{d}{dt}_{|t=0} E_\varepsilon(v_\varepsilon) = -\langle \nabla E_\varepsilon(u_\varepsilon(0)), \partial_t v_\varepsilon(0)\rangle_{X_\varepsilon}$, equations (II.2) and (II.3)) imply

$$\begin{aligned}
\|\nabla F(u)\|^2_Y &\le \frac{\delta'}{2} - \langle \nabla E_\varepsilon(u_\varepsilon(0)), \partial_t v_\varepsilon(0)\rangle_{X_\varepsilon} + o(1) \\
&\le \frac{\delta'}{2} + \frac{1}{2}\left(\|\nabla E_\varepsilon(u_\varepsilon)\|^2_{X_\varepsilon} + \|\partial_t v_\varepsilon(0)\|^2_{X_\varepsilon}\right) + o(1) \\
&\le \frac{\delta'}{2} + \frac{1}{2}\left(\|\nabla E_\varepsilon(u_\varepsilon)\|^2_{X_\varepsilon} + \|\partial_t v(0)\|^2_Y + \delta\right) + o(1), \qquad \text{(II.4)}
\end{aligned}$$

Inserting $\|\partial_t v(0)\|_Y = \|\nabla F(u)\|_Y$ yields the desired result. □

Proof of Theorem 1: Let u_ε be a family of solutions of the gradient flow $\partial_t u_\varepsilon = -\nabla E_\varepsilon(u_\varepsilon)$ satisfying the hypothesis of Theorem 1. Then condition 1) is satisfied and from the previous lemma 2') is satisfied too. Since u_ε is initially well prepared $E_\varepsilon(0) = F(0)+o(1)$ and $E_\varepsilon(t) = F(t) + D_\varepsilon(t) + o(1)$, where $E_\varepsilon(t) := E_\varepsilon(u_\varepsilon(t))$, $F(t) := F(u(t))$. Thus

$$E_\varepsilon(0) - E_\varepsilon(t) = F(0) - F(t) - D_\varepsilon(t) + o(1). \tag{II.5}$$

On the other hand, since $\partial_t u_\varepsilon = -\nabla E_\varepsilon(u_\varepsilon)$,
(II.6)

$$E_\varepsilon(0) - E_\varepsilon(t) = -\int_0^t \langle \nabla E_\varepsilon(u_\varepsilon(s)), \partial_t u_\varepsilon(s)\rangle_{X_\varepsilon}\, ds = \frac{1}{2}\int_0^t \|\nabla E_\varepsilon(u_\varepsilon(s))\|^2 + \|\partial_t u_\varepsilon(s)\|^2\, ds.$$

Inserting (I.2), (I.4), we find

$$\text{(II.7)}\qquad E_\varepsilon(0)-E_\varepsilon(t)\geq \frac{1}{2}\int_0^t \|\nabla F(u(s))\|_Y^2+\|\partial_t u\|_Y^2-(g(s)+f(s))D(s)\,ds-o(1).$$

Combining with (II.5) yields

$$\text{(II.8)}\quad F(0)-F(t)-D_\varepsilon(t)+o(1)\geq \frac{1}{2}\int_0^t \|\nabla F(u(s))\|_Y^2+\|\partial_t u\|_Y^2-(g(s)+f(s))D(s)\,ds.$$

Then it follows from (II.8) and

$$\text{(II.9)}\qquad \frac{1}{2}\int_0^t \|\nabla F(u(s))\|_Y^2+\|\partial_t u\|_Y^2\,ds\geq \int_0^t \langle -\nabla F(u(s)),\partial_t u\rangle_Y\,ds=F(0)-F(t)$$

that

$$D_\varepsilon(t)\leq \int_0^t (g(s)+f(s))D(s)\,ds.$$

Since $D(0)=0$ by assumption and from Gronwall's lemma, after passing to the limit we find $D(t)=0$ for $t\in[0,T]$, i.e. “well-prepared initial data remains well-prepared in time”. Returning to (II.8), (II.9) and inserting $D(t)=0$ we conclude that

$$\int_0^t \|\nabla F(u(s))+\partial_t u\|_Y^2\,ds\leq 0,$$

hence $\partial_t u=-\nabla F(u)$ a.e in $[0,T]$.

To prove the last assertion of Theorem 1, note that all the inequalities in this proof have now to be equalities, hence

$$\text{(II.10)}\qquad \lim_{\varepsilon\to 0}\int_0^T \|\nabla E_\varepsilon(u_\varepsilon)\|_{X_\varepsilon}^2\,dt=\int_0^T \|\nabla F(u)\|_Y^2\,dt.$$

Moreover if condition 2) is satisfied, using the notations of Theorem 1, we must have equality in (II.3) i.e. for each t_0,

$$\text{(II.11)}\qquad \lim_{\varepsilon\to 0}\frac{d}{dt}_{|t=t_0}E_\varepsilon(v_\varepsilon^{t_0})=\lim_{\varepsilon\to 0}\langle \nabla E_\varepsilon(u_\varepsilon(t_0)),v_\varepsilon'(t_0)\rangle_{X_\varepsilon}=-\|\nabla F(u(t_0))\|_Y^2$$

and $\lim_{\varepsilon\to 0}\|\partial_t v_\varepsilon'(t_0)\|_{X_\varepsilon}^2=\|\nabla F(u(t_0))\|_Y^2$. Combining this with (II.10), passing to the limit in

$$\int_0^T \|\nabla E_\varepsilon(u_\varepsilon)+v_\varepsilon'(t)\|_{X_\varepsilon}^2\,dt=\int_0^T \|\nabla E_\varepsilon(u_\varepsilon(t))\|_{X_\varepsilon}^2+\|v_\varepsilon'(t)\|_{X_\varepsilon}^2+2\langle \nabla E_\varepsilon(u_\varepsilon(t)),v_\varepsilon'(t)\rangle_{X_\varepsilon}\,dt,$$

we are led to

$$\lim_{\varepsilon\to 0}\int_0^T \|\nabla E_\varepsilon(u_\varepsilon)+v_\varepsilon'(t)\|_{X_\varepsilon}^2\,dt=0$$

which proves (I.5).

If the hypotheses are satisfied with $f = 0$ and $g = 0$, we obtain in place of (II.8), even without assuming $D(0) = 0$, that

$$D(0) - D(t) \geq 0.$$

This can be applied on any subinterval of $[0, T]$, thus $D(t)$ decreases in time. □

Proof of Proposition I.1: For the time-rescaled version, note that $\partial_t u_\varepsilon = -\lambda_\varepsilon \nabla E_\varepsilon(u_\varepsilon)$ solves the gradient flow for the structure X'_ε where $\|.\|_{X'_\varepsilon} = \frac{1}{\sqrt{\lambda_\varepsilon}}\|.\|_{X_\varepsilon}$. It follows that $\int_0^T \|\partial_t u_\varepsilon\|^2_{X'_\varepsilon}\, dt = \int_0^T \|\nabla_{X'} E_\varepsilon(u_\varepsilon)\|^2_{X'}\, dt$, where ∇_X and $\nabla_{X'}$ denote respectively the gradients with respect to the X and X' structures, remains bounded. Thus if $\lambda_\varepsilon = o(1)$, condition 1) implies

$$\int_0^s \|\partial_t u\|^2_Y\, dt \leq \liminf_{\varepsilon \to 0} \int_0^s \|\partial_t u_\varepsilon\|^2_{X_\varepsilon}\, dt \leq \lambda_\varepsilon \int_0^s \|\partial_t u_\varepsilon\|^2_{X'_\varepsilon}\, dt + o(1) \leq o(1).$$

Thus $\partial_t u = 0$ a.e. which proves the result in the case $\lambda_\varepsilon \ll 1$. Moreover, we notice that $\|\nabla_X E_\varepsilon(u_\varepsilon)\|_X = \frac{1}{\sqrt{\lambda_\varepsilon}}\|\nabla_{X'} E_\varepsilon(u_\varepsilon)\|_{X'}$. Hence if $\lambda_\varepsilon \gg 1$, using Lemma II.1 and condition 2'), we have for all $t \in [0, T_1)$,

$$\int_0^t \|\nabla F(u)\|^2_Y \leq \frac{1}{\lambda_\varepsilon} \int_0^t \|\nabla_{X'} E_\varepsilon(u_\varepsilon)\|^2_{X'} \leq o(1).$$

Thus, $\nabla F(u(t)) = 0$ a.e. in t.

Remark II.1 *If (I.2) is satisfied with $f(t) = 0$, and (I.3) with $g(t) = 0$, then for any $u_\varepsilon \stackrel{S}{\rightharpoonup} u \in \mathcal{N}$,*

$$\liminf_{\varepsilon \to 0} \|\nabla E_\varepsilon(u_\varepsilon)\|_{X_\varepsilon} \geq \|\nabla F(u)\|_Y$$

implying that critical points of E_ε converge to critical points of F. Also then, for any solution of $\partial_t v_\varepsilon = -\nabla E_\varepsilon(u_\varepsilon)$ (not necessarily well-prepared), $D(t)$ decreases in time.

Remark II.2 *For almost every $t \in [0, T]$ we have $\limsup_{\varepsilon \to 0} \|\nabla E_\varepsilon(u_\varepsilon(t))\|_{X_\varepsilon} < \infty$. If this implied that $D(t) = 0$, we could deduce that $D(t) = 0$ in $]0, T]$, i.e. that any energy-excess disappears instantaneously.*

III Ginzburg-Landau without magnetic field

The main result (Theorem 2) is a consequence of the following result which we prove in the next sections:

Theorem 4 *Under the hypotheses of Theorem 2, the conclusions hold on some time-interval $[0, T)$ with $T > 0$.*

III.1 Preliminaries : definition of vortex-trajectories

We use the notation (I.6) for the Ginzburg-Landau energy. We need the following.

Proposition III.1 *Let $u_\varepsilon(t,x)$ be defined over $[0,T]\times\Omega$ (with $\Omega\subset\mathbb{R}^2$) and be such that*

$$\forall t\in[0,T],\ F_\varepsilon(u_\varepsilon(t,.))\le C|\log\varepsilon|,\qquad \int_{[0,T]\times\Omega}|\partial_t u_\varepsilon|^2\le C|\log\varepsilon|.$$

Then after extraction $\operatorname{curl}(iu_\varepsilon(t),\nabla u_\varepsilon(t))\to\mu(t)$ *for every* $t\in[0,T]$*, in the dual of* $C_c^{0,\gamma}(\Omega)$ *for every* $\gamma>0$*. Moreover* $\mu(t)$ *is of the form* $2\pi\sum_{i=1}^{n(t)}d_i(t)\delta_{a_i(t)}$*, where* $d_i\in\mathbb{Z}$*. Finally* $t\to\langle\mu(t),\zeta\rangle$ *is in* $H^1((0,T))$ *for any* $\zeta\in C_c^1(\Omega)$*, where* $\langle\mu,\zeta\rangle:=\int\zeta\,d\mu$*.*

Proof: For the convergence of $\frac{1}{2}\operatorname{curl}(iu_\varepsilon,\nabla u_\varepsilon)$ see [SS6], Theorem 3. That $\mu(t)$ is of the form $2\pi\sum_{i=1}^{n(t)}d_i(t)\delta_{a_i(t)}$ with $d_i(t)\in\mathbb{Z}$ results from the upper bound on the energy (see [BBH], [JS2], [SS6]). From [SS6], Theorem 3, there exists a measure-valued vector field V such that $\partial_t\mu=-\operatorname{div}V$. Moreover the components of V are in $L^2([0,T],\mathcal{M})$ where $\mathcal{M}$ is the set of bounded Radon measures on Ω. Now let $\zeta\in C_c^1(\Omega)$ and $\varphi\in C_c^\infty((0,T))$. Since ζ is independent of time and $\partial_t\mu=-\operatorname{div}V$, we get

$$\int_0^T\langle\mu(t),\zeta\rangle\varphi'(t)\,dt=\int_0^T\langle\mu(t),\partial_t(\zeta\varphi)\rangle\,dt=\int_0^T\langle V,\nabla(\zeta\varphi)\rangle\,dt.$$

Then, since $\nabla(\zeta\varphi)=(\nabla\zeta)\varphi$,

$$\int_0^T\langle\mu(t),\zeta\rangle\varphi'(t)\,dt\le C\left(\int_0^T|\varphi(t)|^2\,dt\right)^{\frac{1}{2}},$$

where $C=\|\nabla\zeta\|_{L^\infty}\left(\int_0^T\|V\|^2dt\right)^{\frac{1}{2}}$. Hence $t\to\langle\mu(t),\zeta\rangle$ is in $H^1((0,T))$ as claimed. $\square$

Proposition III.2 *Assume that* $\mu(t)$ *is a measure of the form* $\sum_{i=1}^{n(t)}d_i(t)\delta_{a_i(t)}$ *for every* $t\in[0,T)$*, with* $d_i(t)\in\mathbb{Z}$ *and* $a_i(t)\in\Omega$*, and that* $t\to\langle\mu(t),\zeta\rangle$ *is in* $H^1((0,T))$ *for any* $\zeta\in C_c^1(\Omega)$*. Assume moreover that* $\sum_i|d_i(t)|\le\sum_i|d_i(0)|$ *for every* t*, that* $d_i(0)\in\{+1,-1\}$ *and that* $\{a_i(0)\}$ *are distinct points.*

Then there exists a time $0<T^*\le T$ *and* $n=n(0)$ *functions* $a_i(t)\in H^1((0,T^*),\mathbb{R}^2)$ *such that for all* $t\in[0,T^*)$ *the points* $\{a_i(t)\}_i$ *are distinct and* $\mu(t)=\sum_i d_i(0)\delta_{a_i(t)}$*. Moreover, if* $T^*<T$*, as* t *tends to* T^**, either there exists* i *such that* $a_i(t)$ *tends to* $\partial\Omega$ *or there exists* $i\neq j$ *such that* $a_i(t)$ *and* $a_j(t)$ *tend to the same limit.*

Proof: Let $B_i=B(a_i(0),r_i)$ be disjoint balls and φ_i be a smooth function compactly supported in B_i and equal to 1 in a neighborhood of $a_i(0)$. Then since $f_i(t)=\langle\mu(t),\varphi_i\rangle$ is in H^1 hence is continuous, then for t close to 0 the function $f(t)$ is close to $f(0)=d_i(0)$. Therefore if t is small enough, $\mu(t)$ has a Dirac mass in B_i. Since this is true for every i and the total degree is decreasing, there is exactly one Dirac mass in each ball. Thus, relabeling if necessary, $\mu(t)=\sum_i d_i(t)\delta_{a_i(t)}$ with $a_i(t)\in B_i$. Then $f(t)=d_i(t)$ and continuity implies that $d_i(t)$ is constant. Since $x_i(t)=\langle\mu(t),x_i\rangle$ is in H^1 — where x_i is a smooth function compactly supported in B_i and equal to the coordinate x in a neighborhood of $a_i(0)$ — and a similarly defined $y_i(t)$ also, then $a_i(t)\in H^1((0,T^*),\mathbb{R}^2)$.

It is clear that the process can be repeated by applying the above reasoning until two points collide or one exits the domain. □

III.2 Proof of Theorem 4

First, multiplying (I.10) by $\partial_t u_\varepsilon$ and integrating in $\Omega\times[t,s]$ we find

$$F_\varepsilon(u_\varepsilon(t))-F_\varepsilon(u_\varepsilon(s))=\frac{1}{|\log\varepsilon|}\int_{[t,s]\times\Omega}|\partial_t u_\varepsilon|^2. \tag{III.1}$$

It follows that $F_\varepsilon(u_\varepsilon(t))$ is decreasing. Then from (I.11), we deduce $F_\varepsilon(u_\varepsilon(t))\le C|\log\varepsilon|$ for all $t\ge 0$.

Lemma III.1 *Assume u_ε satisfies the hypothesis of Theorem 2. There exists $T_0>0$ such that u_ε satisfies the hypotheses of Proposition III.1 on $[0,T_0]$.*

Proof: Let us first rescale time and consider $v_\varepsilon(x,|\log\varepsilon|t)=u_\varepsilon(x,t)$. This way, v_ε is a solution of

$$\partial_t v=\Delta v+\frac{v}{\varepsilon^2}(1-|v|^2). \tag{III.2}$$

Assume by contradiction that there exists some subsequence of ε for which there exists $\lambda_\varepsilon\ll|\log\varepsilon|$ with

$$\int_0^{\lambda_\varepsilon}\int_\Omega|\partial_t v_\varepsilon|^2\,dt=1 \tag{III.3}$$

Multiplying (III.2) by $\partial_t v_\varepsilon$ and integrating in $\Omega\times[0,\lambda_\varepsilon]$, we find

$$1=\int_0^{\lambda_\varepsilon}\int_\Omega|\partial_t v_\varepsilon|^2\,dt=F_\varepsilon(v_\varepsilon(0))-F_\varepsilon(v_\varepsilon(\lambda_\varepsilon)). \tag{III.4}$$

On the other hand, rescaling again in time and considering $w_\varepsilon(x,t)=v_\varepsilon(x,\lambda_\varepsilon t)$, we have

$$\int_0^1\int_\Omega|\partial_t w_\varepsilon|^2\,dt=\lambda_\varepsilon\int_0^{\lambda_\varepsilon}\int_\Omega|\partial_t v_\varepsilon|^2\,dt=\lambda_\varepsilon\ll|\log\varepsilon|. \tag{III.5}$$

Applying Theorem 3 of [SS6] to w_ε, we deduce from (III.5) (and the bound $\int_\Omega|\nabla u_\varepsilon|^2\le F_\varepsilon(u_\varepsilon)\le C|\log\varepsilon|$) that for every function f, compactly supported in $\Omega\times[0,1]$, and every vector field X compactly supported on $\Omega\times[0,1]$, we have

$$\left|\int_{\Omega\times[0,1]}fV\cdot X\right|\le\liminf_{\varepsilon\to0}\frac{1}{|\log\varepsilon|}\left(\int_{\Omega\times[0,1]}|X\cdot\nabla w_\varepsilon|^2\int_{\Omega\times[0,1]}f^2|\partial_t w_\varepsilon|^2\right)^{\frac12}=0. \tag{III.6}$$

Here V is the limiting velocity associated to the limiting vorticity measure μ, in such a way that $\partial_t \mu + \operatorname{div}\ V = 0$, $\mu(t)$ being the limit of $\operatorname{curl}(iw_\varepsilon, \nabla w_\varepsilon)(t)$. From (III.6), we find $V = 0$, and thus $\mu(t) = \mu(0) = 2\pi \sum_{i=1}^n d_i \delta_{a_i^0}$. Returning to the previous time-scaling, we find that $\operatorname{curl}(iv_\varepsilon, \nabla v_\varepsilon)(t) \rightharpoonup \mu(0)$ for every $t \in [0, \lambda_\varepsilon]$, i.e the vortices have not moved. Thus, from the Γ-convergence (relation (I.9)), we deduce that

$$F_\varepsilon(v_\varepsilon(\lambda_\varepsilon)) \geq \pi n |\log\,\varepsilon| + nC_0 + W(a_i^0, \mathbf{d}) + o(1) = F_\varepsilon(v_\varepsilon(0)) + o(1).$$

Plugging this back into (III.4), we get $1 \leq o(1)$, a contradiction. We thus deduce the existence of a $T_0 > 0$ such that

$$\int_0^{T_0 |\log\,\varepsilon|} \int_\Omega |\partial_t v_\varepsilon|^2 \, dt \leq 1,$$

that is after rescaling $\int_0^{T_0} \int_\Omega |\partial_t u_\varepsilon|^2 \leq |\log\,\varepsilon|$ for all ε. □

We can thus apply Proposition III.1 in $[0, T_0]$. It yields that $\mu(t)$, the limit of $\operatorname{curl}(iu_\varepsilon(t), \nabla u_\varepsilon(t))$, is of the form $2\pi \sum_{i=1}^{n(t)} d_i(t) \delta_{a_i(t)}$. From (I.11), (I.9) and the energy-decrease, it follows that $\sum_i |d_i(t)| \leq \sum_i |d_i(0)|$ and therefore Proposition III.2 applies on $[0, T_0]$.

Thus there exists $T_1 > 0$ and trajectories $a_1(t), \ldots, a_n(t)$ in $H^1((0, T_1))$ such that for any $t < T_1$, the points $a_i(t)$ are distinct and $\mu(t) = 2\pi \sum_i d_i \delta_{a_i(t)}$. The degrees d_i are constant, equal to ± 1 and T_1 is the smallest of T^* (defined in Theorem 2) and T_0.

We let from now on

(III.7) $$\mathbf{d} = (d_1, \ldots, d_n),$$

and $\mathbf{d}$ is fixed. Letting $\mathcal{B} = H^1(\Omega, \mathbb{C})$ we define $\mathcal{M}$ either by $\mathcal{M} = \{u \in \mathcal{B} \mid u = g \text{ on } \partial\Omega\}$ or $\mathcal{M} = \{u \in \mathcal{B}\} \mid \partial_\nu u = 0 \text{ on } \partial\Omega\}$, according to whether we are interested in the Dirichlet or Neumann problem. The space $\mathcal{B}$ embeds into $X_\varepsilon = L^2(\Omega)$ that we equip with the norm

$$\|v\|_{X_\varepsilon}^2 = \frac{1}{|\log\,\varepsilon|} \int_\Omega |v|^2.$$

We let $E_\varepsilon = F_\varepsilon - \pi n |\log\,\varepsilon|$ (recall that n is the initial number of vortices). On the limiting side, we let $\mathcal{N} = \Omega_*^n$, by which we mean the set of n-uples of *distinct* points in Ω. It is a subset of $Y = (\mathbb{R}^2)^n$ that we equip with the norm

$$\|v\|_Y^2 = \pi \sum_{i=1}^n |v_i|^2.$$

The limiting functional will be, as suggested by (I.9), for any $u \in \Omega_*^n$,

(III.8) $$F(u) = W(u, \mathbf{d}) + nC_0.$$

By $u_\varepsilon \overset{S}{\rightharpoonup} u = (a_1, \ldots, a_n)$ we will mean that $\operatorname{curl}(iu_\varepsilon, \nabla u_\varepsilon) \to 2\pi \sum_i d_i \delta_{a_i}$ in the sense of distributions.

Noting that (I.10) and (I.13) are respectively the gradient flow of E_ε for the structure X_ε and the gradient flow of F for the structure Y, it is a simple verification to check that Theorem 1 applied in this framework on any interval $[0,T]$, with $T < T_1$, proves the result of Theorem 4 i.e. the limiting dynamics until T_1 (the minimum of the collision time and the exit time and T_0). In order to deduce Theorem 4, it remains to prove that the hypothesis of Theorem 1 are satisfied.

It is proved in [BBH], and we will see it below, that F defined by (III.8) is a smooth function in Ω_*^n. Together with (I.9), this implies that E_ε and F satisfy the hypothesis of Theorem 1. It remains to check hypothesis 1) and 2). Hypothesis 1) is provided by Corollary 7 of [SS6], which states that

$$\forall 0 \le t_1 < t_2 < T^*, \quad \liminf_{\varepsilon\to 0} \frac{1}{|\log \varepsilon|} \int_{\Omega\times[t_1,t_2]} |\partial_t u_\varepsilon|^2 \ge \pi \sum_i \int_{t_1}^{t_2} |\partial_t a_i|^2 \, dt. \tag{III.9}$$

It remains to prove that 2) is satisfied, which is done in paragraph III.4. Note that with Proposition I.1, one retrieves the results of [Li1, JS1] for the other time scalings.

III.3 Proof of Theorem 2

We prove here that Theorem 2 follows from Theorem 4. Let T be the supremum of the times until which the result holds, and assume that $T < T^*$ (minimum of collision and exit times). Then, for all $t \in [0,T)$, $\operatorname{curl}(iu_\varepsilon(t), \nabla u_\varepsilon(t)) \rightharpoonup 2\pi sum_i d_i \delta_{a_i(t)}$ where the trajectories $a_i(t)$ remain distinct and solve (I.13). Moreover, since $T < T^*$, $a_i(t) \to a_i(T)$ as $t \to T$ where the $a_i(T)$'s are distinct points in Ω. We claim that $\operatorname{curl}(iu_\varepsilon(T), \nabla u_\varepsilon(T)) \rightharpoonup 2\pi \sum_i d_i \delta_{a_i(T)}$ and that (I.12) holds at the time T. Then, Theorem 4 can be applied starting at time T and this contradicts the maximality of T.

Note that from (III.1) and (I.12), for every $t < T$ we have

$$\frac{1}{|\log \varepsilon|} \int_{[0,t]\times\Omega} |\partial_t u_\varepsilon|^2 = F_\varepsilon(u_\varepsilon(t)) - F_\varepsilon(u_\varepsilon(0)) \longrightarrow W(a_i(t), \mathbf{d}) - W(a_i^0, \mathbf{d}).$$

Passing to the limit $t \to T$ and using the fact that the points $a_i(T)$ are distinct, we find

$$\int_{[0,T]\times\Omega} |\partial_t u_\varepsilon|^2 \le C|\log \varepsilon|.$$

We can then apply Theorem 3 of [SS6], to say that for all $t \in [0,T]$, $\operatorname{curl}(iu_\varepsilon(t), \nabla u_\varepsilon(t)) \rightharpoonup 2\pi sum_i d_i \delta_{a_i(t)}$ with $a_i(t)$ continuous on $[0,T]$. Therefore, $\lim_{\varepsilon\to 0} \operatorname{curl}(iu_\varepsilon(T), \nabla u_\varepsilon(T))$ coincides with $2\pi \sum_i d_i \delta_{a_i(T)}$. It remains to prove that (I.12) holds at time T. From (III.1), we have $F_\varepsilon(u_\varepsilon(T)) \le F_\varepsilon(u_\varepsilon(t))$ for all $t < T$. Hence, using (I.12) at time t,

$$F_\varepsilon(u_\varepsilon(T)) \le F_\varepsilon(u_\varepsilon(t)) \le \pi n |\log \varepsilon| + nC_0 + W(a_i(t), \mathbf{d}) + o(1).$$

Therefore,

$$\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon(T)) - \pi n |\log \varepsilon| - nC_0 \le W(a_i(t), \mathbf{d}),$$

and passing to the limit $t \to T$ proves the result.

III.4 Construction

In this section, we prove that 2) is satisfied via a construction which consists, as mentioned in the introduction, in "pushing" the vortices along a given direction, while controlling $\|\partial_t v_\varepsilon\|_{X_\varepsilon}$ and the variation of $E_\varepsilon(v_\varepsilon(t))$. We prove

Proposition III.3 *Let u_ε satisfy $E_\varepsilon(u_\varepsilon) \le C$, $\|\nabla E_\varepsilon(u_\varepsilon)\|_{X_\varepsilon} \le C$ and $u_\varepsilon \overset{S}{\rightharpoonup} u \in \mathcal{N}$, with $u = (a_1, \dots a_n)$. Considering any $V \in (\mathbb{R}^2)^n$ and any $v(t)$ satisfying*

$$\text{(III.10)} \qquad \begin{cases} v(0) = u \\ \partial_t v(0) = V, \end{cases}$$

we can find a $v_\varepsilon \in \mathcal{M}$ such that

$$\text{(III.11)} \qquad v_\varepsilon(0) = u_\varepsilon(0)$$

$$\text{(III.12)} \qquad \|\partial_t v_\varepsilon(0)\|^2_{X_\varepsilon} = \frac{1}{|\log\,\varepsilon|}\int_\Omega |\partial_t v_\varepsilon|^2(0) = \|\partial_t v(0)\|^2_Y + o(1)$$

$$\text{(III.13)} \qquad \lim_{\varepsilon\to 0} \tfrac{d}{dt}_{|t=0} E_\varepsilon(v_\varepsilon(t)) = \tfrac{d}{dt}_{|t=0} F(v(t)) + g(u) D_\varepsilon,$$

where g is locally bounded on $\mathcal{N}$.

Remark III.1 *In a more abstract manner, what could be proved based on this proposition is that there exists a linear embedding $\mathcal{I}_\varepsilon : (\mathbb{R}^2)^n = T_u\mathcal{N} \to T_{u_\varepsilon}\mathcal{M}$ (with $\mathcal{I}_\varepsilon(V) = \partial_t v_\varepsilon(0)$ above) which is an "almost isometry" in the sense that $\lim_{\varepsilon\to 0}\|\mathcal{I}_\varepsilon(V)\|_{X_\varepsilon} = \|V\|_Y$ and which satisfies*

$$\lim_{\varepsilon\to 0} \mathcal{I}^*_\varepsilon \nabla E_\varepsilon(u_\varepsilon) = \nabla F(u)$$

in the sense that $\lim_{\varepsilon\to 0}\langle \nabla E_\varepsilon(u_\varepsilon), \mathcal{I}_\varepsilon(V)\rangle_{X_\varepsilon} = \langle \nabla F(u), V\rangle_Y$. Then, one easily deduces that $\liminf_{\varepsilon\to 0}\|\nabla E_\varepsilon(u_\varepsilon)\|_{X_\varepsilon} \ge \|\nabla F(u)\|_Y$ holds.

Recall that $\mathcal{M}$ is defined differently when dealing with the Dirichlet or Neumann problem. There are really two constructions but we do them in parallel.

The proof requires to go into the definition of W introduced in [BBH]. We will write $W(u)$ instead of $W(u, \mathbf{d})$ since the degrees are now fixed. First we define Φ by

$$\text{(III.14)} \qquad \begin{cases} \Delta\Phi = 2\pi \sum_i d_i \delta_{a_i} & \text{in } \Omega \\ \dfrac{\partial \Phi}{\partial n} = (ig, \dfrac{\partial g}{\partial \tau}) & \text{on } \partial\Omega \quad (\text{resp. } \Phi = 0 \text{ on } \partial\Omega \text{ for Neumann}). \end{cases}$$

We define as in [BBH], R by

$$\text{(III.15)} \qquad R(x) = \Phi(x) - \sum_j d_j \log|x - a_j|.$$

Then, the renormalized energy W associated to $u = (a_1, \cdots, a_n)$ is defined by

$$W(u) = -\pi \sum_{i \neq j} d_i d_j \log |a_i - a_j| + \frac{1}{2} \int_{\partial\Omega} \Phi(ig, \partial_\tau g) - \pi \sum_i R(a_i), \tag{III.16}$$

respectively $W(u) = -\pi \sum_{i\neq j} d_i d_j \log |a_i - a_j| - \pi \sum_i R(a_i)$ for Neumann.

We will need the following results, the proof of which is postponed until the end of the section.

Lemma III.2 *Let $u_\varepsilon \overset{S}{\rightharpoonup} u(a_1, \ldots, a_n)$ and $E_\varepsilon(u_\varepsilon) \leq F(u) + D_\varepsilon$, with D_ε bounded, and let Φ be as in (III.14). Then for every $\rho > 0$ such that the $B(a_i, \rho)$ are disjoint,*

$$\frac{1}{2} \int_{B(a_i,\rho)} |\nabla u_\varepsilon|^2 = \pi |\log \varepsilon| + O(1) \tag{III.17}$$

$$\frac{1}{2} \int_{\Omega \setminus \cup_i B(a_i,\rho)} |\nabla |u_\varepsilon||^2 + \frac{1}{2\varepsilon^2}(1 - |u_\varepsilon|^2)^2 \leq D_\varepsilon \tag{III.18}$$

$$\frac{1}{2} \int_{\Omega \setminus \cup_i B(a_i,\rho)} |\nabla u_\varepsilon - iu_\varepsilon \nabla^\perp \Phi|^2 \leq D_\varepsilon + o_\varepsilon(1), . \tag{III.19}$$

Proof of Proposition III.3: We have $u = (a_1, \ldots, a_n)$ and recall $d_i = \pm 1$. Let $\rho > 0$ be small enough so that the balls $B_i = B(a_i, \rho)$ are disjoint and included in Ω. We wish to "push" the vortices along the direction V, so we will simply translate the balls B_i along this direction, and then study how the energy varies.

For every $1 \leq i \leq n$, we can find smooth compactly supported vector fields in Ω, X_{i1} and X_{i2} such that

$$X_{i1}(x) = (1, 0) \text{ and } \; X_{i2} = (0, 1) \quad \text{in } B_i$$
$$X_{i1} = X_{i2}(x) = (0, 0) \quad \text{in } B_j, \; j \neq i.$$

Then, for any family of vectors $V = (V_1, \cdots, V_n)$, we can define X_V to be

$$X_V = \sum_{i=1}^n \sum_{j=1,2} V_{ij} X_{ij}.$$

Then, X_V depends linearly on V (and in a one-to-one fashion, see [S3]) and clearly $X_V(x) \equiv V_i$ in each B_i. We then define χ_t be the C^1 one-parameter family of diffeomorphisms of Ω, $\chi_t(x) = x + tX_V(x)$ defined in a small interval around 0. Thus,

$$\chi_t(x) = x + tV_i \quad \text{in each } B_i, \tag{III.20}$$

i.e. χ_t is a translation of vector V_i in B_i. Also χ_t and such that $\|\nabla \chi_t\|_\infty$, $\|\partial_t \chi_t\|_\infty$ seen as functions of $(a_1, \ldots, a_n)$ are locally bounded on $\mathcal{N}$.

Let Φ_t solve

$$\text{(III.21)} \qquad \begin{cases} \Delta \Phi_t = 2\pi \sum_i d_i \delta_{a_i(t)} & \text{in } \Omega \\ \dfrac{\partial \Phi_t}{\partial n} = (ig, \dfrac{\partial g}{\partial \tau}) & \text{on } \partial\Omega \quad (\text{resp. } \Phi_t = 0 \text{ on } \partial\Omega), \end{cases}$$

and

$$\text{(III.22)} \qquad R_t(x) = \Phi_t(x) - \sum_j d_j \log |x - a_j(t)|.$$

R_t is a smooth harmonic function in Ω, and we recall that the renormalized energy W associated to $u(t) = (a_1(t), \cdots, a_n(t))$ is defined by

$$\text{(III.23)} \qquad W(u(t)) = -\pi \sum_{i \neq j} d_i d_j \log |a_i(t) - a_j(t)| + \frac{1}{2} \int_{\partial\Omega} \Phi_t(ig, \partial_\tau g) - \pi \sum_{i=1}^n R_t(a_i(t)).$$

We can also consider $\tilde{R}_t$, the conjugate harmonic function of R_t. We then denote by θ_t^j the polar coordinate centered at $a_j(t)$, and define

$$\text{(III.24)} \qquad \psi_t = \sum_{j=1}^n d_j \theta_t^j \circ \chi_t - \sum_{j=1}^n d_j \theta_0^j + \tilde{R}_t \circ \chi_t - \tilde{R}_0.$$

One can check that ψ_t is a smooth function in Ω, the singularities at $a_i(0)$ in fact cancelling out, and that it is smooth in space-time. It follows from (III.22) that

$$\text{(III.25)} \qquad \nabla^\perp \Phi_t = \nabla^\perp R_t + \nabla^\perp \sum_j d_i \log |x - a_j(t)| = \nabla \tilde{R}_t + \sum_j d_j \nabla \theta_t^j,$$

and hence in view of (III.24),

$$\text{(III.26)} \qquad \nabla^\perp \Phi_0 + \nabla \psi_t = \nabla \left(\sum_j d_j \theta_t^j \circ \chi_t + \tilde{R}_t \circ \chi_t \right).$$

Since χ_t keeps $\partial\Omega$ fixed, we deduce that $\frac{\partial \psi_t}{\partial \tau} = \frac{\partial \Phi_t}{\partial n} - \frac{\partial \Phi_0}{\partial n} = 0$ on $\partial\Omega$. In the Dirichlet case, we can change the harmonic conjuguate by a constant so that $\psi_t = 0$ on $\partial\Omega$. Similarly $\frac{\partial \psi_t}{\partial n} = \frac{\partial \Phi_t}{\partial \tau} - \frac{\partial \Phi_0}{\partial \tau} = 0$ on $\partial\Omega$ for the Neumann case.

We then define $v_\varepsilon(x,t)$ as follows:

$$\text{(III.27)} \qquad v_\varepsilon(\chi_t(x), t) = u_\varepsilon(x) e^{i\psi_t(x)}.$$

Let us check that v_ε satisfies the desired properties.

First, $\psi_t = 0$ on $\partial\Omega$ (resp $\frac{\partial \psi_t}{\partial n} = 0$ on $\partial\Omega$) thus v_ε satisfies the right boundary conditions. In addition, v_ε is C^1 in time and clearly $v_\varepsilon(0) = u_\varepsilon$. Second, the leading order of the energy of u_ε is concentrated in the balls B_i (see (III.17)–(III.18)), and χ_t is a translation of vector V_i there, thus, applying Corollary 4 of [SS6], we have

$$\frac{1}{|\log \varepsilon|}\int_\Omega |\partial_t v_\varepsilon|(0)^2 = \frac{1}{|\log \varepsilon|}\sum_i \int_{B_i} |V_i\cdot \nabla u_\varepsilon|^2 + o(1) = \sum_i \pi |V_i|^2 + o(1). \tag{III.28}$$

We then evaluate $\frac{d}{dt}_{|t=0} E_\varepsilon(v_\varepsilon)$. In view of the definition of v_ε, with the change of variables $y=\chi_t(x)$, we have

$$\begin{aligned} E_\varepsilon(v_\varepsilon) &= \frac{1}{2}\int_\Omega \left(|(\nabla v_\varepsilon)\circ \chi_t(x)|^2 + \frac{1}{2\varepsilon^2}(1-|v_\varepsilon(\chi_t(x))|^2)^2\right)|Jac\ \chi_t|(x)\,dx\\ &= \frac{1}{2}\int_\Omega \left(|D\chi_t^{-1}\nabla(v_\varepsilon\circ\chi_t)|^2 + \frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\right)|Jac\ \chi_t|\\ &= \frac{1}{2}\int_\Omega \left(|D\chi_t^{-1}\nabla(u_\varepsilon e^{i\psi_t})|^2 + \frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\right)|Jac\ \chi_t|. \end{aligned}\tag{III.29}$$

First, observing that $\frac{d}{dt}_{|t=0}|Jac\ \chi_t| = 0$ in $\cup_i B_i$, and is bounded otherwise by $g(u)$ (locally bounded in $\mathcal{N}$), we have, in view of (III.18),

$$\frac{d}{dt}_{|t=0}\int_\Omega \frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2|Jac\ \chi_t| \le g(u)\frac{1}{2\varepsilon^2}\int_\Omega (1-|u_\varepsilon|^2)^2 \le g(u)D_\varepsilon(0). \tag{III.30}$$

In what follows $g(u)$ will always denote some function of u locally bounded in $\mathcal{N}$ (but possibly changing). We then have

(III.31)
$$\begin{aligned} \frac{d}{dt}_{|t=0}\frac{1}{2}\int_\Omega |D\chi_t^{-1}\nabla(u_\varepsilon e^{i\psi_t})|^2|Jac\ \chi_t| &= \frac{d}{dt}_{|t=0}\frac{1}{2}\int_\Omega |D\chi_t^{-1}(\nabla u_\varepsilon + iu_\varepsilon\nabla\psi_t)|^2|Jac\ \chi_t|\\ &= \int_\Omega (\frac{d}{dt}_{|t=0}D\chi_t^{-1}\nabla u_\varepsilon)\cdot\nabla u_\varepsilon + iu_\varepsilon\frac{d}{dt}_{|t=0}\nabla\psi_t\cdot\nabla u_\varepsilon + \frac{1}{2}|\nabla u_\varepsilon|^2\frac{d}{dt}_{|t=0}|Jac\ \chi_t| \end{aligned}$$

Observe that when $\limsup \|\nabla E_\varepsilon(u_\varepsilon)\|_{X_\varepsilon} < \infty$, which we can assume to be true,

$$\int_\Omega \left|\Delta u_\varepsilon + \frac{u_\varepsilon}{\varepsilon^2}(1-|u_\varepsilon|^2)\right|^2 \le \frac{C}{|\log \varepsilon|}$$

but

$$\left(iu_\varepsilon, \Delta u_\varepsilon + \frac{u_\varepsilon}{\varepsilon^2}(1-|u_\varepsilon|^2)\right) = (iu_\varepsilon,\Delta u_\varepsilon) = \text{div }(iu_\varepsilon,\nabla u_\varepsilon)$$

hence div $(iu_\varepsilon,\nabla u_\varepsilon)\to 0$ strongly in L^2. Since we also have

$$\text{curl }\ \left((iu_\varepsilon,\nabla u_\varepsilon) - \nabla^\perp\Phi_0\right) \rightharpoonup 0$$

by definition of $\Phi_0=\Phi$ (see (III.21)), we deduce

$$(iu_\varepsilon,\nabla u_\varepsilon) \rightharpoonup \nabla^\perp\Phi_0 + cst, \tag{III.32}$$

in the sense of distributions, and the constant vector is 0 thanks to the boundary condition. Now let $W_\varepsilon = \nabla u_\varepsilon - iu_\varepsilon \nabla^\perp \Phi_0$. In view of (III.19), we have $\int_{\Omega\setminus\cup_i B(a_i,\rho)} |W_\varepsilon|^2 \le 2D_\varepsilon + o(1)$. In view of (III.32) we also have $(W_\varepsilon, iu_\varepsilon) \rightharpoonup 0$.

Observing that ψ_t is smooth and C^1 in time, we deduce from (III.32) that

$$\text{(III.33)}\qquad \int_\Omega \frac{d}{dt}_{|t=0} \nabla\psi_t \cdot (iu_\varepsilon, \nabla u_\varepsilon) = \int_\Omega \frac{d}{dt}_{|t=0} \nabla\psi_t \cdot \nabla^\perp \Phi_0 + o(1).$$

Also, since $\frac{d}{dt}_{|t=0} D\chi_t^{-1} = \frac{d}{dt}_{|t=0} |Jac\ \chi_t| = 0$ in $\cup_i B_i$, using (III.18) and (III.19), we get

$$\text{(III.34)}\qquad \frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 \frac{d}{dt}_{|t=0} |Jac\ \chi_t| = \frac{1}{2}\int_\Omega \frac{d}{dt}_{|t=0} \frac{1}{2}|\nabla^\perp\Phi_0|^2 \frac{d}{dt}_{|t=0}|Jac\ \chi_t| + g(u)D_\varepsilon,$$

and

$$\begin{aligned}\text{(III.35)}\qquad \int_\Omega \frac{d}{dt}_{|t=0} D\chi_t^{-1}\nabla u_\varepsilon\cdot\nabla u_\varepsilon &= \int_\Omega \frac{d}{dt}_{|t=0} D\chi_t^{-1}(W_\varepsilon + iu_\varepsilon\nabla^\perp\Phi_0)\cdot(W_\varepsilon + iu_\varepsilon\nabla^\perp\Phi_0)\\
&= \int_\Omega \frac{d}{dt}_{|t=0} D\chi_t^{-1} W_\varepsilon\cdot W_\varepsilon + 2\frac{d}{dt}_{|t=0} D\chi_t^{-1}\nabla^\perp\Phi_0\cdot(iu_\varepsilon, W_\varepsilon)\\
&\quad + |u_\varepsilon|^2 \frac{d}{dt}_{|t=0} D\chi_t^{-1}\nabla^\perp\Phi_0\cdot\nabla^\perp\Phi_0\\
&= O(g(u)D_\varepsilon) + \int_\Omega \frac{d}{dt}_{|t=0} D\chi_t^{-1}\nabla^\perp\Phi_0\cdot\nabla^\perp\Phi_0\end{aligned}$$

where we have used the properties of W_ε. Inserting (III.33), (III.34) and (III.35) in (III.31), we have

$$\begin{aligned}\text{(III.36)}\qquad &\frac{d}{dt}_{|t=0}\frac{1}{2}\int_\Omega |D\chi_t^{-1}\nabla(u_\varepsilon e^{i\psi_t})|^2 |Jac\ \chi_t|\\
&= \int_\Omega \frac{d}{dt}_{|t=0} D\chi_t^{-1}\nabla^\perp\Phi_0\cdot\nabla^\perp\Phi_0 + \frac{d}{dt}_{|t=0}\nabla\psi_t\cdot\nabla^\perp\Phi_0 + \frac{1}{2}|\nabla^\perp\Phi_0|^2\frac{d}{dt}_{|t=0}|Jac\ \chi_t| + g(u)D_\varepsilon.\end{aligned}$$

Using again the fact that $\frac{d}{dt}_{|t=0} D\chi_t^{-1} = \frac{d}{dt}_{|t=0}|Jac\ \chi_t| = 0$ in $\cup_i B_i$, with $\psi_0 = 0$, we deduce that, for any $0 < r < \rho$,

$$\begin{aligned}\text{(III.37)}\qquad &\frac{d}{dt}_{|t=0}\int_\Omega |D\chi_t^{-1}\nabla(u_\varepsilon e^{i\psi_t})|^2 |Jac\ \chi_t|\\
&= \frac{d}{dt}_{|t=0}\int_{\Omega\setminus\cup_j B(a_j,r)} |D\chi_t^{-1}(\nabla^\perp\Phi_0 + \nabla\psi_t)|^2 |Jac\ \chi_t| + g(u)D_\varepsilon + o_r(1)\\
&= \lim_{r\to 0}\frac{d}{dt}_{|t=0}\int_{\Omega\setminus\cup_j B(a_j,r)} |D\chi_t^{-1}(\nabla^\perp\Phi_0 + \nabla\psi_t)|^2 |Jac\ \chi_t| + g(u)D_\varepsilon.\end{aligned}$$

Inserting (III.26) into (III.37) and doing a change of variables, we are led to

$$\text{(III.38)} \quad \frac{d}{dt}_{|t=0} \int_\Omega |D\chi_t^{-1} \nabla(u_\varepsilon e^{i\psi_t})|^2 |Jac\ \chi_t|$$
$$= \lim_{r\to 0} \frac{d}{dt}_{|t=0} \int_{\Omega\setminus\cup_j B(a_j(t),r)} |\nabla(\sum_j d_j\theta_t^j + \tilde{R}_t)|^2 + g(u)D_\varepsilon = \lim_{r\to 0} \frac{d}{dt}_{|t=0} \int_{\Omega\setminus\cup_j B(a_j(t),r)} |\nabla^\perp \Phi_t|^2 + g(u)D_\varepsilon,$$

where we have used (III.25). We then introduce $S_t^j(x) = \Phi_t(x) - d_j \log|x - a_j(t)|$, smooth harmonic function in a neighborhood of a_j, also C^1 in time. As in [BBH], p. 22, we have $S_t^j(a_j(t)) = R_t(a_j(t)) + \sum_{k\neq j} d_k \log|a_j(t) - a_k(t)|$ and

(III.39)

$$\frac{d}{dt}_{|t=0} \int_{\Omega\setminus\cup_j B(a_j(t),r)} |\nabla\Phi_t|^2 = -\frac{d}{dt}_{|t=0} \left(\sum_j \int_{B(a_j(t),r)} |\nabla S_t^j|^2 + 2\pi d_j S_t^j(a_j(t)) + 2\pi d_j^2 \log r \right)$$
$$= \frac{d}{dt}_{|t=0} \left(-\sum_j \int_{B(a_j(t),r)} |\nabla S_t^j|^2 - 2\pi \sum_j d_j R_t(a_j(t)) - 2\pi \sum_{j\neq k} d_j d_k \log|a_j(t) - a_k(t)| \right)$$
$$= -\frac{d}{dt}_{|t=0} \sum_j \int_{B(a_j(t),r)} |\nabla S_t^j|^2 + 2\frac{d}{dt}_{|t=0} W(a_1(t), \cdots, a_n(t)).$$

But, $\lim_{r\to 0} \frac{d}{dt}_{|t=0} \sum_j \int_{B(a_j(t),r)} |\nabla S_t^j|^2 = 0$, because S^j is a smooth function in a neighborhood of a_j, C^1 in time, thus taking the limit $r \to 0$ in (III.39) and combining it with (III.38), we find

$$\text{(III.40)} \qquad \frac{d}{dt}_{|t=0} \int_\Omega |D\chi_t^{-1} \nabla(u_\varepsilon e^{i\psi_t})|^2 |Jac\ \chi_t| = 2\frac{d}{dt}_{|t=0} W(a_1(t), \cdots, a_n(t)) + g(u)D_\varepsilon.$$

Combining this with (III.29) and (III.30), we conclude that

$$\lim_{\varepsilon\to 0} \frac{d}{dt}_{|t=0} E_\varepsilon(v_\varepsilon(x,t)) = \frac{d}{dt}_{|t=0} W(a_1(t), \cdots, a_n(t)) + g(u)D_\varepsilon,$$

hence the desired result. □

Remark III.2 *In [S4] it will be proved that $\|\nabla E_\varepsilon(u_\varepsilon)\|_{X_\varepsilon} \leq C$ implies $D_\varepsilon = o(1)$. Thus Proposition III.3 combined with Lemma II.1 yield the estimate*

$$\liminf_{\varepsilon\to 0} \left(|\log\ \varepsilon| \int_\Omega |\Delta u_\varepsilon + \frac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2)|^2 \right) \geq \pi \sum_{i=1}^n |\partial_i W(a_1, \cdots, a_n)|^2 = \pi |\nabla W(u)|^2$$

(already proved in [Li1]). It also implies the result of [BBH] that critical points of E_ε converge to critical points of the renormalized energy.

Proof of Lemma III.2 : By assumption, we have the a priori upper bound

$$\frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1-|u_\varepsilon|^2)^2 \le \pi n |\log\varepsilon| + nC_0 + W(u) + D_\varepsilon. \tag{III.41}$$

By the usual lower bounds of [BBH], it is easy to check that this implies

$$\frac{1}{2}\int_{\Omega\setminus\cup_i B(a_i,\rho)} |\nabla u_\varepsilon|^2 - |\nabla|u_\varepsilon||^2 \ge \pi n \log\frac{1}{\rho} + W(u) + o_\rho(1), \tag{III.42}$$

while

$$\frac{1}{2}\int_{\cup_i B(a_i,\rho)} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1-|u_\varepsilon|^2)^2 \ge \pi n |\log\rho/\varepsilon| + nC_0 + o_\rho(1), \tag{III.43}$$

(see Chapter 8 of [BBH]). Summing the two equations and comparing to (III.41), we must have

$$\frac{1}{2}\int_{\Omega\setminus\cup_i B(a_i,\rho)} |\nabla|u_\varepsilon||^2 + \frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 \le D_\varepsilon + o_\rho(1), \tag{III.44}$$

and also, keeping ρ fixed and using the fact that $D_\varepsilon = O(1)$,

$$\frac{1}{2}\int_{\cup_i B(a_i,\rho)} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1-|u_\varepsilon|^2)^2 = \pi n |\log\varepsilon| + O_\varepsilon(1), \tag{III.45}$$

which yields (III.17). Going back to (III.44), we find that, for ρ_0 fixed, letting $\rho\to 0$,

$$\frac{1}{2}\int_{\Omega\setminus\cup_i B(a_i,\rho_0)} |\nabla|u_\varepsilon||^2 + \frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 \le \lim_{\rho\to 0}\frac{1}{2}\int_{\Omega\setminus\cup_i B(a_i,\rho)} |\nabla|u_\varepsilon||^2 + \frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 \le D_\varepsilon, \tag{III.46}$$

which is (III.18).
Let us turn to (III.19). Let Ψ_ρ be defined as in [BBH] by

$$\begin{cases} \Delta\Psi_\rho = 0 & \text{in } \Omega\setminus\cup_i B(a_i,\rho) \\ \Psi_\rho = const = c_i & \text{on } \partial B(a_i,\rho) \\ \displaystyle\int_{\partial B(a_i,\rho)} \frac{\partial\Psi_\rho}{\partial n} = 2\pi d_i & \text{for all } i \\ \displaystyle\frac{\partial\Psi_\rho}{\partial n} = (ig,\partial_\tau g) & (\text{resp. } \Psi_\rho = 0 \text{ for Neumann}) \text{ on } \partial\Omega \\ \displaystyle\int_{\partial\Omega}\Psi_\rho = 0. \end{cases} \tag{III.47}$$

We will use two results from [BBH]: first that $\Psi_\rho - \Phi \to 0$ in $C^k_{loc}(\Omega\setminus\cup_i\{a_i\})$, second that as $\rho\to 0$,

$$\frac{1}{2}\int_{\Omega\setminus\cup_i B(a_i,\rho)} |\nabla\Psi_\rho|^2 = \pi n\log\frac{1}{\rho} + W(u) + O(\rho). \tag{III.48}$$

We deduce that

$$\int_{\Omega\setminus\cup_i B(a_i,\rho)} |\nabla u_\varepsilon|^2 \leq \int_{\Omega\setminus\cup_i B(a_i,\rho)} |\nabla \Psi_\rho|^2 + o_\rho(1) + 2D_\varepsilon. \tag{III.49}$$

Then,

$$\int_{\Omega\setminus\cup_i B(a_i,\rho)} |\nabla u_\varepsilon - iu_\varepsilon \nabla^\perp \Psi_\rho|^2 = \int_{\Omega\setminus\cup_i B(a_i,\rho)} |\nabla u_\varepsilon|^2 + (|u_\varepsilon|^2-2)|\nabla\Psi_\rho|^2 - 2\left((iu_\varepsilon,\nabla u_\varepsilon) - \nabla^\perp\Psi_\rho\right)\cdot\nabla^\perp\Psi_\rho. \tag{III.50}$$

But, $\int_{\Omega\setminus\cup_i B(a_i,\rho)} |\nabla u_\varepsilon|^2$ is bounded by a constant independent of ε hence there exists an S^1-valued map u with the same degrees d_i on $\partial B(a_i,\rho)$ such that $u_\varepsilon \rightharpoonup u$ weakly in $H^1(\Omega\setminus \cup_i B(a_i,\rho))$, and $\operatorname{curl}(iu_\varepsilon,\nabla u_\varepsilon) \rightharpoonup \operatorname{curl}(iu,\nabla u) = 0$ in $\mathcal{D}'(\Omega\setminus\cup_i B(a_i,\rho))$. We deduce with the definition of Ψ_ρ (cf (III.47)) that

$$\begin{aligned}\int_{\Omega\setminus\cup_i B(a_i,\rho)} &\left((iu_\varepsilon,\nabla u_\varepsilon) - \nabla^\perp\Psi_\rho\right)\cdot\nabla^\perp\Psi_\rho \\ = -\int_{\Omega\setminus\cup_i B(a_i,\rho)} \operatorname{curl}(iu_\varepsilon,\nabla u_\varepsilon)\Psi_\rho &- \sum_i \int_{\partial B(a_i,\rho)} \left(\left(iu_\varepsilon, \frac{\partial u_\varepsilon}{\partial\tau}\right) - \frac{\partial \Psi_\rho}{\partial n}\right)\Psi_\rho \to 0 \quad \text{as } \varepsilon\to 0.\end{aligned} \tag{III.51}$$

Inserting this in (III.50), with (III.49), and using the fact that $|u_\varepsilon|\to 1$ strongly in L^2, we are led to

$$\int_{\Omega\setminus\cup_i B(a_i,\rho)} |\nabla u_\varepsilon - iu_\varepsilon\nabla^\perp\Psi_\rho|^2 \leq 2D_\varepsilon + o_\rho(1) + o_\varepsilon(1). \tag{III.52}$$

Considering now ρ_0 to be fixed, we deduce that for $\rho\to 0$,

$$\int_{\Omega\setminus\cup_i B(a_i,\rho_0)} |\nabla u_\varepsilon - iu_\varepsilon\nabla^\perp\Psi_\rho|^2 \leq 2D_\varepsilon + o_\rho(1) + o_\varepsilon(1). \tag{III.53}$$

But we also know that $\nabla^\perp\Psi_\rho \to \nabla^\perp\Phi$ uniformly in every compact subset of $\Omega\setminus\cup_i\{a_i\}$, thus, passing to the limit $\rho\to 0$ in (III.52) yields

$$\int_{\Omega\setminus\cup_i B(a_i,\rho_0)} |\nabla u_\varepsilon - iu_\varepsilon\nabla^\perp\Phi|^2 \leq 2D_\varepsilon + o_\varepsilon(1), \tag{III.54}$$

which is the desired result (III.19). The same would work for the Neumann boundary condition. $\square$

IV Ginzburg-Landau with magnetic field

In this section we apply Theorem 1 to the full Ginzburg-Landau functional (I.7) to prove Theorem 3. Just like in the case without magnetic field, it suffices to prove that the conclusions hold on some small time-interval $[0,T]$.

We assume (I.15) is satisfied and define ξ_0 and J_0 as in (I.18), (I.19).

IV.1 Preliminary results

Most of the statements on Ginzburg-Landau without magnetic field can be translated into a gauge-invariant version. For instance the equivalent of the Jacobian will be

$$\mu_\varepsilon = \operatorname{curl}\left((iu_\varepsilon, \nabla_{A_\varepsilon} u_\varepsilon) + A_\varepsilon\right) = \operatorname{curl}\,(iu_\varepsilon, \nabla_{A_\varepsilon} u_\varepsilon) + h_\varepsilon. \tag{IV.1}$$

we also let

$$J_f(u, A) = \frac{1}{2}\int_\Omega |\nabla u - iAu|^2 + |\operatorname{curl} A|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2, \tag{IV.2}$$

be the Ginzburg-Landau functional with no applied field. We will need the following result [Sa, J1, SS4, JS2, ASS, SS6].

Lemma IV.1 *Assume $J(u_\varepsilon, A_\varepsilon) \le C|\log\,\varepsilon|^2$. Then there exists a family (depending on ε) of disjoint balls $B(a_i, r_i)$ such that for any $q > 0$, $\sum_i r_i \le C_q |\log\,\varepsilon|^{-q}$, $|u_\varepsilon| \ge 1 - |\log\,\varepsilon|^{-q}$ in $\Omega \setminus \cup_i B(a_i, r_i)$, and*

$$\frac{1}{2}\int_{B(a_i, r_i)} |\nabla_{A_\varepsilon} u_\varepsilon|^2 \ge \pi |d_i| |\log\,\varepsilon| (1 - o(1)) \tag{IV.3}$$

where $d_i = deg(u_\varepsilon, \partial B(a_i, r_i))$ if $B(a_i, r_i) \subset \Omega$ (0 otherwise). Moreover,

$$\mu_\varepsilon - 2\pi \sum_i d_i \delta_{a_i} \to 0, \tag{IV.4}$$

in the dual of $C_c^{0,\gamma}(\Omega)$, for any $\gamma > 0$.

We also recall the following result from [ASS, SS4, SS5, SS6].

Lemma IV.2 *Assume $J(u_\varepsilon, A_\varepsilon) \le CN_\varepsilon |\log\,\varepsilon|$, where $N_\varepsilon \le C|\log\,\varepsilon|^q$, with q arbitrary. Then there exists a bounded Radon measure $\mu \in \mathcal{M}(\Omega)$ such that, after extraction $\frac{1}{N_\varepsilon}\mu_\varepsilon \rightharpoonup \mu$ weakly in the dual of $C_c^{0,\gamma}$, for any $\gamma > 0$ and*

$$J(u_\varepsilon, A_\varepsilon) - h_{\rm ex}^2 J_0 \ge \lambda |\log\,\varepsilon| \int_\Omega \xi_0 \mu_\varepsilon + J_f(u_\varepsilon, {A_\varepsilon}') + o(1), \tag{IV.5}$$

$$\liminf_{\varepsilon \to 0} \frac{J(u_\varepsilon, A_\varepsilon) - h_{\rm ex}^2 J_0}{N_\varepsilon |\log\,\varepsilon|} \ge \frac{1}{2}\int_\Omega |\mu| + \lambda \int_\Omega \xi_0 \, d\mu, \tag{IV.6}$$

where λ is defined in (I.15), and ${A_\varepsilon}' = A_\varepsilon - h_{\rm ex}\nabla^\perp \xi_0$.

Proof : For the convergence of μ_ε, see [SS1, JS2, ASS, SS6]. We reproduce the proof of the lower bound (IV.5) (see [SS1, SS5]) for the convenience of the reader.

We let $h' = \operatorname{curl} A'$ and $j' = (iu, \nabla_{A'} u)$ (we dropped the subscript ε). Then $\nabla_A u = \nabla_{A'} u - iuh_{\rm ex}\nabla^\perp \xi_0$, and from (I.18), we find $h := \operatorname{curl} A = h' + h_{\rm ex}(\xi_0 + 1)$. Replacing in (I.7), we are led to

$$J(u,A) = J_f(u,A') + \frac{h_{\rm ex}^2}{2}\int_\Omega |u|^2|\nabla\xi_0|^2 + |\xi_0|^2 + h_{\rm ex}\int_\Omega -\nabla^\perp\xi_0 \cdot j' + \xi_0 h'. \tag{IV.7}$$

From the energy upper bound and Cauchy-Schwarz inequality, we have $\int_\Omega ||u|^2 - 1| \le C\varepsilon|\log\varepsilon|^{q/2}$ therefore $|u|^2|\nabla\xi_0|^2$ tends to $|\nabla\xi_0|^2$ in L^1. Integrating by parts the last term in (IV.7) and noting that from (I.18) it follows that $\operatorname{curl} j' + h' = \operatorname{curl}(iu, \nabla_A u) + h = \mu_\varepsilon$, we get

$$J(u,A) = h_{\rm ex}^2 J_0 + h_{\rm ex}\int_\Omega \xi_0\mu_\varepsilon + J_f(u,A') + o(1), \tag{IV.8}$$

which (IV.5). Moreover, the ball construction of Lemma IV.1 applied to (u, A') implies

$$\frac{1}{2}\int_{\cup_i B(a_i,r_i)} |\nabla_{A'} u|^2 \ge \pi\sum_i |d_i||\log\varepsilon|(1 - o(1)), \tag{IV.9}$$

where $(\operatorname{curl} j' + h') - 2\pi\sum_i d_i\delta_{a_i}$ converges to 0 hence $\frac{1}{N_\varepsilon}2\pi\sum_i d_i\delta_{a_i} \rightharpoonup \mu$. In view of (IV.8), we deduce

$$J(u,A) \ge h_{\rm ex}^2 J_0 + h_{\rm ex}\int_\Omega \xi_0\mu_\varepsilon + \frac{|\log\varepsilon|}{2}\sum_i |d_i|(1 - o(1)) + o(1), \tag{IV.10}$$

and (IV.6) follows by dividing by $N_\varepsilon|\log\varepsilon|$ and passing to the limit $\varepsilon \to 0$. □

IV.2 Boundedness of the vorticity

As in the case without magnetic field we need to show that, starting from a configuration (u, A) with n vortices of degree $+1$ or -1, and which is well-prepared, the number of vortices remains constant for some time and sufficiently regular trajectories may be defined. We recall the following notations

$$\mathcal{E} = -\partial_t A - \nabla\Phi, \quad \mathcal{F} = \partial_t u + iu\Phi, \tag{IV.11}$$

We will need the following variant of Proposition III.1.

Proposition IV.1 *Let $(u_\varepsilon, A_\varepsilon, \Phi_\varepsilon)$ be defined on the time interval $[0,T]$ and be such that*

$$\forall t \in [0,T],\ J_f(u_\varepsilon, A_\varepsilon) \le CN_\varepsilon|\log\varepsilon| \qquad \int_{\Omega\times[0,T]} |\mathcal{E}_\varepsilon|^2 + |\mathcal{F}_\varepsilon|^2 \le CN_\varepsilon|\log\varepsilon|, \tag{IV.12}$$

with $N_\varepsilon \le C|\log\varepsilon|$. Then $\frac{\mu_\varepsilon}{N_\varepsilon}$ converges to a measure μ for every $t \in [0,T]$ in the dual of $C_c^{0,\gamma}(\Omega)$ for every $\gamma > 0$. Moreover $t \to \langle\mu(t), \zeta\rangle$ is in $H^1([0,T])$ for any $\zeta \in C_c^1(\Omega)$ and

$$|\langle\mu(t_2), \zeta\rangle - \langle\mu(t_1), \zeta\rangle| \le C\sqrt{t_2 - t_1}\liminf_{\varepsilon\to0}\frac{1}{\sqrt{N_\varepsilon|\log\varepsilon|}}\left(\int_{\Omega\times[t_1,t_2]} |\mathcal{F}_\varepsilon|^2\right)^{\frac{1}{2}},$$

for any $[t_1, t_2] \subset [0,T]$.

Proof: The proposition is a consequence of Theorem 3 of [SS6], when working in a suitable gauge. Since the above results are invariant under gauge transformations (I.17) we may work in the Lorentz gauge $\Phi_\varepsilon = -\mathrm{div}\ A_\varepsilon$ in $\Omega\times[0,T]$, with $A_\varepsilon\cdot\nu = 0$ on $\partial\Omega\times[0,T]$ and $\mathrm{div}\ A_\varepsilon = 0$ at time $t=0$. Then, in view of (IV.11),

$$\partial_t A_\varepsilon - \Delta A_\varepsilon = -\mathcal{E}_\varepsilon.$$

The fact that A_ε satisfies $A_\varepsilon\cdot\nu = 0$ on $\partial\Omega$ and $\mathrm{div}\ A_\varepsilon = 0$ initially implies that $\|A_\varepsilon\|_{H^1(\Omega)} \le CN_\varepsilon|\log\,\varepsilon|$ initially and then standard parabolic estimates give for all time

$$\|A_\varepsilon\|^2_{H^1(\Omega)} \le CN_\varepsilon|\log\,\varepsilon|, \quad \|\Phi_\varepsilon\|^2_{H^1(\Omega)} \le CN_\varepsilon|\log\,\varepsilon| \tag{IV.13}$$

Now we apply Theorem 3 of [SS6] to u_ε. From (IV.12)–(IV.13), its hypothesis are satisfied. Moreover the defect measures of L^2 convergence of

$$\frac{|X\cdot\nabla u_\varepsilon|}{\sqrt{N_\varepsilon|\log\,\varepsilon|}}, \quad \frac{f|\partial_t u_\varepsilon|}{\sqrt{N_\varepsilon|\log\,\varepsilon|}},$$

coincide with those of their gauge equivalents, where ∇u_ε is replaced with $\nabla_{A_\varepsilon}u_\varepsilon$ and $\partial_t u_\varepsilon$ with $\partial_t u_\varepsilon + iu\Phi_\varepsilon$. Also, $(iu_\varepsilon,\nabla_{A_\varepsilon}u_\varepsilon)+A_\varepsilon = (iu_\varepsilon,\nabla u_\varepsilon)+(1-|u_\varepsilon|^2)A_\varepsilon$ and $(1-|u_\varepsilon|^2)A_\varepsilon$ tends to 0 in the sense of distributions from (IV.12), (IV.13). Therefore $\mu_\varepsilon - \mathrm{curl}\,(iu_\varepsilon,\nabla u_\varepsilon)$ tends to zero as a distribution. We then find that $\frac{\mu_\varepsilon}{N_\varepsilon}$ converges to a measure μ for every $t\in[0,T]$ in the dual of $C_c^{0,\gamma}(\Omega)$ for every $\gamma>0$ and that there exists a vector V with components in $L^2([0,T],\mathcal{M}(\Omega))$ such that $\partial_t\mu + \mathrm{div}\ V = 0$, where $\mathcal{M}$ denotes the set of bounded Radon measures in Ω. Moreover, for any $[t_1,t_2]\subset[0,T]$, any $X\in C_c^0([t_1,t_2]\times\Omega,\mathbb{R}^2)$, we have (see [SS6], Theorem 3)

$$\liminf_{\varepsilon\to 0}\frac{1}{N_\varepsilon|\log\,\varepsilon|}\left(\int_{\Omega\times[t_1,t_2]}|X\cdot\nabla_{A_\varepsilon}u_\varepsilon|^2\int_{\Omega\times[t_1,t_2]}|\partial_t u_\varepsilon + iu_\varepsilon\Phi_\varepsilon|^2\right)^{\frac12} \ge \frac12\left|\int_{\Omega\times[t_1,t_2]}V\cdot X\right|. \tag{IV.14}$$

The proposition follows by taking $X=\nabla\zeta$, using (IV.12) and reasoning as in Proposition III.1. □

We will need

Lemma IV.3 *Assume $(u_\varepsilon,A_\varepsilon,\Phi_\varepsilon)$ solve* (I.16) *in $\Omega\times\mathbb{R}_+$. Then*

$$J(u_\varepsilon,A_\varepsilon)(0) - J(u_\varepsilon,A_\varepsilon)(T) = \int_{[0,T]\times\Omega}|\mathcal{F}_\varepsilon|^2 + |\mathcal{E}_\varepsilon|^2. \tag{IV.15}$$

The proof is a direct computation. We are now in a position to prove

Proposition IV.2 *Assume $(u_\varepsilon,A_\varepsilon,\Phi_\varepsilon)$ solve* (I.16) *in $\Omega\times[0,T]$. Assume moreover that $\mu_\varepsilon(0)$ converges to a finite measure $\mu(0) = 2\pi\sum_{i=1}^{n(0)}d_i(0)\delta_{a_i(0)}$ with $d_i(0)\in\mathbb{Z}$ and that $(u_\varepsilon,A_\varepsilon)$ is well prepared initially, i.e.*

$$J(u_\varepsilon(0),A_\varepsilon(0)) \le h_{\mathrm{ex}}^2 J_0 + \frac{|\log\,\varepsilon|}{2}\int_\Omega|\mu(0)| + \lambda|\log\,\varepsilon|\int_\Omega\xi_0 d\mu(0) + o(|\log\,\varepsilon|).$$

Then $\mu_\varepsilon(t)$ converges to a finite measure $\mu(t)$ of the form $2\pi\sum_{i=1}^{n(t)} d_i(t)\delta_{a_i(t)}$ with $d_i(t)\in\mathbb{Z}$ for every $t\geq 0$, and there exists $T_0>0$ independent of the particular solution taken such that if $t\leq T_0$ then $\sum_{i=1}^{n(t)}|d_i(t)|\leq\sum_{i=1}^{n(0)}|d_i(0)|$.

Proof: From Lemma IV.2,

$$J(u_\varepsilon(t),A_\varepsilon(t))-h_{\mathrm{ex}}^2J_0\geq\lambda|\log\varepsilon|\int_\Omega\xi_0\mu_\varepsilon+J_f(u_\varepsilon(t),A_\varepsilon(t)')+o(1),$$

where $A'=A-h_{\mathrm{ex}}\nabla^\perp\xi_0$. Together with the well-preparedness of $(u_\varepsilon(0),A_\varepsilon(0))$ this yields

(IV.16)
$$J(0)-J(t)\leq\lambda|\log\varepsilon|\int_\Omega(\mu_\varepsilon(0)-\mu_\varepsilon(t))\xi_0+\frac{|\log\varepsilon|}{2}\int_\Omega|\mu(0)|-J_f(u_\varepsilon(t),{A_\varepsilon}'(t))+o(|\log\varepsilon|),$$

where $J(t)$ stands for $J(u_\varepsilon(t),A_\varepsilon(t))$. We let

$$N_\varepsilon=\sup_{t\in[0,T]}\frac{J_f(u_\varepsilon(t),{A_\varepsilon}'(t))}{|\log\varepsilon|}.$$

Then we may apply Theorem 3 of [SS6] to $(u_\varepsilon,{A_\varepsilon}')$ to find

(IV.17)
$$\begin{aligned}\left(\int_\Omega(\mu_\varepsilon(0)-\mu_\varepsilon(t))\xi_0\right)^2&\leq\frac{1}{|\log\varepsilon|^2}\int_{\Omega\times[0,t]}|\nabla_{A_\varepsilon'}u_\varepsilon|^2|\nabla\xi_0|^2\int_{\Omega\times[0,t]}|\partial_tu_\varepsilon+iu_\varepsilon\Phi_\varepsilon|^2(1+o(1))\\&\leq\frac{C}{|\log\varepsilon|^2}\int_0^tJ_f(u_\varepsilon,A_\varepsilon')\,dt\int_{\Omega\times[0,t]}|\mathcal{F}_\varepsilon|^2\end{aligned}$$

and thus

$$\left(\int_\Omega(\mu_\varepsilon(0)-\mu_\varepsilon(t))\xi_0\right)^2\leq C\frac{tN_\varepsilon}{|\log\varepsilon|}\int_{\Omega\times[0,t]}|\mathcal{F}_\varepsilon|^2\leq C\frac{tN_\varepsilon}{|\log\varepsilon|}(J(0)-J(t)).\tag{IV.18}$$

Thus, in view of (IV.16) and letting $\Delta(t)=\int_\Omega(\mu_\varepsilon(0)-\mu_\varepsilon(t))\xi_0$, we obtain

$$\frac{\Delta(t)^2}{CtN_\varepsilon}\leq\lambda\Delta(t)+\frac12\int_\Omega|\mu(0)|-\frac{1}{|\log\varepsilon|}J_f(u_\varepsilon(t),{A_\varepsilon}'(t))+o(1).\tag{IV.19}$$

It follows that

$$\Delta(t)^2\leq C(1+tN_\varepsilon).\tag{IV.20}$$

Replacing in the previous inequality we find $|\log\varepsilon|^{-1}J_f(u_\varepsilon(t),{A_\varepsilon}'(t))\leq C(1+tN_\varepsilon)$ and, taking the supremum over $t\in[0,T]$, $N_\varepsilon\leq C(1+TN_\varepsilon)$. Thus, for T smaller than some T_0, N_ε is bounded independently of ε. Inserting (IV.18) into (IV.16), we also obtain $J(0)-J(t)\leq C|\log\varepsilon|^{\frac12}(J(0)-J(t))^{\frac12}+C|\log\varepsilon|$, thus $\int_{[0,T]\times\Omega}|\mathcal{E}_\varepsilon|^2+|\mathcal{F}_\varepsilon|^2=J(0)-J(t)\leq C|\log\varepsilon|$ and the hypotheses of Proposition IV.1 are satisfied with $N_\varepsilon=1$.

The convergence of $\mu_\varepsilon(t)$ to a measure of the form $2\pi\sum_{i=1}^{n(t)} d_i(t)\delta_{a_i(t)}$ with $d_i(t)\in\mathbb{Z}$ for every $t\le T_0$ follows from the upper bound on $J_f(u_\varepsilon(t), {A_\varepsilon}'(t))$ together with (IV.3)-(IV.4) applied to $(u_\varepsilon(t), {A_\varepsilon}'(t))$. Returning to (IV.20) we find that $\Delta(t)$ is bounded in $[0,T_0]$, and with (IV.19) that $\lim_{t\to 0}\Delta(t)=0$. Since Lemma IV.2 and (IV.3)–(IV.4) imply

$$0\le J(0)-J(t)\le \Delta(t)+\pi|\log\varepsilon|\left(\sum_{i=1}^{n(0)}|d_i(0)|-\sum_{i=1}^{n(t)}|d_i(t)|\right)+o(|\log\varepsilon|),$$

we deduce, taking T_0 smaller if necessary, that $\sum_{i=1}^{n(t)}|d_i(t)|\le\sum_{i=1}^{n(0)}|d_i(0)|$ for $t\le T_0$. $\square$

IV.3 Proof of Theorem 3

As mentioned earlier, we need only prove the local version. The framework of Section II applies as follows. Assume $(u_\varepsilon, A_\varepsilon, \Phi_\varepsilon)$ satisfy the hypothesis of Theorem 3. Then Proposition IV.2 applies together with Proposition III.2. Thus there exists $T_0>0$ and trajectories $a_1(t),\dots,a_n(t)$ in $H^1((0,T_0))$ such that for any $t<T_0$ the points $a_i(t)$ are distinct, continuous, and such that $\mu_\varepsilon(t)\to\mu(t)=2\pi\sum_i d_i\delta_{a_i(t)}$ in the sense of distributions. The degrees d_i are constant and equal to ±1.

Letting $\mathcal{B}=H^1(\Omega,\mathbb{C})\times H^1(\Omega,\mathbb{R}^2)$ we define $\mathcal{M}$ to be the set of $(u,A)\in\mathcal{B}$ satisfying the boundary conditions in (I.16). Writing $w=(u,A)$ we let

$$E_\varepsilon(w)=\frac{J(u,A)-h_{\rm ex}^2J_0-\pi n|\log\varepsilon|}{|\log\varepsilon|}. \tag{IV.21}$$

The space $\mathcal{B}$ embeds into $X_\varepsilon=L^2(\Omega,\mathbb{C})\times L^2(\Omega,\mathbb{R}^2)$ that we equip with the norm

$$\|\delta w\|_{X_\varepsilon}^2=\frac{1}{|\log\varepsilon|}\int_\Omega|\delta u|^2+|\delta A|^2.$$

One may check that the gradient-flow of E_ε for this structure is a solution of (I.16) satisfying the temporal gauge condition $\Phi=0$.

For the limiting functional, we define $\mathcal{N}$ to be as in the case without magnetic field the set Ω_*^n of n-uples of distinct points in Ω. It embeds into $Y=(\mathbb{R}^2)^n$ on which we use the norm $\|v\|_Y^2=\pi\sum_i|v_i|^2$. We say that a family w_ε in $\mathcal{M}$ converges to $w=(a_1,\dots,a_n)\in\mathcal{N}$ if $\mu_\varepsilon=\operatorname{curl}(iu_\varepsilon,\nabla_{A_\varepsilon}u_\varepsilon)+h_\varepsilon$ converges to $\mu=2\pi\sum_i d_i\delta_{a_i}$ in the sense of distributions. The limiting functional is

$$F(w)=2\pi\lambda\sum_{i=1}^n d_i\xi_0(a_i)=\lambda\int_\Omega\xi_0\,d\mu, \tag{IV.22}$$

and we let $D_\varepsilon=E_\varepsilon(w_\varepsilon)-F(w)$. Under the constraint that the limit in the sense S of w_ε is in $\mathcal{N}$ (i.e. is a configuration of n distinct vortices of degrees d_i), E_ε Γ-converges to F in the sense of Definition 1, as seen in Lemma IV.2, and E_ε, F satisfy the hypothesis of Theorem 1.

There remains to check conditions 1) and 2) of Theorem 1, which is done in the next section.

IV.4 Lower and upper bounds

We start with the lower bound. The result is essentially the one of (III.9) but requires some more careful application of Theorem 3 of [SS6] because the hypothesis $\frac{1}{2}\int_{B(a_i,r_i)}|\nabla u|^2 = \pi|\log\,\varepsilon|(1+o(1))$ does not a priori have to be satisfied for all time in the present case. First we need the following

Lemma IV.4 *Let $(u_\varepsilon, A_\varepsilon)$ be as before, $B(a_i,r_i)$ be given by Lemma IV.1 for each time with $r_i \le C|\log\,\varepsilon|^{-3}$, X be a smooth vector field on $\Omega\times[0,T]$, we have, for every i and every $[t_1,t_2]\subset[0,T]$,*

$$\text{(IV.23)}\quad \int_{t_1}^{t_2}\int_{B(a_i(t),r_i(t))}|\nabla_{A_\varepsilon}u_\varepsilon\cdot X|^2\,dt \le \frac{1}{2}\int_{t_1}^{t_2}\int_{B(a_i(t),r_i(t))}|X|^2|\nabla_{A_\varepsilon}u_\varepsilon|^2\,dt + C|\log\,\varepsilon|\int_{t_1}^{t_2}D_\varepsilon(t)\|X\|^2_{L^\infty(\Omega)}(t)\,dt + o(1).$$

Proof: For any (u,A), we let

$$\begin{aligned}\partial_u J(u,A) &= \nabla_A^2 u + \tfrac{u}{\varepsilon^2}(1-|u|^2)\\ \partial_A J(u,A) &= \nabla^\perp h + j.\end{aligned}$$

As in [SS4], we introduce the stress-energy tensor associated to the energy J:

$$\text{(IV.24)}\quad T = \frac{1}{2}\begin{pmatrix}|\partial_1^A u|^2-|\partial_2^A u|^2 & 2(\partial_1^A u,\partial_2^A u)\\ 2(\partial_1^A u,\partial_2^A u) & |\partial_2^A u|^2-|\partial_1^A u|^2\end{pmatrix} + \left(\frac{h^2}{2}-\frac{(1-|u|^2)^2}{4\varepsilon^2}\right)\begin{pmatrix}1&0\\0&1\end{pmatrix}.$$

Here $\partial_j^A = \partial_j - iA_j$. A direct calculation (see for example [Sp]) yields

$$\text{(IV.25)}\qquad \mathrm{div}\ T := \begin{pmatrix}\partial_1 T_{11}+\partial_2 T_{12}\\ \partial_1 T_{21}+\partial_2 T_{22}\end{pmatrix} = (\nabla_A u,\partial_u J(u,A)) - h(\partial_A J(u,A))^\perp.$$

Since (u,A) is here assumed to be a solution of (I.16), the right-hand side is equal to $(\nabla_A u,\mathcal{F})+h\mathcal{E}^\perp$. We will write temporarily $V=\frac{1}{2}(|\partial_1^A u|^2-|\partial_2^A u|^2, 2(\partial_1^A u,\partial_2^A u))$. Let f be a $C_0^\infty(\Omega)$ function and let us multiply this relation (IV.25) by the vector field $(f,0)$, and integrate. We are led to

$$\text{(IV.26)}\qquad \left|\int_\Omega f\mathrm{div}\ V - \partial_1 f\left(\frac{h^2}{2}-\frac{(1-|u|^2)^2}{4\varepsilon^2}\right)\right| \le \int_\Omega |f|\left(|\nabla_A u||\mathcal{F}|+|h||\mathcal{E}|\right).$$

Since the vortex balls remain well-separated in time, we can find for each i an $f\in C_0^\infty(\Omega)$ which has support in $\Omega\setminus\cup_{j\neq i}B(a_j,r_j)$ and such that

$$\begin{cases}\nabla f = (1,0) \quad \text{in } B(a_i,r_i)\\ \|\nabla f\|_{L^\infty(\Omega)}\le C\\ \|f\|_{L^\infty(\Omega)} \le r_i \le \frac{C}{|\log\,\varepsilon|^3}\\ |supp\ f| \le Cr_i \le \frac{C}{|\log\,\varepsilon|^3},\end{cases}$$

where $|supp\ f|$ denotes the area of the support of f. This may be achieved by a function of the form $\xi(x_1)\chi(x_2)$. Then from (IV.26),

$$\left|\int_{B(a_i,r_i)} V_1\right| \le \int_{\Omega\setminus\cup_j B(a_j,r_j)} |\nabla f\cdot V| + \int_\Omega |\nabla f|\left|\frac{h^2}{2} - \frac{(1-|u|^2)^2}{4\varepsilon^2}\right| + \int_\Omega |f|\left(|\nabla_A u||\mathcal{F}| + |h||\mathcal{E}|\right). \tag{IV.27}$$

But from Lemma IV.2, letting $A' = A - h_{\rm ex}\nabla^\perp\xi_0$,

$$J_f(u,A') \le |\log\varepsilon|(\pi n + D_\varepsilon + o(1)), \tag{IV.28}$$

while (see (IV.9))

$$\frac{1}{2}\int_{\cup_j B(a_j,r_j)} |\nabla_{A'}u|^2 \ge \pi n|\log\varepsilon|(1-o(1)), \tag{IV.29}$$

therefore

$$\int_{\Omega\setminus\cup_j B(a_j,r_j)} |\nabla u - iA'u|^2 \le 2D_\varepsilon|\log\varepsilon|(1+o(1)).$$

Moreover, since ξ_0 is C^∞ and the support of f has measure less than $\frac{C}{|\log\varepsilon|^3}$, we deduce

$$\int_{\Omega\setminus\cup_j B(a_j,r_j)\cap supp\ f} |V_1| \le C\int_{(\Omega\setminus\cup_j B(a_j,r_j))\cap supp\ f} |\nabla u - iAu|^2 \le CD_\varepsilon|\log\varepsilon|(1+o(1)) + o(1).$$

Using also the fact that $\|\nabla f\|_{L^\infty} \le C$, we may finally deduce that $\int_{\Omega\setminus\cup_j B(a_j,r_j)} |\nabla f\cdot V| \le CD_\varepsilon|\log\varepsilon| + o(1)$. By the same argument, from (IV.28),

$$\int_{(\Omega\setminus\cup_j B(a_j,r_j))\cap supp\ f} |h|^2 \le CD_\varepsilon|\log\varepsilon| + o(1),$$

and, with the help of (IV.28) again,

$$\int_\Omega |\nabla f|\left|\left(\frac{h^2}{2} - \frac{(1-|u|^2)^2}{4\varepsilon^2}\right)\right| \le CD_\varepsilon|\log\varepsilon| + o(1).$$

Finally, (IV.27) becomes

$$\left|\int_{B(a_i,r_i)} V_1\right| \le CD_\varepsilon|\log\varepsilon| + o(1) + \frac{C}{|\log\varepsilon|^3}\int_\Omega |\nabla_A u||\mathcal{F}| + |h||\mathcal{E}|, \tag{IV.30}$$

and the same result holds for V_2. Let now X be a continuous vector field. Then

$$\begin{aligned}\int_{B(a_i,r_i)} |X\cdot\nabla_A u|^2 &= \int_{B(a_i,r_i)} |X|^2\frac{|\nabla_A u|^2}{2} + \frac{X_1^2 - X_2^2}{2}(|\partial_1^A u|^2 - |\partial_2^A u|^2) + 2X_1X_2(\partial_1^A u, \partial_2^A u)\\ &= \int_{B(a_i,r_i)} |X|^2\frac{|\nabla_A u|^2}{2} + \int_{B(a_i,r_i)} (X_1^2 - X_2^2)V_1 + 2X_1X_2V_2.\end{aligned}$$

We deduce that

$$\text{(IV.31)}\quad \int_{B(a_i,r_i)} |X\cdot\nabla_A u|^2 \le \int_{B(a_i,r_i)} |X|^2\frac{|\nabla_A u|^2}{2} + \|X\|^2_{L^\infty(\Omega)}\left(CD_\varepsilon|\log\varepsilon| + o(1) + \frac{C}{|\log\varepsilon|^3}\int_\Omega |\nabla_A u||\mathcal{F}| + |h||\mathcal{E}|\right).$$

But, since $(u_\varepsilon, A_\varepsilon)$ is a solution of (I.16) it satifies (IV.12), thus $\int_{\Omega\times[t_1,t_2]}|\nabla_A u||\mathcal{F}| \le C|\log\varepsilon|^2$ and $\int_{t_1}^{t_2}|h||\mathcal{E}|\le C|\log\varepsilon|^2$. Integrating in time (IV.31), we get (IV.23). $\square$

Proposition IV.3 *If $w_\varepsilon = (u_\varepsilon, A_\varepsilon)$ is a solution of the gradient flow $\partial_t w_\varepsilon = -\nabla E_\varepsilon(w_\varepsilon)$, such that $w_\varepsilon \overset{S}{\rightharpoonup} w(t)\in\mathcal{N}$ on $[0,T)$, there exists $f\in L^1(\mathbb{R})$ such that for all $[t_1,t_2]\subset[0,T]$,*

$$\text{(IV.32)}\qquad \liminf_{\varepsilon\to0}\int_{t_1}^{t_2}\|\partial_t w_\varepsilon\|^2_{X_\varepsilon}\,dt \ge \int_{t_1}^{t_2}\left(\|\partial_t w\|^2_Y - f(t)D(t)\right)\,dt.$$

Proof: If we combine (IV.9) and (IV.10) with the definition of E_ε and F, we find that $\forall i,\ \frac12\int_{B(a_i,r_i)(t)}|\nabla_{A'}u|^2 \le (\pi + CD_\varepsilon + o(1))|\log\varepsilon|$ and thus $\int_{B(a_i,r_i)(t)}|\nabla_A u|^2\le(\pi+CD_\varepsilon + o(1))|\log\varepsilon|$. Then, combining this with Lemma IV.4, we find

$$\text{(IV.33)}\quad \int_{t_1}^{t_2}\int_{\cup_i B(a_i(t),r_i(t))}|\nabla_{A_\varepsilon}u_\varepsilon\cdot X|^2 \le \int_{t_1}^{t_2}\pi\sum_i|X(a_i(t),t)|^2|\log\varepsilon|\,dt + \int_{t_1}^{t_2}C\|X\|^2_{L^\infty(\Omega)}(t)D_\varepsilon(t)|\log\varepsilon|\,dt + o(1).$$

We may plug this into the proof of Theorem 1 of [SS6] to obtain as an alternate of Proposition IV.1, the following result

$$\text{(IV.34)}\quad \liminf_{\varepsilon\to0}\left(\frac{1}{|\log\varepsilon|}\int_{\Omega\times[t_1,t_2]}|\mathcal{F}_\varepsilon|^2\right)\left(\int_{t_1}^{t_2}\pi\sum_i|X(a_i(t),t)|^2 + C\|X\|^2_{L^\infty(\Omega)}D_\varepsilon(t)\,dt\right) \ge \left|\frac12\int_{\Omega\times[t_1,t_2]}V\cdot X\right|^2,$$

where V is such that $\partial_t\mu + \text{div }V = 0$. Since the $a_i(t)$'s remain distinct (we work before collision) and continuous in time, we may work in open sets U_i which contain only one $a_i(t)$ for t ranging in a small interval $[t_1,t_2]$. Applying (IV.34) on U_i for $X(x,t) = \nabla\zeta(x,t)$, and considering $D(t) = \limsup_{\varepsilon\to0}D_\varepsilon(t)$, we are led to

$$\text{(IV.35)}\quad \frac12\left|\int_{U_i\times[t_1,t_2]}V\cdot\nabla\zeta\right| \le \left(\int_{t_1}^{t_2}\pi|\nabla\zeta(a_i(t),t)|^2 + C\|\nabla\zeta\|^2_{L^\infty(U_i)}D(t)\,dt\right)^{\frac12}\liminf_{\varepsilon\to0}\frac{\left(\int_{U_i\times[t_1,t_2]}|\mathcal{F}_\varepsilon|^2\right)^{\frac12}}{|\log\varepsilon|^{\frac12}}.$$

But, one may check that in the sense of distributions, div $V=$ div $(2\pi\sum_i d_i\partial_t a_i(t)\delta_{a_i(t)})$, hence (IV.35) becomes

$$\text{(IV.36)}\quad \left|\int_{t_1}^{t_2}\pi\partial_t a_i(t)\cdot\nabla\zeta(a_i(t),t)\,dt\right|$$
$$\leq\left(\int_{t_1}^{t_2}\pi\left(|\nabla\zeta(a_i(t),t)|^2+C\|\nabla\zeta\|^2_{L^\infty(U_i)}D(t)\right)\,dt\right)^{\frac{1}{2}}\liminf_{\varepsilon\to0}\frac{\left(\int_{U_i\times[t_1,t_2]}|\mathcal{F}_\varepsilon|^2\right)^{\frac{1}{2}}}{|\log\,\varepsilon|^{\frac{1}{2}}}.$$

For any vector-valued $X(t)$ we may choose ζ's such that $\|\nabla\zeta(x,t)\|_{L^\infty(U_i)}=|\nabla\zeta(a_i(t),t)|$ and $\nabla\zeta(a_i(t),t)=X(t)$. Then (IV.36) rewrites

$$\left|\int_{t_1}^{t_2}\pi\partial_t a_i(t)\cdot X(t)\,dt\right|\leq\left(\int_{t_1}^{t_2}\pi|X(t)|^2(1+CD(t))\,dt\right)^{\frac{1}{2}}\liminf_{\varepsilon\to0}\frac{\left(\int_{U_i\times[t_1,t_2]}|\mathcal{F}_\varepsilon|^2\right)^{\frac{1}{2}}}{|\log\,\varepsilon|^{\frac{1}{2}}}.$$

By a duality argument, we deduce that $\partial_t a_i\in L^2_{1+CD}([t_1,t_2])$, where L^2_ρ denotes the weighted L^2 Lebesgue space with weight ρ, and that

$$\text{(IV.37)}\qquad \pi\int_{t_1}^{t_2}\frac{|\partial_t a_i|^2}{1+CD(t)}\,dt\leq\liminf_{\varepsilon\to0}\frac{1}{|\log\,\varepsilon|}\int_{U_i\times[t_1,t_2]}|\mathcal{F}_\varepsilon|^2.$$

Using the identity $\frac{1}{1+CD}\geq1-2CD$, the fact that $D(t)$ is bounded by $E_\varepsilon(0)$, and summing up over i and over small time intervals, we get that $|\partial_t a_i|^2\in L^1([0,T])$ and that for all $[t_1,t_2]\subset[0,T]$,

$$\text{(IV.38)}\quad \liminf_{\varepsilon\to0}\frac{1}{|\log\,\varepsilon|}\int_{\Omega\times[t_1,t_2]}|\mathcal{F}_\varepsilon|^2\geq\pi\int_{[t_1,t_2]}\sum_i|\partial_t a_i|^2(1-2CD(t))\,dt$$
$$=\pi\int_{[t_1,t_2]}\sum_i|\partial_t a_i|^2-f(t)D(t)\,dt,$$

where $f(t)=2C\sum_i|\partial_t a_i|^2\in L^1([0,T])$. Choosing the gauge $\Phi=0$, this is the desired result. $\square$

We turn to the proof of upper bound result, i.e. condition 2) of Theorem 1. As in Lemma II.1, it is enough to prove the result for $w_\varepsilon=(u_\varepsilon,A_\varepsilon)$ satisfying $\|\nabla E_\varepsilon(w_\varepsilon)\|_{X_\varepsilon}\leq C$.

Proposition IV.4 *Let $(u_\varepsilon,A_\varepsilon)\overset{S}{\rightharpoonup}w=(a_1,\cdots,a_n)$ be such that $\|\nabla E_\varepsilon(w_\varepsilon)\|_{X_\varepsilon}\leq C$ and $D=\limsup_{\varepsilon\to0}E_\varepsilon(w_\varepsilon)-F(w)$. For any family of vectors $V_1,\cdots,V_n$, letting $(b_1,\cdots,b_n)(t)$ satisfy*

$$\text{(IV.39)}\qquad \begin{cases}\partial_t b_i(0)=V_i\\ b_i(0)=a_i,\end{cases}$$

there exists $(v_\varepsilon, A_\varepsilon)$ of class C^1 in a neighborhood of 0*, equal to $(u_\varepsilon, A_\varepsilon)(0)$ at $t=0$, and a locally bounded (on $\mathcal{N}$) $g(w)$ such that*

$$\limsup_{\varepsilon\to 0} \frac{1}{|\log \varepsilon|}\int_\Omega |\partial_t A_\varepsilon(0)|^2 + |\partial_t v_\varepsilon(0)|^2 \le \pi \sum_i |V_i|^2(1+g(w)D) + g(w)D \tag{IV.40}$$

$$\liminf_{\varepsilon\to 0}\left(-\frac{d}{dt}_{|t=0} E_\varepsilon(v_\varepsilon, A_\varepsilon)\right) \ge -\frac{d}{dt}_{|t=0} 2\pi\lambda \sum_i d_i \xi_0(b_i(t)) - g(w)D. \tag{IV.41}$$

Proof: We define a family of diffeomorphisms χ_t as in Proposition III.3, satisfying (III.20) and such that $\|\nabla\chi_t\|_\infty$, $\|\partial_t\chi_t\|_\infty$ seen as functions of $(a_1,\dots,a_n)$ are locally bounded on $\mathcal{N}$. In view of (IV.28)-(IV.29), we may find $\rho_\varepsilon \ll |\log\varepsilon|^{-3}$ such that

$$\frac12 \int_{\Omega\setminus\cup_i B(a_i,\rho_\varepsilon)} |\nabla u - iuA'|^2 + \frac12\int_\Omega |\mathrm{curl}\, A'|^2 + \frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 \le (D_\varepsilon + o(1))|\log\varepsilon|. \tag{IV.42}$$

We then define v_ε as follows

$$v_\varepsilon(\chi_t(x)) = u_\varepsilon(x). \tag{IV.43}$$

We assume that A_ε is in the Coulomb gauge and is equal to $h_{\mathrm{ex}}\nabla^\perp\xi_0 + \nabla^\perp\xi$ with $\xi=0$ on $\partial\Omega$, (ξ depends implicitely on ε). We observe that from (IV.42), $\int_\Omega |\Delta\xi|^2 \le 2D|\log\varepsilon|$ and since $\xi = 0$ on $\partial\Omega$, by elliptic regularity we have $\|\xi\|^2_{H^2} \le CD|\log\varepsilon|$. We then take at time $t\ge 0$,

$$A_\varepsilon(t) = h_{\mathrm{ex}}\nabla^\perp\xi_0 + \nabla^\perp(\xi\circ\chi_t^{-1}). \tag{IV.44}$$

Let us first prove that (IV.40) is satisfied. First,

$$\frac{1}{|\log\varepsilon|}\int_\Omega |\partial_t v_\varepsilon|^2(0) = \frac{1}{|\log\varepsilon|}\int_\Omega |\nabla u_\varepsilon\circ\chi_t^{-1}\cdot\partial_t\chi_t^{-1}|^2.$$

Then, using the fact that $\partial_t\chi_t^{-1} = -V_i$ in each B_i and that χ_t^{-1} is a translation in each B_i, we have

$$\begin{aligned}
\frac{1}{|\log\varepsilon|}\int_\Omega |\partial_t v_\varepsilon|^2(0) &\le \frac{g(w)}{|\log\varepsilon|}\int_{\Omega\setminus\cup_i B(a_i,\rho_\varepsilon)} |\nabla u_\varepsilon|^2 + \frac{1}{|\log\varepsilon|}\sum_i \int_{B(a_i,\rho_\varepsilon)} |\nabla u_\varepsilon\cdot V_i|^2 \\
&\le \frac{g(w)}{|\log\varepsilon|}\left(\int_{\Omega\setminus\cup_i B(a_i,\rho_\varepsilon)} |\nabla u_\varepsilon - iu_\varepsilon\nabla^\perp\xi|^2 + \int_\Omega |\nabla\xi|^2\right) \\
&\quad + \frac{1}{|\log\varepsilon|}\sum_i \int_{B(a_i,\rho_\varepsilon)} |\nabla u_\varepsilon\cdot V_i|^2 \\
&\le g(w)D + \frac{1}{|\log\varepsilon|}\sum_i \int_{B(a_i,\rho_\varepsilon)} |\nabla u_\varepsilon\cdot V_i|^2 + o(1),
\end{aligned}$$

where we have used (IV.42)-(IV.44). For the remaining term, we use $\|\nabla E_\varepsilon(w_\varepsilon(0))\|_{X_\varepsilon} \le C$, that is

$$\frac{1}{|\log\,\varepsilon|}\int_\Omega |\nabla^2_{A_\varepsilon} u_\varepsilon + \frac{u_\varepsilon}{\varepsilon^2}(1-|u_\varepsilon|^2)|^2 + |-\nabla^\perp \mathrm{curl}\, A_\varepsilon - j|^2 \le C,$$

or $\frac{1}{|\log\,\varepsilon|}\int_\Omega |\partial_u J(u,A)|^2 + |\partial_A J(u,A)|^2 \le C$. Applying the method of Lemma IV.4, this implies that

$$\int_{B(a_i,\rho_\varepsilon)} |\nabla_{A_\varepsilon} u_\varepsilon \cdot V_i|^2 \le \frac{1}{2}\int_{B(a_i,\rho_\varepsilon)} |\nabla_{A_\varepsilon} u_\varepsilon|^2 |V_i|^2 + C|V_i|^2 D_\varepsilon(0)|\log\,\varepsilon| + o(1).$$

Finally, we will be able to conclude as in (IV.23) that

$$\frac{1}{|\log\,\varepsilon|}\int_\Omega |\partial_t v_\varepsilon|^2(0) \le \pi \sum_i |V_i|^2 + g(w)D(1+\sum_i |V_i|^2) + o(1).$$

Meanwhile, in view of (IV.44),

$$\frac{1}{|\log\,\varepsilon|}\int_\Omega |\partial_t A_\varepsilon|^2(0) \le C\frac{1}{|\log\,\varepsilon|}\int_\Omega |D^2\xi|^2 \le \frac{g(w)}{|\log\,\varepsilon|}\|\xi\|^2_{H^2(\Omega)} \le g(w)D + o(1).$$

Thus (IV.40) is satisfied. Let us then evaluate $J(v_\varepsilon, A_\varepsilon)$. As in (IV.7)-(IV.8) we have

$$\text{(IV.45)}\quad \begin{aligned} J(v_\varepsilon, A_\varepsilon) &= J_f(v, \nabla^\perp(\xi\circ\chi_t^{-1})) + \frac{h_{\rm ex}^2}{2}\int_\Omega |v|^2|\nabla\xi_0|^2 + |\xi_0|^2 \\ &\quad + h_{\rm ex}\int_\Omega -\nabla^\perp\xi_0\cdot(iv,\nabla v - iv\nabla^\perp(\xi\circ\chi_t^{-1}) + \xi_0\Delta(\xi\circ\chi_t^{-1}) \\ &= h_{\rm ex}^2 J_0 + J_f(v,\nabla^\perp(\xi\circ\chi_t^{-1})) + \frac{h_{\rm ex}^2}{2}\int_\Omega (|v|^2-1)|\nabla\xi_0|^2 - h_{\rm ex}\int_\Omega \nabla^\perp\xi_0\cdot(iv,\nabla v) \\ &\quad + h_{\rm ex}\int_\Omega (|v|^2-1)\nabla\xi_0\cdot\nabla(\xi\circ\chi_t^{-1}). \end{aligned}$$

We will deal separately with the time derivative at $t=0$ of all the terms in the right-hand side. First, with a change of variables $y=\chi_t(x)$ as in Proposition III.3, from (IV.43) we have

$$\text{(IV.46)}\quad \begin{aligned} I &:= \frac{1}{2}\int_\Omega (|v_\varepsilon|^2-1)\nabla\xi_0\cdot(\nabla\xi_0 + 2\nabla(\xi\circ\chi_t^{-1})) + |\Delta(\xi\circ\chi_t^{-1})|^2 + \frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2 \\ &= \frac{1}{2}\int_\Omega \Big((|u_\varepsilon|^2-1)\nabla\xi_0\circ\chi_t\cdot(\nabla\xi_0\circ\chi_t + 2\nabla(\xi\chi_t^{-1})\circ\chi_t) \\ &\quad + |(\Delta(\xi\circ\chi_t^{-1}))\circ\chi_t|^2 + \frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)|Jac\ \chi_t|. \end{aligned}$$

Differentiating with respect to t, we deduce with the a priori estimates and (IV.42) that

$$\text{(IV.47)}\quad \left|\frac{d}{dt}_{|t=0} I\right| \le g(w)\left(\int_\Omega |D^2\xi|^2 + \frac{(1-|u_\varepsilon|^2)^2}{2\varepsilon^2} + o(1)\right) \le g(w)|\log\,\varepsilon|(D+o(1)).$$

Arguing again as in Proposition III.3, we have

(IV.48)
$$\frac{d}{dt}_{|t=0}\frac{1}{|\log\,\varepsilon|}\int_\Omega |\nabla v_\varepsilon - iv_\varepsilon\nabla^\perp(\xi\circ\chi_t^{-1})|^2 = \frac{d}{dt}_{|t=0}\frac{1}{|\log\,\varepsilon|}\int_\Omega |D\chi_t^{-1}\cdot(\nabla u_\varepsilon - iu_\varepsilon\nabla^\perp\xi)|^2|Jac\;\chi_t|$$
$$\leq \frac{g(w)}{|\log\,\varepsilon|}\int_{\Omega\setminus\cup_i B(a_i,r_i)} |\nabla u_\varepsilon - iu_\varepsilon\nabla^\perp\xi|^2 \leq g(w)(D+o(1)),$$

where we have used (IV.42). There remains to differentiate the cross-term. For that we first write (with the same change of variables),

$$\frac{d}{dt}_{|t=0}\int_\Omega \nabla^\perp\xi_0\cdot(iv_\varepsilon,\nabla v_\varepsilon) = \frac{d}{dt}_{|t=0}\int_\Omega (\nabla^\perp\xi_0)\circ\chi_t\cdot D\chi_t^{-1}(iu_\varepsilon,\nabla u_\varepsilon)|Jac\;\chi_t|. \tag{IV.49}$$

Let us now introduce Θ_0 the harmonic conjugate of the solution of

$$\begin{cases} \Delta\Phi = 2\pi\sum_i d_i\delta_{a_i} & \text{on } \Omega \\ \Phi = 0 & \text{on } \partial\Omega. \end{cases}$$

Θ_0 is not uni-valued but satisfies $\mathrm{curl}\,(\nabla\Theta_0) = 2\pi\sum_i d_i\delta_{a_i}$. Hence, in view of Lemma IV.1, and the fact that we are in the Coulomb gauge, we have

$$\mathrm{curl}\,((iu_\varepsilon,\nabla u_\varepsilon) - \nabla\Theta_0) \rightharpoonup 0 \quad \text{in } \mathcal{D}'(\Omega).$$

Therefore, after extraction, there exists $H\in\mathcal{D}'(\Omega)$ such that

$$((iu_\varepsilon,\nabla u_\varepsilon) - \nabla\Theta_0 - \nabla H) \rightharpoonup 0 \quad \text{in } \mathcal{D}'(\Omega). \tag{IV.50}$$

Inserting this into (IV.49), we are led to

$$\begin{aligned}
\frac{d}{dt}_{|t=0}\int_\Omega \nabla^\perp\xi_0\cdot(iv_\varepsilon,\nabla v_\varepsilon) &= \frac{d}{dt}_{|t=0}\int_\Omega (\nabla^\perp\xi_0)\circ\chi_t\cdot D\chi_t^{-1}(\nabla\Theta_0+\nabla H)|Jac\;\chi_t| + o(1) \\
&= \frac{d}{dt}_{|t=0}\int_\Omega \nabla^\perp\xi_0\cdot\nabla((\Theta_0+H)\circ\chi_t^{-1}) + o(1) \\
&= -\frac{d}{dt}_{|t=0}\int_\Omega \xi_0\;\mathrm{curl}\,\nabla(\Theta_0\circ\chi_t^{-1}) + o(1) \\
&= -\frac{d}{dt}_{|t=0}\left(2\pi\sum_i d_i\xi_0(b_i(t))\right) + o(1). \qquad \text{(IV.51)}
\end{aligned}$$

Combining (IV.45)–(IV.47)–(IV.48)–(IV.49) and (IV.51), we deduce that

$$\frac{d}{dt}_{|t=0}\frac{J(v_\varepsilon,A_\varepsilon)}{|\log\,\varepsilon|} = \frac{d}{dt}_{|t=0}F(b_i(t)) + g(w)(D+o(1)). \tag{IV.52}$$

which yields the desired result (IV.41). $\square$

Finally, the result of this last proposition applied to $V_i = -2d_i\lambda\nabla\xi_0(a_i)$, allows to apply Theorem 1 and conclude with Theorem 3. (Indeed, it is enough to apply the proof of Lemma II.1.) The last statement of Theorem 3 follows from the last of Theorem 1 and the construction above (bearing in mind that $D = 0$).

References

[ASS] A. Aftalion, E. Sandier and S. Serfaty, Pinning Phenomena in the Ginzburg-Landau Model of Superconductivity, *J. Math. Pures Appl., 80, No 3,* (2001), 339-372.

[AGS] L. Ambrosio, N. Gigli and G. Savaré, *Gradient Flows in Metric Spaces and in the Wasserstein Spaces of Probability Measures*, forthcoming.

[AS] L. Ambrosio and H.M. Soner, A measure-theoretic approach to higher codimension mean curvature flows, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 25, No 1-2,* (1997), 27–49.

[BBH] F. Bethuel, H. Brezis and F. Hélein, *Ginzburg-Landau Vortices*, Birkhäuser, (1994).

[BOS] F. Bethuel, G. Orlandi and D. Smets, Convergence of the parabolic Ginzburg-Landau equation to motion by mean-curvature, preprint.

[BR1] F. Bethuel and T. Rivière, Vortices for a Variational Problem Related to Superconductivity, *Annales IHP, Analyse non linéaire, 12*, (1995), 243-303.

[BR2] F. Bethuel and T. Rivière, Vorticité dans les modèles de Ginzburg-Landau de la supraconductivité, *Séminaire X EDP*, exposé XVI, École Polytechnique, Palaiseau, (1994).

[Bra] A. Braides, *Γ-convergence for Beginners*, Oxford Lecture Series, vol 22, Oxford. Univ. Press, (2002).

[Bre] H. Brezis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North Holland, (1973).

[CRS] S. J. Chapman, J. Rubinstein, and M. Schatzman, A Mean-field Model of Superconducting Vortices, *Eur. J. Appl. Math., 7, No. 2,* (1996), 97-111.

[Ch] X. Chen, Generation and propagation of interfaces for reaction-diffusion equations, *J. Diff. Eq., 96,* (1992), 116-141.

[CGG] Y.G. Chen, Y. Giga and S. Goto, Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations, *J. Diff. Geom. 33*, (1991), 749-786.

[CJ1] J. Colliander and R. Jerrard, Vortex dynamics for the Ginzburg-Landau-Schrödinger equation, *Inter. Math. Res. Notices*, (1998), No 7, 333-358.

[CJ2] J. Colliander and R. Jerrard, Ginzburg-Landau vortices: weak stability and Schrödinger equations dynamics, *J. Anal. Math., 77*, (1999), 129-205.

[dMS] P. De Mottoni and M. Schatzman, Development of interfaces in $\mathbb{R}^N$, *Proc. Roy. Soc. Edinburgh, Sec A, 116,* (1990), 207-220.

[Du] Q. Du, Global existence and uniqueness of solutions of the time-dependent Ginzburg-Landau model for superconductivity, *Appl. Anal. 53*, (1994), 1-17.

[E] W. E, Dynamics of vortices in Ginzburg-Landau theories with applications to superconductivity, *Phys. D, 77,*(1994), 383-404.

[ESS] L. C. Evans, H. M. Soner and P. E. Souganidis, Phase transitions and generalized motion by mean curvature, *Comm. Pure Appl. Math., 45, No 9,* (1992), 1097–1123.

[F] I. Fatkullin, PhD thesis, New York University, (2003).

[I] T. Ilmanen, Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature, *J. Diff. Geom., 38*, (1993), 417-461.

[J1] R.L. Jerrard, Lower Bounds for Generalized Ginzburg-Landau Functionals, *SIAM Journal Math. Anal., 30, No 4*, (1999), 721-746.

[J2] R.L. Jerrard, Vortex dynamics for the Ginzburg-Landau wave equation, *Calc. Var. PDE, 9*, (1999), 1-30.

[JS1] R.L. Jerrard and H.M. Soner, Dynamics of Ginzburg-Landau vortices, *Arch. Rational Mech. Anal. 142, No 2,* (1998), 99–125.

[JS2] R.L. Jerrard and H.M. Soner, The Jacobian and the Ginzburg-Landau functional, *Calc. Var. PDE. 14, No 2*, (2002), 151-191.

[Li1] F.H. Lin, Some Dynamical Properties of Ginzburg-Landau Vortices, *Comm. Pure Appl. Math., 49*, (1996), 323-359.

[Li2] F.H. Lin, Vortex Dynamics for the Nonlinear Wave Equation, *Comm. Pure Appl. Math., 52*, (1999), 737-761.

[LR] F. H. Lin and T. Rivière, A quantization property for moving line vortices, *Comm. Pure Appl. Math. 54, No 7,* (2001), 826–850.

[LX] F. H. Lin and J. X. Xin, On the incompressible fluid limit and the vortex motion law of the nonlinear Schrödinger equation, *Comm. Math. Phys. 200*, (1999), no. 2, 249–274.

[PR] L. Pismen and J. Rubinstein, Motion of vortex-lines in the Ginzburg-Landau model, *Phys. D, 64*, (1993), 299-309.

[RuSt] J. Rubinstein and P. Sternberg, On the slow motion of vortices in the Ginzburg-Landau heat flow, *SIAM J. Math. Anal. 26, no 6* (1995), 1452–1466.

[O] F. Otto, The geometry of dissipative evolution equations: the porous medium equation, *Comm. PDE, 26*, (2001), 101-174.

[Sa] E. Sandier, Lower Bounds for the Energy of Unit Vector Fields and Applications, *J. Functional Analysis, 152, No 2,* (1998), 379-403.

[SS1] E. Sandier and S. Serfaty, Global Minimizers for the Ginzburg-Landau Functional below the First Critical Magnetic Field, *Annales IHP, Analyse non linéaire. 17,* 1, (2000), 119-145.

[SS2] E. Sandier and S. Serfaty, On the Energy of Type-II Superconductors in the Mixed Phase, *Reviews in Math. Phys., 12, No 9,* (2000), 1219-1257.

[SS3] E. Sandier and S. Serfaty, A Rigorous Derivation of a Free-Boundary Problem Arising in Superconductivity, *Annales Sci. Ecole Normale Sup.,* 4e série, 33, (2000), 561-592.

[SS4] E. Sandier and S. Serfaty, Limiting Vorticities for the Ginzburg-Landau Equations, *Duke Math. J., 117, No 3,* (2003), 403-446.

[SS5] E. Sandier and S. Serfaty, Ginzburg-Landau Minimizers Near the First Critical Field Have Bounded Vorticity, *Calc. Var. PDE, 17, No 1*, (2003), 17-28.

[SS6] E. Sandier and S. Serfaty, A product estimate for Ginzburg-Landau and corollaries, to appear in *J. Func. Anal.*

[SS7] E. Sandier and S. Serfaty, *Vortices in the Magnetic Ginzburg-Landau Model*, in preparation.

[S1] S. Serfaty, Local Minimizers for the Ginzburg-Landau Energy near Critical Magnetic Field, part I, *Comm. Contemp. Math., 1 , No. 2,* (1999), 213-254. part II, *Comm. Contemp. Math., 1, No. 3,* (1999), 295-333.

[S2] S. Serfaty, Stable Configurations in Superconductivity: Uniqueness, Multiplicity and Vortex-Nucleation, *Arch. for Rat. Mech. Anal., 149,* (1999), 329-365.

[S3] S. Serfaty, Stability in 2D Ginzburg-Landau Passes to the Limit, preprint.

[S4] S. Serfaty, Vortex Collision and Energy Dissipation Rates in the Ginzburg-Landau Heat Flow, in preparation.

[Sp] D. Spirn, Vortex dynamics for the full time-dependent Ginzburg-Landau equations, *Comm Pure Appl Math, 55, No 5*, (2002), 537-581.

[T] M. Tinkham, *Introduction to Superconductivity, 2d edition*, McGraw-Hill, (1996).