Ginzburg-Landau Minimizers Near the First Critical Field Have Bounded Vorticity

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Abstract

We prove that for fields close enough to the first critical field, minimizers of the Ginzburg-Landau functional have a number of vortices bounded independently from the Ginzburg-Landau parameter. This generalizes a result proved in [SS1] and shows that locally minimizing solutions of the Ginzburg-Landau equation found in [S1, S3] are actually global minimizers. It also gives a partial answer to a question raised by F. Bethuel and T. Rivière in [BR].

1 Introduction and statement of the results

Consider a simply connected domain Ω in $\mathbb{R}^2$. We are interested in the minimization of the Ginzburg-Landau functional

$$(1)\qquad J(u,A)=\frac{1}{2}\int_\Omega |(\nabla - iA)\,u|^2+\frac{1}{2}\int_\Omega |\text{curl } A-h_{\text{ex}}|^2+\frac{\kappa^2}{2}\left(1-|u|^2\right)^2$$

which arises in the physics of superconductivity. The unknowns are the so-called order parameter $u\in H^1(\Omega,\mathbb{C})$ and the connection $A\in H^1(\Omega,\mathbb{R}^2)$, while κ and $h_{\rm ex}$ are two positive numbers: the Ginzburg-Landau parameter and the applied magnetic field. Since A is a vector field in $\mathbb{R}^2$, we may see curl A as a real-valued function, called the induced magnetic field and often denoted by the letter h. We refer to [SS1, SS2, SS3] for a discussion of this functional. Our interest is the asymptotic behaviour of minimizers as κ tends to $+\infty$.

The minimizers of (1) exhibit, for certain values of $h_{\rm ex}$, vortices, i.e. points where the order parameter vanishes. It is of interest to determine where these points are located since, in several cases, (see [JT], [M] and [PR]) two critical points of (1) having the same zeroes of the order parameter and same order of vanishing at these points are in fact identical, in other words the zero set determines the critical point. We first give a result proved in [S1].

Let h_0 be the solution to

$$\begin{cases} -\Delta h_0 + h_0 = 0 & \text{in } \Omega \\ h_0 = 1 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

We define

$$\xi_0 = h_0 - 1, \qquad H_{c_1}(\kappa) = \frac{\log \kappa}{2 \max |\xi_0|}, \tag{3}$$

(Note that ξ_0 is negative in Ω.) We also define

$$F_\kappa(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{\kappa^2}{2} \left(1 - |u|^2\right)^2, \tag{4}$$

and let

$$D_M = \{(u, A) \mid F_\kappa(u) < \pi M \log \kappa, \text{ div } A = 0 \text{ and } A.\nu = 0 \text{ on } \partial\Omega\}. \tag{5}$$

The following theorem was proved in [S1].

Theorem. ([S1]) *For any $M > 0$ there exists constants $\kappa_0 > 0$ and C such that*

1. *For any $\kappa > \kappa_0$ and $h_{\rm ex} < H_{c_1} + C - o(1)$ there exists a stable critical point (u, A) of (1) which is minimizing over D_M. Any such critical point is such that u does not vanish.*

2. *For any $\kappa > \kappa_0$ and $h_{\rm ex} > H_{c_1} + C + o(1)$ there exists a stable critical point (u, A) of (1) which is minimizing over D_M. Any such critical point is such that u vanishes.*

In this theorem and in the rest of this paper $o(1)$ denotes a function of κ tending to zero as $\kappa \to +\infty$. The meaning of this result is the following. For reasons that we do not go into but that come from ideas of the book by F. Bethuel, H. Brezis and F. Hélein [BBH], see also [AB], the set D_M can be seen as the set of configurations with less than M vortices. The above theorem then says that if the minimization of (1) is restricted to D_M, the value H_{c_1} of the applied magnetic field is a critical value below which no vortices are present and above which vortices appear in the minimizers. The result is a result on the asymptotics of minimizers as $\kappa \to +\infty$ since the critical value is only known to be $H_{c_1} + C$ up to a $o(1)$.

The minimization over D_M was a technical commodity. As already said, it indirectly bounds the number of vortices and allows to expand as $\kappa \to +\infty$ the energy of a critical point of (1) as a function of the positions and degrees of its vortices, thus reducing the minimization to a problem in a finite number of variables. In [SS1] the authors were able to go around some of the technical difficulties associated to dealing with a number of vortices not a-priori bounded to prove the following.

Theorem. ([SS1]) *There exists constants $\kappa_0, C > 0$ such that for any $\kappa > \kappa_0$ and $h_{\text{ex}} < H_{c_1} - C \log\log\kappa$, the minimizer (u, A) of* (1) *is such that u does not vanish.*

It was proved in [S3] that in this case the minimizer is actually unique.

This theorem gives an information on the global minimizer at the cost of an imprecision on the critical value of the order of $\log\log\kappa$ instead of a $o(1)$. Fortunately this imprecision remains small compared to H_{c_1} which is of the order of $\log\kappa$.

We have until now spoken of the value for which the first vortex appears, but many other results where proved in [S1, S2, S3] in the case where Ω was a ball — but they could easily be extended to a general domain — concerning the values of the external field for which the second, third, and so on vortices appear, and their respective positions. All these results concerned local minimizers of (1) belonging to D_M. We prove in this paper that the set D_M, for M large enough, is the set where the global minimizers of (1) lie, i.e. the local minimizers described in [S1, S2, S3] are actually global minimizers. Our main result is

Theorem 1. *For any $K > 0$, there exist positive constants κ_0, M such that for any $\kappa > \kappa_0$ and any $h_{\text{ex}} < H_{c_1} + K \log\log\kappa$, the global minimizers of* (1) *are gauge equivalent to an element of D_M.*

Two configurations (u, A) and (v, B) are said to be gauge equivalent if there exists a function $f \in H^2(\Omega, \mathbb{R})$ such that $u = ve^{if}$, $A = B + \nabla f$. Two gauge equivalent configurations correspond to the same physical configuration and the Ginzburg-Landau functional J is invariant under gauge transformations.

This result does look technical, but it is obtained as a consequence of a more meaningful one that we explain now. First we recall the construction of vortex balls (see [J, Sa, SS1, SS2]).

Proposition 1. *For any $K > 0$, there exists positive constants κ_0, C such that for any $\kappa > \kappa_0$, $h_{\text{ex}} < K \log \kappa$, and any (u, A) satisfying the a-priori bound $J(u, A) < K(\log \kappa)^2$ the following is true.*

There exists a finite family of disjoint balls $\{B_i\}$, where $B_i = B(a_i, r_i)$ such that

1. $\{|u| < 1 - 1/(\log \kappa)^2\} \subset \cup_i B_i$,
2. $\sum_i r_i < (\log \kappa)^{-10}$,
3. *Writing* $u = \rho e^{i\varphi}$,

$$\frac{1}{2} \int_{B_i} \rho^2 |\nabla \varphi - A|^2 + |h - h_{\text{ex}}|^2 \geq \pi |d_i| (\log \kappa - C \log \log \kappa), \tag{6}$$

where d_i is the degree of the map $u/|u|$ restricted to ∂B_i if $B_i \subset \Omega$ and $d_i = 0$ otherwise.

We say that $\{(a_i, d_i)\}$ is a family of vortices associated to (u, A), and we call $\{B_i\}$ the family of vortex balls. It is easily seen by testing J with the configuration $(u \equiv 1, A \equiv 0)$ that a minimizer (u, A) of (1) satisfies the a-priori bound $J(u, A) < \frac{1}{2} |\Omega| {h_{\text{ex}}}^2$, and therefore can be associated a family of vortices by the above proposition whenever $h_{\text{ex}} < K \log \kappa$. We may now state

Theorem 2. *For any $K > 0$, there exist positive constants κ_0, C, α such that for any $\kappa > \kappa_0$ and any $h_{\text{ex}} < H_{c_1} + K \log \log \kappa$, if (u, A) is a global minimizer of (1) and $\{(a_i, d_i)\}$ is an associated family of vortices then the following holds.*

1. $\forall i, \quad d_i \geq 0$,
2. $\sum_i d_i < C$.

3. dist $(a_i, \Lambda) < C(\log\kappa)^{-\alpha}$ *for any* i *such that* $d_i \neq 0$,

where Λ *is the subset of* Ω *where the function* h_0 *defined in* (2) *attains its minimum.*

It turns out that the fact that Ω is simply connected implies that Λ consists of a finite set of points, and a single point if Ω is convex.

Recall that the previous theorem somehow implies Theorem 1, proving that the minimizers we describe coincide with the local minimizers of [S1, S2, S3] which in turn provides more precise information on the vortices, for instance that the degrees are all equal to one. Another point to be noted is that our definition of the vortices is different from the one used in [S1, S2, S3]. Our vortices are seen at a larger scale: their radii are a negative power of $\log\kappa$ while the vortices in [S1, S2, S3] or in [BBH, BR] are much smaller. Hence the results of [S1, S2, S3] will give information on the much finer structure of vortices.

In [BR] the following question is raised. Choose $\varepsilon > 0$ and for any $\ell \in \mathbb{Z}$ choose ℓ points a_i such that $|a_i - a_j| > 2\varepsilon$ for $i \neq j$ and $d(a_i, \partial\Omega) \geq 2\varepsilon$. Finally choose for every $1 \leq i \leq \ell$ a degree $d_i \in \{-1, +1\}$. Then define the energy of the configuration of points and degrees to be

$$W = \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla\phi|^2 + 2\pi h_{\text{ex}} \sum_i d_i \xi_0(a_i),$$

where $\Omega_\varepsilon = \cup_i B(a_i, \varepsilon)$ and ϕ is the solution to

$$\begin{cases} \Delta\phi = 2\pi \sum_{i=1}^{\ell} d_i \delta_{a_i} & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

For what configuration of points and degrees is W minimal ? In particular, is their number bounded independently of ε ? The approach we use allows, even though we do not explicitely adress this problem, to answer as follows: if $h_{\text{ex}} < H_{c_1}(1/\varepsilon) + C\log|\log\varepsilon|$ and ε is small enough, the points of the minimizing configuration can be grouped into clusters of size like a negative power of $|\log\varepsilon|$ so that the sum of the degrees in any cluster is nonnegative, the number of clusters of positive degree is bounded independently of ε, and the clusters of positive degree are located near the set Λ mentioned above.

In the following we denote by C any positive constant depending only on the domain Ω.

2 Expansion of the energy

The proof of Theorem 2 — from which Theorem 1 follows — relies on an expansion of the energy of a minimizer in terms of the positions and degrees of its vortices. This expansion is similar to those computed in [S1, S2, SS1] but differs by some details.

Proposition 2. *For any $K > 0$, there exist positive constants κ_0, C such that for any $\kappa > \kappa_0$ and any $h_{\rm ex} < K \log \kappa$, if (u, A) is a critical point of (1) satisfying $J(u, A) < K(\log\kappa)^2$ and $\{(a_i, d_i)\}$ is an associated family of vortices then*

(7)
$$J(u,A) \geq {h_{\rm ex}}^2 J_0 + \pi \left(\sum_i |d_i|\right)(\log\kappa - C\log\log\kappa) + 2\pi h_{\rm ex}\sum_i d_i\xi_0(a_i) + \\ + \frac{1}{2}\int_{\Omega\setminus\cup_i B_i} |\nabla(h - h_{\rm ex}h_0)|^2 + \frac{1}{2}\int_\Omega |h - h_{\rm ex}h_0|^2 - o(1),$$

where we have written $h = curl\ A$ and

$$J_0 = \frac{1}{2}\int_\Omega |\nabla h_0|^2 + |h_0 - 1|^2.$$

It is convenient to modify the configuration (u, A) before proving the proposition. We use the following lemma proved in [SS4] in a slightly different form.

Lemma 1. *Under the assumptions of Proposition 2, there exists $(\tilde{u}, \widetilde{A})$ such that, writing $u = \rho e^{i\varphi}$ and $\tilde{u} = \tilde{\rho} e^{i\tilde{\varphi}}$,*

1. *$|\tilde{u}| = 1$ outside the vortex balls. Moreover $\tilde{u}$ and $u/|u|$ have the same degree as maps from ∂B_i to S^1.*
2. *$\|\rho(\nabla\varphi - A) - \tilde{\rho}(\nabla\tilde{\varphi} - \widetilde{A})\|^2_{L^2(\Omega)} \leq o(1)$.*
3. *$\|curl\ A - curl\ \widetilde{A}\|^2_{H^1(\Omega)} \leq \frac{C}{(\log\kappa)^2}$.*
4. *$J(\tilde{u}, \widetilde{A}) \leq J(u, A) + o(1)$.*

It is clear from this lemma that the vortices of (u, A) are also vortices for $(\tilde{u}, \widetilde{A})$ in the sense of Proposition 1. Then if the lower bound (7) holds for $(\tilde{u}, \widetilde{A})$, using Lemma 1 again, it also holds for (u, A). Thus, in the proof of Proposition 2, we may assume that $|u| = 1$ outside the vortex-balls.

Proof of Lemma 1: Since the quantities involved are gauge invariant we may assume we are working in a Coulomb gauge, i.e. that div $A = 0$ in Ω and $A.\nu = 0$ on $\partial\Omega$. Following [SS4], we define a configuration $(\tilde{u}, \widetilde{A})$ as follows:

$$\tilde{u} = \chi(|u|)\frac{u}{|u|},$$

where $\chi : [0, 1] \to [0, 1]$ is defined by

$$\begin{cases} \chi(x) = x & \text{if } 0 \leq x \leq \frac{1}{2} \\ \chi(x) = 1 & \text{if } x \geq 1 - \frac{1}{(\log\kappa)^2} \\ \chi \text{ is affine between } \frac{1}{2} \text{ and } 1 - \frac{1}{(\log\kappa)^2} \end{cases}$$

and $\widetilde{A}$ is such that the second Ginzburg-Landau equation $-\nabla^{\perp}\tilde{h} = (i\tilde{u}, \nabla_{\widetilde{A}}\tilde{u})$, where $\tilde{h} = \text{curl } \widetilde{A}$, is verified (see [SS4]). In the Coulomb gauge, the following a-priori bounds hold (see [SS1])

$$(8) \qquad ||A||_{H^1(\Omega)} \leq Ch_{\text{ex}}, \qquad ||A||_{L^\infty(\Omega)} \leq Ch_{\text{ex}}, \qquad ||u||_{H^1(\Omega)} \leq Ch_{\text{ex}}.$$

It follows easily from the definition of $\tilde{u}$ that property 1) of the lemma is verified and that

$$(9) \qquad |||u| - |\tilde{u}|||_{L^\infty(\Omega)} \leq \frac{C}{(\log\kappa)^2}.$$

In fact, since $\rho = \tilde{\rho}$ wherever $\rho \leq 1/2$, we may write

$$(10) \qquad |||u| - |\tilde{u}|||_{L^\infty(\Omega)} \leq \rho\frac{C}{(\log\kappa)^2}.$$

By construction $\varphi = \tilde{\varphi}$, thus

$$|\rho\nabla\varphi - \tilde{\rho}\nabla\tilde{\varphi}| \leq \frac{C}{(\log\kappa)^2}|\rho\nabla\varphi| \leq \frac{C}{(\log\kappa)^2}|\nabla u|,$$

therefore

$$(11) \qquad ||\rho\nabla\varphi - \tilde{\rho}\nabla\tilde{\varphi}||_{L^2(\Omega)} \leq \frac{C}{\log\kappa},$$

and, since $(iu, \nabla u) = \rho^2\nabla\varphi$,

$$(12) \qquad ||(iu, \nabla u) - (i\tilde{u}, \nabla\tilde{u})||_{L^2(\Omega)} \leq \frac{C}{\log\kappa}.$$

The second Ginzburg-Landau equation may be written

$$\begin{cases} -\Delta A + |u|^2 A = (iu, \nabla u) & A\cdot\nu = 0 \text{ on } \partial\Omega \\ -\Delta \tilde{A} + |\tilde{u}|^2 \tilde{A} = (i\tilde{u}, \nabla \tilde{u}) & \tilde{A}\cdot\nu = 0 \text{ on } \partial\Omega \end{cases}$$

Substracting as in [SS4] we get

(13)

$$-\Delta(\tilde{A} - A) + (\tilde{A} - A) = (i\tilde{u}, \nabla\tilde{u}) - (iu, \nabla u) + (1 - |\tilde{u}|^2)\tilde{A} + (1 - |u|^2)A.$$

From (12), the right-hand side of (13) is bounded in $L^2(\Omega)$ by $C/\log\kappa$, from which it follows that $\|A - \widetilde{A}\|_{H^2(\Omega)} \le C/\log\kappa$. This, together with (11) proves properties 2) and 3) in the Lemma. To prove 4), it suffices to check that

$$\frac{1}{2}\int_\Omega |\nabla\tilde{\rho}|^2 + \frac{\kappa^2}{2}(1-\tilde{\rho}^2)^2 \le \frac{1}{2}\int_\Omega |\nabla\rho|^2 + \frac{\kappa^2}{2}(1-\rho^2)^2 + o(1),$$

which follows easily from the definition of $\tilde{\rho}$.

Proof of Proposition 2: As already mentioned, we may assume that $|u| = 1$ in $\Omega \setminus \cup_i B_i$, where $\{B_i\}_i$ is the family of vortex balls, and that $-\nabla^\perp h = (iu, \nabla_A u)$ in Ω. We recall the following lemma from [SS4, ASS]

Lemma 2. *For any $q < 2$,*

$$\left\| -\Delta h + h - 2\pi \sum_i d_i \delta_{a_i} \right\|_{W^{-1,q}(\Omega)} = o(1)$$

Let us now prove (7). It follows from $-\nabla^\perp h = (iu, \nabla_A u)$ and $|u| \le 1$ that $|\nabla_A u|^2 \ge |\nabla h|^2$. Thus,

$$J(u, A) \ge J_{\cup_i B_i}(u, A) + \frac{1}{2}\int_{\widetilde{\Omega}} |\nabla h|^2 + |h - h_{\rm ex}|^2, \tag{14}$$

where $\widetilde{\Omega} = \Omega \setminus \cup_i B_i$. Also, from Proposition 1,

$$J_{\cup_i B_i}(u, A) \ge \pi \sum_i |d_i| \left(\log\kappa - C\log\log\kappa\right) \tag{15}$$

and, letting $h = h_{\rm ex} h_0 + f$

(16)

$$\frac{1}{2}\int_{\widetilde{\Omega}} |\nabla h|^2 + |h - h_{\rm ex}|^2 \ge {h_{\rm ex}}^2 J_0 + \frac{1}{2}\|f\|^2_{H^1(\widetilde{\Omega})} + h_{\rm ex}\int_{\widetilde{\Omega}} \nabla f . \nabla(h_0 - 1) + f(h_0 - 1).$$

Since the measure of $\cup_i B_i$ is less than $C(\log\kappa)^{-20}$,

$$(17)\qquad {h_{\rm ex}}^2 \int_{\cup_i B_i} (h_0-1)^2 = o(1).$$

Moreover, $f = h_{\rm ex}h_0 - h$ and both h and $h_{\rm ex}h_0$ are bounded in H^1 norm and therefore in L^4 norm for instance, by $C\log\kappa$. Then, by Hölder's inequality,

$$(18)\qquad \left(\int_{\cup_i B_i} f^2\right)^2 \le \left(\sum_i |B_i|\right) \int_{\cup_i B_i} |f|^4 = o(1).$$

Also,

$$(19)\qquad \int_{\cup_i B_i} \nabla f.\nabla(h_0-1) + f(h_0-1) \le C h_{\rm ex} \int_{\cup_i B_i} |\nabla f| + |f| = o(1).$$

From (14) – (19) we get

(20)

$$J(u,A) \ge \pi \sum_i |d_i| \left(\log\kappa - C\log\log\kappa\right) + {h_{\rm ex}}^2 J_0 + \frac{1}{2}\int_{\tilde\Omega} |\nabla f|^2 + \frac{1}{2}\int_\Omega f^2 +$$
$$+ h_{\rm ex}\int_\Omega \nabla f.\nabla(h_0-1) + f(h_0-1).$$

Moreover, noting that $-\Delta f + f = -\Delta h + h$ and using Lemma 2,

$$\int_\Omega \nabla f.\nabla(h_0-1) + f(h_0-1) = 2\pi\sum_i d_i(h_0-1)(a_i) + o(1),$$

which, together with (20), proves (7).

3 Proof of Theorem 2

Lemma 3. *For any $\kappa, h_{\rm ex} > 0$, a minimizer (u,A) of (1) satisfies $J(u,A) \le {h_{\rm ex}}^2 J_0$.*

The proof is immediate, it suffices to let $A_0 = \nabla^\perp h_0$ and to notice that curl $A_0 = \Delta h_0 = h_0$, using (2). Then

$$J(1, h_{\rm ex}A_0) = \frac{1}{2}\int_\Omega |h_{\rm ex}\nabla h_0|^2 + |h_{\rm ex}h_0 - h_{\rm ex}|^2 = {h_{\rm ex}}^2 J_0.$$

The lemma is proved ☐

We also have

Lemma 4. *The set of critical points of the function* ξ_0 *defined in* (3) *is a finite set of points* $\{p_1, \dots, p_k\}$. *In particular the set* Λ *where* ξ_0 *attains its minimum is finite and there exist* $\delta, N > 0$ *such that* $\xi_0(a) \geq \min_\Omega \xi_0 + \delta \, dist\,(a, \Lambda)^N$ *for every* $a \in \Omega$.

Proof. First note that ξ_0 is a real analytic function in Ω, which implies directly the second assertion of the lemma. For the first assertion, first note that by the maximum principle $\nabla\xi_0.\nu > 0$ on $\partial\Omega$. Since $|\nabla\xi_0|^2$ is real analytic in Ω, the set $\mathcal{A} = \{|\nabla\xi_0|^2 = 0\}$ is the union of a finite number of points and analytic arcs. If $\mathcal{A}$ actually contained analytic arcs, these could not have endpoints on the boundary of Ω by the above remark, and there would thus be a piecewise analytic curve γ on which $\nabla\xi_0 = 0$. Since we have assumed Ω to be simply connected, this curve would be the boundary of a nonempty open subset $\omega \subset \Omega$. The function ξ_0 being constant on $\partial\omega$, the maximum principle would imply that ξ_0 is constant in ω and then in Ω by analyticity. Hence a contradiction, and thus $\mathcal{A}$ is the union of finitely many points. ☐

In the rest of this section, (u, A) is a minimizer of (1) and $\{(a_i, d_i)\}_{i\in I}$ are associated vortices. We also assume that $h_{\rm ex} < H_{c_1} + K \log\log\kappa$. Using expansion (7) and Lemma 3 we have

(21)
$$\pi\left(\sum_i |d_i|\right)(\log\kappa - C\log\log\kappa) + 2\pi h_{\rm ex}\sum_i d_i\xi_0(a_i) + \\ + \frac{1}{2}\int_{\Omega\setminus\cup_i B_i} |\nabla(h - h_{\rm ex}h_0)|^2 + \frac{1}{2}\int_\Omega |h - h_{\rm ex}h_0|^2 \leq o(1).$$

We let

(22)
$$D = \sum_{i\in I} |d_i|.$$

Step 1: Vortices have mostly positive degrees. We let

(23)
$$I_- = \{i \in I \mid d_i < 0\}, \quad I_+ = I \setminus I_-, \quad D_- = \sum_{i\in I_-} |d_i|, \quad D_+ = \sum_{i\in I_+} |d_i|,$$

so that $D = D_- + D_+$. The function ξ_0 is negative in Ω thus

$$\sum_i d_i \xi_0(a_i) \geq D_+ \min_\Omega \xi_0 = -D_+ \max_\Omega |\xi_0|.$$

Since $h_{\rm ex} \leq H_{c_1} + C \log\log\kappa$, in view of (3) this implies, replacing in (21)

$$\pi D(\log\kappa - C\log\log\kappa) - D_+(\pi\log\kappa + C\log\log\kappa) \leq o(1)$$

from which we deduce

$$D_- \leq CD\frac{\log\log\kappa}{\log\kappa} + o(1).$$

Hence

$$D_- \leq CD_+\frac{\log\log\kappa}{\log\kappa} + o(1). \tag{24}$$

Step 2: Vortices are mostly close to Λ. Let

$$I_0 = \{i \in I \mid \text{dist}\,(a_i, \Lambda) < (\log\kappa)^{-\frac{1}{2N}}\}, \quad D_0 = \sum_{i\in I_0} |d_i| \tag{25}$$

where N is defined in Lemma 4. If $i \notin I_0$, Lemma 4 yields

$$\xi_0(a_i) \geq -\max_\Omega |\xi_0| + \frac{C}{\sqrt{\log\kappa}}, \tag{26}$$

while, if $i \in I_0$, we have the obvious inequality $\xi_0(a_i) \geq -\max|\xi_0|$. Replacing in (21) yields

$$\pi D(\log\kappa - C\log\log\kappa) - 2\pi(D-D_0)\left(H_{c_1}(\kappa) + C\log\log\kappa\right)\left(\max_\Omega|\xi_0| - \frac{C}{\sqrt{\log\kappa}}\right)$$
$$- 2\pi D_0\left(H_{c_1}(\kappa) + C\log\log\kappa\right)\max_\Omega|\xi_0| \leq o(1).$$

From this, we get

$$-CD\log\log\kappa + C(D-D_0)\sqrt{\log\kappa} \leq o(1)$$

and then

$$D - D_0 \leq CD\frac{\log\log\kappa}{\sqrt{\log\kappa}} + o(1). \tag{27}$$

Step 3. To conclude the proof of Theorem 2, we make use of the last term in the expansion (7). We let $f = h - h_{\text{ex}} h_0$, $\widetilde{\Omega} = \Omega \setminus \cup_{i\in I} B_i$ and try to bound from below the term

$$\int_{\widetilde{\Omega}} |\nabla f|^2 + \int_{\Omega} f^2. \tag{28}$$

Let C_t be a circle of radius t lying entirely in $\widetilde{\Omega}$, i.e. not intersecting the vortex balls, and bounding a ball B_t. In $\widetilde{\Omega}$ we have $|u| = 1$ thus we may write $u = e^{i\varphi}$. The two following equations hold

$$\begin{cases} -\nabla^{\perp} h = \nabla\varphi - A & \text{in } \widetilde{\Omega} \\ -\Delta h_0 + h_0 = 0 & \text{in } \Omega. \end{cases} \tag{29}$$

Combining these equations we get

$$\int_{C_t} \frac{\partial h_0}{\partial \nu} = -\int_{B_t} h_0, \quad \int_{C_t} \frac{\partial h}{\partial \nu} = \int_{C_t} \frac{\partial \varphi}{\partial \tau} - A.\tau = \int_{C_t} \frac{\partial \varphi}{\partial \tau} - \int_{B_t} h,$$

where ν is the inward unit normal vector to B_t and (τ, ν) is a direct frame. We deduce

$$\int_{C_t} \frac{\partial f}{\partial \nu} + \int_{B_t} f = 2\pi d_t,$$

where d_t is the winding number of $u : C_t \to S^1$. Equivalently, d_t is the sum of the degrees of vortices included in B_t. Using the Cauchy-Schwarz inequality on each integral and squaring we find

$$\frac{1}{2} \int_{C_t} \left(\frac{\partial f}{\partial \nu}\right)^2 + \frac{t}{4} \int_{B_t} f^2 \geq \frac{\pi}{2t} {d_t}^2. \tag{30}$$

Now, Λ being the set where the function ξ_0 attains its minimum, we have $\Lambda = \{p_1, \dots, p_n\}$ where p_i belongs to Ω. Let $1 > \alpha > 0$ be such that for all $1 \leq i \neq j \leq n$

$$\alpha < \frac{|p_i - p_j|}{2}, \qquad \alpha < \text{dist}\,(p_i, \partial\Omega).$$

We denote by $C_{i,t}$ the circle with center p_i and radius t and let E be the set of $0 < t < \alpha$ such that for all $1 \leq i \leq n$ we have $C_{i,t} \subset \widetilde{\Omega}$. For $t \in E$, we let

$d_{i,t}$ be the winding number of u restricted to $C_{i,t}$. From (30) we have

$$\frac{1}{2}\int_{C_{i,t}} |\nabla f|^2 + \frac{t}{4}\int_{B_{i,t}} f^2 \geq \frac{\pi}{2t}{d_{i,t}}^2,$$

where $B_{i,t}$ is the ball bounded by $C_{i,t}$. Summing over $1 \leq i \leq n$ we find

$$\frac{1}{2}\int_{\cup_i C_{i,t}} |\nabla f|^2 + \frac{t}{4}\int_{\cup_i B_{i,t}} f^2 \geq \frac{\pi}{2t}\sum_i {d_{i,t}}^2. \tag{31}$$

Now we know from (27) that most of the vortices are close to Λ in a sense precise enough to imply that if $t \geq (\log\kappa)^{\frac{-1}{2N}}$ then

$$\sum_{i=1}^{n} |d_{i,t}| = D(1+o(1)) + o(1).$$

But since from (24) the degrees of the vortices are mostly positive,

$$\sum_{i=1}^{n} d_{i,t} = \sum_{i=1}^{n} |d_{i,t}|(1+o(1)) + o(1).$$

It follows that if $t \geq (\log\kappa)^{\frac{-1}{2N}}$ then

$$\sum_{i=1}^{n} d_{i,t} = D(1+o(1)) + o(1).$$

Using Cauchy-Schwarz's inequality, and noting that the number of points n depends only on Ω, we way write

$$\frac{1}{2}\int_{\cup_i C_{i,t}} |\nabla f|^2 + \frac{t}{4}\int_{\cup_i B_{i,t}} f^2 \geq \frac{C}{t}D^2 + o(1) \tag{32}$$

for all $t \in E$, $t \geq (\log\kappa)^{\frac{-1}{2N}}$. We now wish to integrate this inequality with respect to t. Since the sum of the radii of the vortex balls is less than $(\log\kappa)^{-10}$ and the function $1/t$ is decreasing,we may bound from below the integral of the right-hand side of (32) over $t \in E$

$$\int_{\substack{t\in E\\ t\geq(\log\kappa)^{-\frac{1}{2N}}}} \frac{dt}{t} \geq \int_{(\log\kappa)^{-\frac{1}{2N}}+(\log\kappa)^{-10}}^{\alpha} \frac{dt}{t}.$$

This yields the lower bound

$$\frac{1}{2}\int_{\tilde{\Omega}} |\nabla f|^2 + \frac{1}{2}\int_{\Omega} f^2 \geq CD^2 \log\log\kappa + o(1). \tag{33}$$

Using (33) in expansion (21) together with the fact that $\xi_0(a_i) \leq -\max_\Omega |\xi_0|$ we find

$$-CD\log\log\kappa + CD^2\log\log\kappa \leq o(1),$$

yielding a uniform bound on D. The bounds (24) and (27) prove the other assertions of Theorem 2.

4 Proof of Theorem 1

We have the identity

$$J_\kappa(u,A) = F_\kappa(u) + \frac{1}{2}\int_\Omega |u|^2|A|^2 - 2\left((A.\nabla)u, iu\right) + |h - h_{\mathrm{ex}}|^2. \tag{34}$$

The minimum of $J_\kappa(1,A)$ is achieved for $A = h_{\mathrm{ex}}A_0$ and is exactly ${h_{\mathrm{ex}}}^2 J_0$ (see [S1]). This yields

$$\frac{1}{2}\int_\Omega |A|^2 + |h - h_{\mathrm{ex}}|^2 \geq {h_{\mathrm{ex}}}^2 J_0.$$

Combining with the a-priori bound $J_\kappa(u,A) \leq {h_{\mathrm{ex}}}^2 J_0$ we find

$$F_\kappa(u) \leq \frac{1}{2}\int_\Omega \left(1 - |u|^2\right)|A|^2 + 2\left((A.\nabla)u, iu\right). \tag{35}$$

Assume we are working in the Coulomb gauge. Then, the H^2 norm of A is less then Ch_{ex} while the L^2 norm of $1-|u|^2$ is less than $Ch_{\mathrm{ex}}/\kappa^2$. This implies that the first term in the right-hand side of (35) goes to zero as $\kappa \to +\infty$. It remains to prove that

$$\int_\Omega \left((A.\nabla)u, iu\right) \leq C\log\kappa. \tag{36}$$

We proceed as follows. Denote by $\{B_i = B(a_i, r_i)\}_{i\in I}$ the vortex balls of (u,A). Using Lemma 1 we may assume again that $|u| = 1$ outside $\cup_i B_i$.

Moreover, since we are in the Coulomb gauge, we may write $A = \nabla^\perp \zeta$, where $\zeta : \Omega \to \mathbb{R}$ is zero on $\partial\Omega$. Furthermore,

$$\|\zeta\|_{H^3(\Omega)} \leq C h_{\text{ex}}. \tag{37}$$

We divide the integral of (36) in two. On the one hand, using Hölder's inequality

$$\int_{\cup_i B_i} ((A.\nabla)u, iu) \leq |\cup_i B_i|^{1/4} \|A\|_{L^4(\Omega)} \|u\|_{H^1(\Omega)} = o(1) \tag{38}$$

while on the other hand integration by parts yields, writing $u = e^{i\varphi}$ on $\widetilde{\Omega}$,

$$\int_{\widetilde{\Omega}} ((A.\nabla)u, iu) = \int_{\widetilde{\Omega}} \nabla^\perp \zeta . \nabla \varphi = \sum_{i \in I} \int_{\partial B_i} \zeta \frac{\partial \varphi}{\partial \tau}. \tag{39}$$

Arguing as in [SS1], Lemma II.3 or [ASS], Lemma II.1, we can prove that

$$\sum_{i \in I} \int_{\partial B_i} \zeta \frac{\partial \varphi}{\partial \tau} = \sum_{i \in I} 2\pi d_i \zeta(a_i) + o(1).$$

We conclude by noting that $\|\zeta\|_\infty \leq C h_{\text{ex}}$ and by using the uniform bound we have on $\sum_{i \in I} |d_i|$ to get

$$\int_{\widetilde{\Omega}} ((A.\nabla)u, iu) \leq C h_{\text{ex}} \leq C \log \kappa.$$

The theorem is proved.

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