Limiting Vorticities for the Ginzburg-Landau Equations

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Abstract

We study the asymptotic limit of solutions of the Ginzburg-Landau equations in two dimensions with or without magnetic field. We first study the Ginzburg-Landau system with magnetic field describing a superconductor in an applied magnetic field, in the "London limit" of a Ginzburg-Landau parameter κ tending to infinity. We examine the asymptotic behavior of the "vorticity-measures" associated to the vortices of the solution, and prove that passing to the limit in the equations (via the "stress-energy tensor") yields a criticality condition on the limiting measures. This condition allows to describe the possible locations and densities of the vortices. We establish analogue results for the Ginzburg-Landau equation without magnetic field.

I Introduction

I.1 The full Ginzburg-Landau equations

We are interested in studying the asymptotic limit of the following Ginzburg-Landau equations of superconductivity, referred to as (G.L):

$$-\nabla_A^2 u = \kappa^2 u(1-|u|^2) \quad \text{in } \Omega \tag{I.1}$$

$$-\nabla^\perp h = (iu, \nabla_A u) \quad \text{in } \Omega \tag{I.2}$$

$$h = h_{\text{ex}} \quad \text{on } \partial\Omega \tag{I.3}$$

$$(\nabla u - iAu)\cdot \nu = 0 \quad \text{on } \partial\Omega. \tag{I.4}$$

The solutions of this system are the critical points of the following Ginzburg-Landau energy :

$$J(u,A) = \frac{1}{2}\int_\Omega |\nabla_A u|^2 + \frac{\kappa^2}{2}\left(1-|u|^2\right)^2 + |h-h_{\text{ex}}|^2. \tag{I.5}$$

This energy-functional was introduced by the physicists Ginzburg and Landau in the 50s as a model for superconductivity. Here, we have performed some rescalings of the original functional, and the unit length is the "penetration depth". The simplification made here, which is common, consists in restricting to a two-dimensional model, corresponding to a infinite cylindrical domain of section $\Omega \subset \mathbb{R}^2$ (smooth and *simply connected*), when the applied field is parallel to the axis of the cylinder, and all the quantities are translation-invariant. Here, κ is a dimensionless constant (the Ginzburg-Landau parameter) depending only on characteristic lengths of the material and of temperature. $h_{\text{ex}} > 0$ is the intensity of the applied magnetic field (it is just a real parameter), $A : \Omega \mapsto \mathbb{R}^2$ is the vector-potential, and the induced magnetic field in the material is the real-valued function $h = \operatorname{curl} A = -\partial_2 A_1 + \partial_1 A_2$. $\nabla_A = \nabla - iA$ is the associated covariant derivative. The complex-valued function u is called the "order-parameter". It is a pseudo-wave function that indicates the local state of the material. There can be essentially two phases in a superconductor: $|u(x)| \simeq 0$ is the normal phase, $|u(x)| \simeq 1$, the superconducting phase. The Ginzburg-Landau model was based on Landau's theory of phase-transitions. Since then, the model has been justified by the microscopic theory of Bardeen-Cooper-Schrieffer (BCS theory). $|u(x)|^2$ is then understood as the local density of superconducting electron pairs, called "Cooper pairs", responsible for the superconductivity phenomenon. For a more detailed physical presentation, we refer to the physics literature [T, DeG] (one can also see our previous papers [SS1, SS2, SS3]).

We are interested in the asymptotics of a large κ which corresponds to "extreme-type II" materials. Thus, we set

$$\varepsilon = \frac{1}{\kappa}$$

and will let ε tend to 0, while sometimes writing J_ε.
The Ginzburg-Landau equations and functional are invariant under $U(1)$-gauge-transformations

(it is an abelian gauge-theory) of the type :

$$\text{(I.6)} \qquad \begin{cases} u \mapsto ue^{i\Phi} \\ A \mapsto A + \nabla\Phi \end{cases}$$

The physically relevant quantities are those that are gauge-invariant, such as the energy J, $|u|$, h, etc... This gauge-invariance can be "frozen" by choosing a gauge, for example the Coulomb gauge

$$\text{(I.7)} \qquad \begin{cases} \text{div } A = 0 & \text{in } \Omega \\ A \cdot \nu = 0 & \text{on } \partial\Omega \end{cases}$$

In previous papers [SS1, SS2, SS3] (see also [S1, S2, S3]), we already studied the family of energy-functionals (I.5) as $\varepsilon \to 0$, but we focused on *global minimizers* of the energy. We proved in [SS2, SS3] their convergence to minimizers of a limiting energy, in a suitable regime for the applied field. Here, we wish to address the question of the behavior of *critical points* in general, i.e. the asymptotic behavior, as $\varepsilon \to 0$, of solutions of the Ginzburg-Landau system (I.1)—(I.4), that are not necessarily global or local minimizers. We will restrict to families which satisfy reasonable energy bounds.

Before stating our hypotheses, let us briefly sum up the known results. In the regimes we are interested in, solutions are mainly characterized by the existence (or not) of *vortices*. There have been many mathematical studies of maps with vortices in the Ginzburg-Landau type framework, particularly for the Ginzburg-Landau equation without magnetic field that we examine in Section I.4. Let us say the first main study of vortices for the Ginzburg-Landau equation without magnetic field was the book by Bethuel, Brezis and Hélein [BBH], and this kind of analysis was first adapted to Ginzburg-Landau with magnetic field in the paper of Bethuel and Rivière [BR] (in which a fixed Dirichlet boundary condition was imposed instead of the natural Neumann condition (I.4)).

Vortices are isolated zeros of the complex-valued order parameter u, carrying a nonzero integer-valued winding degree (the topological degree of the map $u/|u|$ around a zero), called the degree of the vortex. In [S1, SS1], we proved that there exists a value H_{c_1} of h_{ex} named by physicists "first critical field", such that for $h_{\text{ex}} \le H_{c_1}$, the only global minimizer of the energy is the unique vortex-free solution of (G.L), and for $h_{\text{ex}} \ge H_{c_1}$, global minimizers of J have vortices. We obtained (see [S1, SS1]), the asymptotic expansion

$$\text{(I.8)} \qquad H_{c_1} = \frac{|\log \varepsilon|}{2 \max(1 - h_0)} + O(1),$$

where h_0 is the solution of the London equation

$$\text{(I.9)} \qquad \begin{cases} -\Delta h_0 + h_0 = 0 & \text{in } \Omega \\ h_0 = 1 & \text{on } \partial\Omega. \end{cases}$$

There exists a second critical field $H_{c_3} = O(\frac{1}{\varepsilon^2})$ above which the only global minimizer (and maybe only critical point) is the normal solution $(u \equiv 0, h \equiv h_{\text{ex}})$ (see [GP]). Between H_{c_1}

and H_{c_3}, it is the "mixed state" where minimizers of the energy exhibit vortices, surrounded by superconducting phase $|u| \sim 1$. These vortices are more and more numerous as $h_{\rm ex}$ increases and tend to arrange in triangular arrays ("Abrikosov lattices"). We proved in [SS2] and [SS3] that global minimizers (for $H_{c_1} \le h_{\rm ex} \ll H_{c_3}$) converge to the minimizer of a limiting energy *depending essentially on the ratio* $\frac{h_{\rm ex}}{|\log\,\varepsilon|}$. We were able to extract a vortex-density measure associated to these minimizers, defined by $\frac{2\pi}{h_{\rm ex}}\sum_i d_i\delta_{a_i}$, a_i being the vortex-centers, and d_i the associated degrees. These measures converge to a uniform density supported in a subdomain of Ω, (which depends again on the ratio $\frac{h_{\rm ex}}{|\log\,\varepsilon|}$). The total vorticity is thus proved to be proportional to the applied field $h_{\rm ex}$ in this mixed state.

I.2 Main results

Let us now describe our assumptions. Let $\varepsilon_n \to 0$, and let (u_n, A_n) denote a sequence of critical points of J_{ε_n}, for an external field $h_{\rm ex}$. h_n will denote $\operatorname{curl} A_n$. Of course, for the reasons we just pointed out, we need to let $h_{\rm ex}$ vary with ε (one can consider it as a function of ε). Here, we deal mainly with intermediate fields $h_{\rm ex} \le C|\log\,\varepsilon|$. $|\log\,\varepsilon|$ is a relevant order of magnitude of $h_{\rm ex}$, it is the order of H_{c_1}, and it allows large numbers of vortices. We could easily extend our results to larger applied fields (see Remark I.1). We make the following hypotheses:

$$h_{\rm ex} \le C|\log\,\varepsilon| \tag{I.10}$$

$$J_\varepsilon(u_\varepsilon, A_\varepsilon) \le C h_{\rm ex}^2. \tag{I.11}$$

As long as $h_{\rm ex} \le \frac{C}{\varepsilon^\beta}$ with β small enough, we can adjust the ball-construction that we did in [SS2] to the present needs and obtain the following result :

Proposition I.1 *If* $h_{\rm ex} \le \frac{C}{\varepsilon^\beta}$*, there exists* ε_0 *such that if* $\varepsilon < \varepsilon_0$ *and* $(u_\varepsilon, A_\varepsilon)$ *satisfies (I.11) and* $\|\nabla_{A_\varepsilon} u_\varepsilon\|_{L^\infty(\Omega)} \le \frac{C}{\varepsilon}$*, there exists a family of balls (depending on* ε*)* $(B_i)_{i\in I_\varepsilon} = (B(a_i, r_i))_{i\in I_\varepsilon}$ *satisfying:*

$$|u_\varepsilon| \ge 1 - \min\left(\frac{1}{h_{\rm ex}^4}, \frac{1}{|\log\,\varepsilon|^4}\right) \quad \textit{in } \Omega\backslash \bigcup_{i\in I_\varepsilon} B(a_i, r_i). \tag{I.12}$$

$$\textit{Card } I_\varepsilon \le C h_{\rm ex}^{20} \tag{I.13}$$

$$r_i \ge C\varepsilon h_{\rm ex}^4 \ \textit{and} \ \sum_{i\in I_\varepsilon} r_i \le \varepsilon^{\frac{1}{2}} \tag{I.14}$$

$$\exists C > 0 \quad \frac{1}{2}\int_{B_i} |\nabla_{A_\varepsilon} u_\varepsilon|^2 \ge C|d_i||\log\,\varepsilon|(1 - o(1)), \tag{I.15}$$

where $d_i = deg(\frac{u_\varepsilon}{|u_\varepsilon|}, \partial B_i)$ *if* $\overline{B_i} \subset \Omega$ *and 0 otherwise.*

Thus, from (I.15), if $h_{\rm ex} \le C|\log \varepsilon|$,

(I.16)
$$N_\varepsilon := 2\pi \sum_{i\in I_\varepsilon} |d_i| \le C h_{\rm ex}.$$

For any such set of balls, we can associate to u_ε the vorticity-measure (an object we had already used in [SS3])

(I.17)
$$\mu_\varepsilon = \frac{2\pi \sum_i d_i \delta_{a_i}}{h_{\rm ex}},$$

which remains bounded as $\varepsilon \to 0$.

Let us now state our main result. We denote by $H^1_1(\Omega)$ the set $\{f \in H^1(\Omega), f - 1 \in H^1_0(\Omega)\}$, and $\mathcal{M}(\Omega)$ the space of bounded Radon measures on Ω.

Theorem 1 *A. Convergence*
Let $\varepsilon_n \to 0$, and (u_n, A_n) be critical points of J_{ε_n} (or, equivalently, solutions of (I.1)—(I.4)), with hypotheses (I.10)-(I.11) satisfied, then, up to extraction of a subsequence,

$$\frac{h_n}{h_{\rm ex}} \rightharpoonup h_\infty \quad \text{in } H^1_1(\Omega),$$

and strongly in $\cap_{p<2} W^{1,p}(\Omega)$.

For any (a_i, d_i) satisfying the results of Proposition I.1, up to extraction

$$\mu_n := \frac{2\pi \sum_i d_i \delta_{a_i}}{h_{\rm ex}} \rightharpoonup \mu_\infty := -\Delta h_\infty + h_\infty \qquad \text{weakly in } \mathcal{M}(\Omega) \text{ i.e. in } (C^0_0(\Omega))'.$$

B. Properties of the limit
h_∞ is stationary with respect to inner variations for the functional

(I.18)
$$\mathcal{L}(h) = \frac{1}{2} \int_\Omega |\nabla h|^2 + h^2$$

defined over $H^1_1(\Omega)$, meaning that for any smooth compactly supported vector field $X : \Omega \to \mathbb{R}^2$, the derivative at $t = 0$ of $L(h_t)$ is zero, where $h_t(x) = h(x + tX(x))$. This is equivalent to

(I.19)
$$\forall i = 1,2, \ \text{div } L_{ij} = 0, \qquad L_{ij} = \frac{h_\infty^2}{2}\delta_{ij} + \frac{1}{2}\begin{pmatrix} (\partial_2 h_\infty)^2 - (\partial_1 h_\infty)^2 & -2\partial_1 h_\infty \partial_2 h_\infty \\ -2\partial_1 h_\infty \partial_2 h_\infty & (\partial_1 h_\infty)^2 - (\partial_2 h_\infty)^2 \end{pmatrix}.$$

A consequence of (I.19) *is that $|\nabla h_\infty| \in C^0(\Omega)$. If we assume moreover that $\nabla h_\infty \in C^0(\Omega)$ and $|\nabla h_\infty| \in BV(\Omega)$ (this is the case if $\mu_\infty \in L^p(\Omega), p > 2$ for example), then,*

(I.20)
$$\begin{cases} \mu_\infty \nabla h_\infty = 0 \\ h_\infty \in C^{1,\alpha}(\Omega) (\forall \alpha < 1) \\ 0 \le h_\infty \le 1 \\ \mu_\infty = h_\infty \mathbf{1}_{|\nabla h_\infty| = 0} \ \textit{thus } \mu_\infty \textit{ is a nonegative } L^\infty \textit{ function.} \end{cases}$$

This theorem provides an interesting result mostly when N_ε is of the order of $h_{\rm ex}$. If it is not the case, then, we can get a more precise result by looking at a better-suited normalization of the vorticity-measure: we define

$$\nu_n := \frac{2\pi \sum_i d_i \delta_{a_i}}{N_{\varepsilon_n}} = \frac{2\pi \sum_i d_i \delta_{a_i}}{2\pi \sum_i |d_i|}. \tag{I.21}$$

ν_n will behave like μ_n when $\frac{h_{\rm ex}}{C} \le N_n \le C h_{\rm ex}$. We also set

$$f_n = h_n - h_{\rm ex} h_0, \tag{I.22}$$

where h_0 is the solution of (I.9).

Theorem 2 *Let $\varepsilon_n \to 0$, and (u_n, A_n) be critical points of J_{ε_n} (or, equivalently, solutions of (I.1)—(I.4)), with hypotheses (I.10)-(I.11) satisfied. Up to extraction of a subsequence,*
- if $N_{\varepsilon_n} \gg h_{\rm ex}$, then $\nu_n \rightharpoonup 0$ weakly in $\mathcal{M}(\Omega)$.
- if $N_{\varepsilon_n} \ll h_{\rm ex}$, then

$$\frac{h_n}{h_{\rm ex}} \rightharpoonup h_0 \quad \text{in } H^1_1(\Omega)$$

$$\frac{f_n}{N_n} \longrightarrow f_\infty \quad \text{in } W_0^{1,p}(\Omega)\ \forall p < 2$$

$$\nu_n := \frac{2\pi \sum_i d_i \delta_{a_i}}{N_n} \rightharpoonup \nu_\infty = -\Delta f_\infty + f_\infty \text{ weakly in } \mathcal{M}(\Omega).$$

In addition,

$$\nu_\infty \nabla h_0 = 0, \tag{I.23}$$

i.e. ν_∞ is supported in the set of critical points of h_0 which is a finite set of points (see [SS5]), and ν_∞ is a finite combination of Dirac masses at these points.

Remark I.1: If $|\log \varepsilon| \ll h_{\rm ex} \le \frac{1}{\varepsilon^\beta}$ where β is some small power, then the natural bound (I.11) on the energy still allows us to construct vortex-balls and Theorem 2 remains valid. But it does not ensure that $\sum_i |d_i| \le C h_{\rm ex}$ i.e. that the vorticity measures μ_ε remain bounded. If we add this as a hypothesis, then our proofs remain valid, and the result of Theorem 1 still holds for these larger fields. More generally, it seems reasonable to believe that, for all fields $h_{\rm ex} \ll \frac{1}{\varepsilon^2}$, if (I.11) is satisfied, then again the same results should hold.

I.3 Interpretation

These theorems provide a general result on the behavior of sequences of solutions of (G.L), under the assumptions (I.10)-(I.11), and these results include in particular, as they should, the case of global minimizers studied in [SS3]. Indeed, let us recall the main results of [SS3]. We assumed that $\lambda = \lim_{\varepsilon \to 0} \frac{|\log \varepsilon|}{h_{\rm ex}}$ exists and is finite and, if $\lambda = 0$, $h_{\rm ex} \ll 1/\varepsilon^2$, then, considered, for every ε, $(u_\varepsilon, A_\varepsilon)$ minimizing J, and $h_\varepsilon = \operatorname{curl} A_\varepsilon$ the associated magnetic field. We proved the existence of balls which are as in Proposition I.1, thus getting vortices (a_i, d_i), and defining the vorticity-measure μ_ε as in (I.17).

Theorem ([SS3]) *If $(u_\varepsilon, A_\varepsilon)$ minimizes J_ε and $\lambda < \infty$, then, as $\varepsilon \to 0$,*

$$\frac{h_\varepsilon}{h_{\text{ex}}} \underset{\varepsilon\to 0}{\rightharpoonup} h_* \quad \textit{weakly in } H^1(\Omega), \qquad \frac{h_\varepsilon}{h_{\text{ex}}} \underset{\varepsilon\to 0}{\longrightarrow} h_* \quad \textit{strongly in } W^{1,p}(\Omega), \forall p < 2,$$

where h_* is the unique minimizer of

$$E(f) = \frac{\lambda}{2}\int_\Omega |-\Delta f + f| + \frac{1}{2}\int_\Omega |\nabla f|^2 + |f-1|^2,$$

and the solution of the free-boundary problem :

$$(\mathrm{P}) \begin{cases} h_* \in H^1_1(\Omega) \\ h_* \geq 1 - \dfrac{\lambda}{2} \text{ in } \Omega \\ \forall v \in H^1_1(\Omega) \text{ such that } v \geq 1 - \dfrac{\lambda}{2}, \quad \displaystyle\int_\Omega (-\Delta h_* + h_*)\,(v - h_*) \geq 0 \end{cases}$$

On the other hand

$$\mu_\varepsilon \rightharpoonup \mu_* = -\Delta h_* + h_* \quad \textit{in } \mathcal{M}(\Omega),$$

and, in addition,

$$\lim_{\varepsilon\to 0} \frac{J(u_\varepsilon, A_\varepsilon)}{h_{\text{ex}}^2} = E(h_*) = \frac{\lambda}{2}\int_\Omega |\mu_*| + \frac{1}{2}\int_\Omega |\nabla h_*|^2 + |h_* - 1|^2.$$

The connection, which turns out to be a duality, between the minimization of $E(f)$ and the free boundary problem (P) is made clear in the recent work of H.Brezis and S.Serfaty [BS]. In this theorem, we derived a limiting energy E (Γ-limit of $\frac{J_\varepsilon}{h_{\text{ex}}^2}$), and proved that minimizers of J_ε converge to minimizers of E. The necessary condition derived in our Theorem 1, that h_∞ is stationary with respect to inner variations for $\mathcal{L}$, is the equivalent condition for limits of critical points. ($\mathcal{L}$ can thus also be seen as a limiting energy for $\frac{J}{h_{\text{ex}}}$.)

Let us return to the theorem of [SS3] we just quoted. A free-boundary problem (P) arose, the associated free-boundary is the boundary of $\omega_\lambda := \{x \in \Omega, h_* = 1 - \frac{\lambda}{2}\}$, which is exactly the support of the limiting vorticity measure $\mu_* = \left(1 - \frac{\lambda}{2}\right)\mathbf{1}_{\omega_\lambda}$. When $\partial\omega_\lambda$ is regular (which is generically true, see [BM]), problem (P) can be rewritten as

$$(\mathrm{I}.24) \qquad \begin{cases} -\Delta h_* + h_* = 0 & \text{in } \Omega\backslash\omega_\lambda \\ h_* = 1 - \dfrac{\lambda}{2} & \text{in } \omega_\lambda \\ \dfrac{\partial h_*}{\partial n} = 0 & \text{on } \partial\omega_\lambda \\ h_* = 1 & \text{on } \partial\Omega. \end{cases}$$

One can picture the domain Ω as split into two regions: a central region ω_λ in which the vortices are scattered with a limiting uniform density, and where the limiting field h_* is constant; and an outer region in which there are essentially no vortices, and h_* satisfies $-\Delta h_* + h_* = 0$. The size of ω_λ depends on λ, hence on the value of the applied field $h_{\rm ex}$. $\omega_\lambda = \varnothing$ for $h_{\rm ex} < H_{c_1}$, then is a point, and then inflates as $h_{\rm ex}$ increases for $h_{\rm ex} > H_{c_1}$, until $\omega_0 = \Omega$. For $\lim_{\varepsilon\to 0} \frac{h_{\rm ex}}{|\log\ \varepsilon|} > \lim_{\varepsilon\to 0} \frac{H_{c_1}}{|\log\ \varepsilon|}$, N_ε is of the order of $h_{\rm ex}$. For more details, we refer to [SS3]. Now this result fits into the result of Theorem 1, taking $h_\infty = h_*$, and $\mu_\infty = \mu_*$. Indeed, h_* and μ_* do satisfy (I.20) since $\{x \in \Omega, |\nabla h_*| = 0\} = \omega_\lambda = Supp\ \mu_*$. The case described in Theorem 1 is of course more general since it can account for more general supports for the limiting measure μ_∞.

In the case of (nonminimizing) critical points, we see from Theorems 1 and 2 that three cases can be distinguished : either $N_\varepsilon \simeq C h_{\rm ex}$, or $N_\varepsilon \gg h_{\rm ex}$, or $N_\varepsilon \ll h_{\rm ex}$. If $N_\varepsilon \gg h_{\rm ex}$, $\nu_\varepsilon \rightharpoonup 0$, thus there should be many vortex dipoles (pairs of positive degree-negative degree vortices close to each other that sort of annihilate) and removing such dipoles, we should be led back to one of the other two cases. If $\frac{h_{\rm ex}}{C} \le N_\varepsilon \le C h_{\rm ex}$, then this allows for vortex regions, i.e. μ_∞ can be a density distributed over a subregion of Ω (with nonzero volume), as in the case of global minimizers. On the contrary, when $N_\varepsilon \ll h_{\rm ex}$, ν_ε can only converge to a sum of Dirac masses at the critical points of h_0 (this set is a finite set of points as seen in [SS5]), i.e. the vortex points all converge to the same few points depending only on the geometry of the domain. (If, for example, the domain is convex, then the set of critical points of h_0 is reduced to its unique minimum point.) For example, if $h_{\rm ex} = C|\log\ \varepsilon|$ and $N_\varepsilon = O(1)$ (finite number of vortices), then as $\varepsilon \to 0$, they all converge to these points. This was already proved for global minimizers : $N_\varepsilon \ll h_{\rm ex}$ when $h_{\rm ex} \le H_{c_1} + O(\log\ |\log\ \varepsilon|)$ (proved in [SS5]), and in this case, the vortices of the global minimizers converge to the finite set of points of minimum of h_0, which we denoted by Λ, as we decribed in [S1, S3, SS1, SS5]. In [S3], some local minimizers of the energy with $N_\varepsilon \le O(1)$ were exhibited and studied. They also corresponded to ν_∞ supported in Λ.

Observe that the result of Theorem 2 excludes many possibilities, such as the case of a two-dimensional lattice of N_ε vortices, $N_\varepsilon \ll h_{\rm ex}$, filling a subregion of Ω. In other words, if there are N_ε vortices with mutual distances $\ge \frac{1}{\sqrt{N_\varepsilon}}$, then necessarily $N_\varepsilon \ge C h_{\rm ex}$.

The relations (I.19), (I.20) tell us that the limiting vorticity measure and the limiting field satisfy some necessary criticality conditions. Let us focus on the relation $\mu_\infty \nabla h_\infty$ in (I.20) or (I.23). (These relations should be compared to a result of [BBH], see Section I.4.) In the case of Theorem 1, when we get a density, or $\mu_\infty \ll dx$, (I.20) can also be understood as : "h_∞ is constant on each connected component of the support of μ_∞." Denoting by ω the support of μ_∞, a model case for (I.20) is that of

$$\text{(I.25)} \qquad \begin{cases} -\Delta h_\infty + h_\infty = 0 & \text{in } \Omega\backslash\omega \\ h_\infty = c & \text{in } \omega \\ \dfrac{\partial h_\infty}{\partial n} = 0 & \text{on } \partial\omega \\ h_\infty = 1 & \text{on } \partial\Omega. \end{cases}$$

This is valid when ω is connected, and smooth enough.

Already, the difference in (I.25) with the case of minimizers (I.24), is that all constant values of h_∞ and μ_∞ in ω can be allowed by (I.20), and this is totally independent of h_{ex}. As was the result of [SS3], this is very reminiscent of the formal model established by J. Chapman, J. Rubinstein and M. Schatzman in [CRS]. In fact, the system of equations they propose in the steady-state case is exactly (I.25).

We conjecture that all solutions of this system (for all constant values c in a certain interval of $[0,1]$), can be achieved as limits of sequences of critical points for any applied fields. We already know from the result of [SS3], that all the solutions of (I.25) with $2\max(1-h_0)+1 \le c \le 1$ are achieved as limits of minimizers with $\lambda = 2(1-c)$. More generally, the question of knowing which h_∞ solving (I.20) can actually be achieved as limits of sequences of critical points of J_ε and for which $h_{\text{ex}}(\varepsilon)$, is an interesting *open problem.*

There are cases in (I.25) where ω is not connected (for certain nonconvex domains, it is already the case for minimizers). In this case, we can expect as many constants as there are connected components of ω. Let us also point out that the vortex-free case is included. In [S3] the existence of vortex-free solutions, for the same regime of applied fields, even for $h_{\text{ex}} \ge H_{c_1}$, was proved. Then the limit of $\frac{h}{h_{\text{ex}}}$ is h_0, and this case is included in (I.25) and (I.20) with $\omega = \varnothing$. In order to include it, we had to allow energies of the order of h_{ex}^2 as we did in hypothesis (I.11).

In the case of Theorem 1, there is unfortunately no way to ensure that $\mu_\infty \ll dx$ is true, unless we know that $\nabla h_\infty \in C^0$ and $|\nabla h_\infty| \in BV$; μ_∞ could be a measure concentrated on points or more likely lines (since it has to belong to H^{-1}). Yet, our result only asserts that $|\nabla h_\infty|$ is continuous, but not necessarily ∇h_∞. There are counter-examples of (h_∞, μ_∞) satisfying these conclusions with ∇h_∞ discontinuous, thus without $\mu_\infty \ll dx$. Here is a counter-example. Let us solve

$$\begin{cases} -\Delta h_1 + h_1 = 0 & \text{in } B(0,R_1) \\ h_1 = 1 & \text{on } \partial B(0,R_1). \end{cases}$$

$$\begin{cases} -\Delta h_2 + h_2 = 0 & \text{in } B(0,R_2)\backslash\overline{B(0,R_1)} \\ h_1 = 1 & \text{on } \partial B(0,R_1) \cup \partial B(0,R_2). \end{cases}$$

Both functions are radial, and we can adjust R_1 and R_2 in such a way that $\frac{\partial h_1}{\partial r}(R_1) = -\frac{\partial h_2}{\partial r}(R_1)$. Now, we can define h as h_1 in $B(0,R_1)$ and h_2 in $B(0,R_2)\backslash B(0,R_1)$, h is in $H^1(B(0,R_2))$, ∇h is discontinuous on $\partial B(0,R_1)$ while $|\nabla h|$ remains continuous. We can check that (I.19) holds because $\left(\frac{\partial h}{\partial n}\right)^2$ is continuous (see the proof of Lemma IV.1). $\mu = -\Delta h + h$ is a positive measure supported on $\partial B(0,R_1)$, thus $\mu \ll dx$ does not hold. Nothing allows us to exclude that there are sequences of critical points converging to such limiting configurations. They would correspond to solutions with vortices of positive degrees concentrating along the circle $\partial B(0,R_1)$. One could imagine many other counter-examples which would not satisfy $\mu \ll dx$, implying ∇h_∞ discontinuous.

I.4 Ginzburg-Landau without magnetic field

The method that we just exposed for the Ginzburg-Landau equations of superconductivity actually also applies to the "simpler" and well-known Ginzburg-Landau equation, i.e. the one without magnetic field. Let us recall a few facts about it. The Ginzburg-Landau functional (without magnetic field) defined over $H^1(\Omega, \mathbb{C})$ is

$$E_\varepsilon(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2, \tag{I.26}$$

and the associated Euler-Lagrange equation is the Ginzburg-Landau equation

$$-\Delta u = \frac{u}{\varepsilon^2}(1 - |u|^2). \tag{I.27}$$

There has been intensive studies of this equation in two-dimensional domains and in the asymptotics of $\varepsilon \to 0$. The asymptotic behavior of minimizers and critical points for a fixed Dirichlet boundary condition was totally described by Bethuel, Brezis and Hélein in [BBH]. The boundary condition is a map $g : \partial\Omega \to S^1$ which has a topological degree (or winding number) d. When $d \neq 0$ (say $d > 0$), solutions of (I.27) have vortices (exactly $|d|$ vortices for the minimizers).

In view of the study of the Ginzburg-Landau equations of superconductivity, it is interesting to ask what happens when the number of vortices, or the total degree, is unbounded when $\varepsilon \to 0$. A first result in that direction was obtained by Sandier and Soret in [SaSo], for the discrete problem on the vortex-points : they consider minimizers of the renormalized energy with a boundary condition that has winding number $n \to \infty$ on a ball, and they prove that the points all go to the boundary of the domain.

A recent paper of Jerrard and Soner [JS2] investigated the Gamma-limit of the Ginzburg-Landau functionals E_ε as $\varepsilon \to 0$, allowing large energies and a total degree that can go to infinity as $\varepsilon \to 0$. They got the analogue results as those we obtained in [SS3] i.e. derived a limiting cost of the vortices, with upper and lower bounds for arbitrary sequences, i.e. not necessarily critical points. What we are able to do here is to derive a characterization of the limiting vorticity for *solutions* of (I.27) using the fact that they are stationary with respect to inner variations for the energy.

For the case of a fixed Dirichlet boundary condition, this was done in Theorem X.5 of [BBH], where a family of critical points u_ε is shown to converge (up to extraction) to a limiting S^1-valued map u_* with vortices a_i, and total degree d, which can be written as

$$u_*(x) = \left(\frac{x - a_1}{|x - a_1|}\right)^{d_1} \cdots \left(\frac{x - a_n}{|x - a_n|}\right)^{d_n} e^{i\varphi(x)},$$

with φ an harmonic function. The limiting vortex-points a_i are not located arbitrarily, they are necessarily critical points of a function of their locations, called the "renormalized energy" and satisfy the "vanishing gradient property" :

$$\forall j,\ \nabla\varphi_j(a_j) = 0, \tag{I.28}$$

where φ_j is defined by $u_*(x)\left(\frac{|x-a_j|}{x-a_j}\right)^{d_j} = e^{i\varphi_j(x)}$. This fact was also formally derived in the case of a single vortex by Fife and Peletier in [FP], and was proved in [BBH]. It corresponds to the fact that u_ε is stationary for E_ε with respect to inner variations, and it was derived by passing to the limit in an equation on the Hopf-differential of u_ε, which amounts to what we do, i.e. passing to the limit in the stress-energy tensor.

The results of Theorem 1 and 2, $\mu_\infty \nabla h_\infty = 0$ or $\nu_\infty \nabla h_0 = 0$ can thus be seen as the analogue of (I.28), and corresponds to the fact that the vortex locations are critical with respect to inner variations in Ω.

Here, the hypotheses are the following: assume we have a family u_ε of solutions of (I.27), for which we do not impose any boundary condition. Assume

$$E_\varepsilon(u_\varepsilon) = \frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 \le \frac{1}{\varepsilon^\beta} \tag{I.29}$$

for some $\beta > 0$ small enough, and $\|\nabla u_\varepsilon\|_{L^\infty(\Omega)} \le \frac{C}{\varepsilon}$. (This is true by a priori estimate at least in each compact of Ω, if it does not hold up to the boundary then one needs to work in any subregion of Ω). Then, writing formally $u_\varepsilon = \rho_\varepsilon e^{i\varphi_\varepsilon}$, it is well-known (see [BBH]) that (I.27) implies (by projecting the equation on iu_ε) that

$$\text{div } (\rho_\varepsilon^2 \nabla \varphi_\varepsilon) = 0. \tag{I.30}$$

Using Poincaré's lemma, Ω being simply connected, we can find a $U_\varepsilon \in H^1(\Omega, \mathbb{R})$ such that

$$\begin{cases} \nabla^\perp U_\varepsilon = \rho_\varepsilon^2 \nabla\varphi_\varepsilon = (iu_\varepsilon, \nabla u_\varepsilon) \\ \int_\Omega U_\varepsilon = 0 \end{cases} \tag{I.31}$$

Thus

$$\Delta U_\varepsilon = \text{curl}\,(iu_\varepsilon, \nabla u_\varepsilon)$$

is the *Jacobian* determinant of u_ε (whose role has been emphasized in [JS1, JS2, ABO]), which basically gives the vorticity of the map u_ε. Thus, U_ε will play the same role as the magnetic field h in the case with magnetic field. $\frac{U_\varepsilon}{\sqrt{E_\varepsilon(u_\varepsilon)}}$ is bounded in $H^1(\Omega)$ and thus has a weak limit U (up to extraction) that we wish to characterize, then ΔU will be the limiting vorticity (in this specific normalization). The result of Proposition I.1 remains true (replacing $(u_\varepsilon, A_\varepsilon)$ by u_ε, $\nabla_{A_\varepsilon} u_\varepsilon$ by ∇u_ε and h_{ex} by $\sqrt{E_\varepsilon(u_\varepsilon)}$) and thus we can isolate disjoint balls $B(a_i, r_i)$ of small radius that contain all the vortices of u_ε, and define the vorticity measure

$$\mu_\varepsilon = 2\pi \sum_i d_i \delta_{a_i}$$

The Jacobian determinant is related to this vorticity measure by the following lemma (the result was in our previous papers, but is included here in Lemma II.2 with a shorter proof):

$$\|\text{curl}\,(iu_\varepsilon, \nabla u_\varepsilon) - \mu_\varepsilon\|_{W^{-1,p}(\Omega)} \le \varepsilon^{\alpha(p)} \qquad \forall p < 2.$$

(A similar result has been proved in [JS1].) Let then again

(I.32) $$N_\varepsilon = 2\pi \sum_i |d_i|.$$

The lower bound of the ball-construction shows that

(I.33) $$N_\varepsilon |\log \varepsilon| \le C E_\varepsilon(u_\varepsilon).$$

Theorem 3 *Let u_ε be a family of solutions of (I.27) such that $E_\varepsilon(u_\varepsilon) \le \frac{1}{\varepsilon^\beta}$ and $\|\nabla u_\varepsilon\|_{L^\infty(\Omega)} \le \frac{C}{\varepsilon}$, the following holds as $\varepsilon \to 0$,*
- if $N_\varepsilon \ll \sqrt{E_\varepsilon(u_\varepsilon)}$, then up to extraction

$$\frac{U_\varepsilon}{\sqrt{E_\varepsilon(u_\varepsilon)}} \rightharpoonup U \quad \text{weakly in } H^1(\Omega) \text{ and strongly in } W^{1,p}_{loc}(\Omega),\ p<2$$

and

$$\Delta U = 0.$$

- if $N_\varepsilon \gg \sqrt{E_\varepsilon(u_\varepsilon)}$ then $\frac{\mu_\varepsilon}{N_\varepsilon} \rightharpoonup 0$ in the weak sense of measures.
- if $N_\varepsilon \sim C\sqrt{E_\varepsilon(u_\varepsilon)} (C>0)$ then, up to extraction,

$$\frac{U_\varepsilon}{\sqrt{E_\varepsilon(u_\varepsilon)}} \rightharpoonup U \quad \text{weakly in } H^1(\Omega) \text{ and strongly in } W^{1,p}_{loc}(\Omega),\ p<2,$$

$$\frac{\mu_\varepsilon}{\sqrt{E_\varepsilon(u_\varepsilon)}} \rightharpoonup \mu = \Delta U \in \mathcal{M}(\Omega) \cap H^{-1}(\Omega) \quad \text{weakly in measures}$$

and the Hopf differential of U

(I.34) $$\omega = (\partial_1 U)^2 - (\partial_2 U)^2 - 2i\partial_1 U \partial_2 U$$

is holomorphic in Ω. This implies that, if $\Omega' \subset \Omega$ is an open subdomain of Ω with $\mu \in L^p(\Omega'), p>2$, then $\mu = 0$ in Ω'.

The interesting case in this theorem is the last case where $\sqrt{E_\varepsilon(u_\varepsilon)}$ and N_ε are of the same order, which implies in view of (I.33) that $E_\varepsilon(u_\varepsilon) \ge C|\log \varepsilon|^2$. In fact, our result seems really particularly relevant when N_ε and $\sqrt{E_\varepsilon(u_\varepsilon)}$ are both of the order of $|\log \varepsilon|$, because in that case (and only in that case) $E_\varepsilon(u_\varepsilon)$ is of the order of $N_\varepsilon|\log \varepsilon|$ as expected. The first case corresponds to the case where the vortices are too few to be "seen" at the limit, the second case tells that if there are too many, they should in fact cancel out as a limiting zero density, like in Theorem 2.

The result on the Hopf differential in the third case is the analogue of the divergence-free tensor result of Theorem 1. It is equivalent to saying that U is stationary with respect to inner variations for the Dirichlet energy $\int_\Omega |\nabla U|^2$. Again here the problem is the regularity of U. $|\nabla U|$ can be proved to be continuous but not ∇U. If ∇U is regular enough (which is the case if $\mu = \Delta U \in L^p, p > 2$), then the fact that ω is holomorphic can be rewritten as $\nabla U \Delta U = 0$ (i.e. $\mu \nabla U = 0$, analogue of the "vanishing gradient property" (I.28)), leading to $\Delta U = 0$. However, ∇U does not need to be regular, for example for $U(x_1, x_2) = |x_1|^2$, the Hopf differential is holomorphic but ∇U is not continuous and $\nabla U \Delta U$ fails to have a meaning. We do not know if such counter-examples can be achieved as limits of solutions of (I.27), this remains an interesting open question. What our result says is that no regular ($L^p (p > 2)$) nonzero measures can be achieved at the limit, which is pretty striking: for example, contrarily to the magnetic field case, it is impossible to get a uniform density of vortices at the limit $\varepsilon \to 0$, $N_\varepsilon \to \infty$. The measure has to concentrate or to be zero, which means that either vortices concentrate, for example on lines, (and not on points because $\mu \in H^{-1}$) — an example is yet to be found — or all go to the boundary as it seems reasonable from the study of [SaSo] (it is at least what should happen for global minimizers in view of the result of [SaSo]). This phenomenon is a major difference between the model without magnetic field and the gauge-invariant model. The "simple" Ginzburg-Landau functional cannot capture "Abrikosov lattices" of vortices that are observed for large enough fields.

The proof of this theorem is essentially the same as that of Theorem 1, therefore, only its main steps are stated, in Section VI of the paper.

I.5 Method of the proof

Let us consider a sequence of critical points (u_n, A_n) with the hypotheses of Theorem 1. We can write $u_n = \rho_n e^{i\varphi_n}$, with $\rho_n = |u_n|$, at least formally. Of course φ_n is not well-defined where u_n vanishes. Since (u_n, A_n) is a critical point or a solution of the (G.L) system, this immediately implies some a priori estimates : first $|u_n| \le 1$ which is standard from the maximum principle, then, $\|\nabla_{A_\varepsilon} u_n\|_{L^\infty(\Omega)} \le \frac{C}{\varepsilon_n}$. (u_n, A_n) is in particular a solution of (I.2), thus we have

$$-\nabla^\perp h_n = \rho_n^2 (\nabla \varphi_n - A_n), \tag{I.35}$$

and this has a meaning everywhere. Then, as we did in our previous papers (see [SS1, SS2, SS3]), we can rewrite the energy the following way, using the previous identity:

$$\begin{aligned} J_{\varepsilon_n}(u_n, A_n) &= \frac{1}{2}\int_\Omega |\nabla |u_n||^2 + |u_n|^2 |\nabla \varphi_n - A_n|^2 + |h - h_{\text{ex}}|^2 + \frac{1}{2\varepsilon^2}(1 - |u_n|^2)^2 \\ &= \frac{1}{2}\int_\Omega |\nabla |u_n||^2 + \frac{|\nabla h_n|^2}{|u_n|^2} + |h_n - h_{\text{ex}}|^2 + \frac{1}{2\varepsilon^2}(1 - |u_n|^2)^2 \\ &\ge \frac{1}{2}\int_\Omega |\nabla |u_n||^2 + |\nabla h_n|^2 + |h_n - h_{\text{ex}}|^2 \frac{1}{2\varepsilon^2}(1 - |u_n|^2)^2. \end{aligned} \tag{I.36}$$

Thus, combining this with the hypothesis (I.11), we deduce the usual a-priori upper bounds:

$$|u_n| \le 1, \quad \frac{|u_n|}{h_{\text{ex}}} \text{ bounded in } H^1(\Omega), \quad \frac{h_n}{h_{\text{ex}}} \text{ bounded in } H^1_1(\Omega).$$

Hence, up to extraction of a subsequence, we can assume that

$$\exists h_\infty \in H_1^1(\Omega), \quad \frac{h_n}{h_{\rm ex}} \rightharpoonup h_\infty \text{ in } H_1^1(\Omega). \tag{I.37}$$

Then, defining μ_n by (I.17), $\int_\Omega |\mu_n|$ remains bounded in view of (I.16); hence, up to extraction again, we can assume that it converges weakly in the sense of measures to a limiting Radon measure μ_∞. Then, we will have, for any choice of (a_i, d_i) satisfying (I.12)—(I.15),

$$\mu_n = \frac{2\pi \sum_{i\in I} d_i \delta_{a_i}}{h_{\rm ex}} \rightharpoonup \mu_\infty = -\Delta h_\infty + h_\infty \quad \text{in } \mathcal{M}(\Omega).$$

The main idea underlying the proof of both theorems is to pass to the limit in the *stress-energy tensor* instead of (G.L). This is similar to the use of the Hopf differential in Chapter VII of [BBH]. Knowing that (u, A) is a critical point of J, we know that the associated stress-energy tensor is divergence-free :

$$\forall i = 1, 2, \quad \sum_j \partial_j T_{ij} = 0,$$

where

$$T_{ij} = \left(\frac{h^2}{2} - \frac{1}{4\varepsilon^2}(1-|u|^2)^2\right)\delta_{ij} + \frac{1}{2}\begin{pmatrix} |\partial_1^A u|^2 - |\partial_2^A u|^2 & 2(\partial_1^A u, \partial_2^A u) \\ 2(\partial_1^A u, \partial_2^A u) & |\partial_2^A u|^2 - |\partial_1^A u|^2 \end{pmatrix},$$

where we have used the notation $\partial_j^A = (\partial_j - iA_j)$, i.e. $\partial_j^A u = \partial_j u - iA_j u$.

For the case without magnetic field, the same holds with

$$T_{ij} = -\frac{1}{4\varepsilon^2}(1-|u|^2)^2\delta_{ij} + \frac{1}{2}\begin{pmatrix} |\partial_1 u|^2 - |\partial_2 u|^2 & 2(\partial_1 u, \partial_2 u) \\ 2(\partial_1 u, \partial_2 u) & |\partial_2 u|^2 - |\partial_1 u|^2 \end{pmatrix}.$$

It is in fact a general property in the calculus of variations, which is only a particular case of Noether's theorem. It comes from writing that (u, A), critical point of the energy-functional, is critical with respect to domain-diffeomorphisms. In other words, let us consider χ_t a one-parameter family of diffeomorphisms of Ω, such that $\chi_0 = Id$, and $\chi_t = Id$ outside of a compact set, then we must have

$$\left.\frac{d}{dt}\right|_{t=0} J(u\circ\chi_t, A\circ\chi_t) = 0,$$

(respectively $\left.\frac{d}{dt}\right|_{t=0} E_\varepsilon(u\circ\chi_t) = 0$ for the case without magnetic field). Then, using the fact that such diffeomorphisms can be generated by smooth vector-fields of Ω, one is led to the divergence-free property of the stress-energy tensor. For further reference, one can see [JT], [He].

Then, the idea is to pass to the limit $n \to \infty$ in the tensors $\frac{1}{h_{\rm ex}^2}(T_{ij}^n)$ associated to (u_n, A_n)

to get a limiting tensor (L_{ij}). One can easily see that, formally, the limiting tensor should be

$$L_{ij} = \frac{h_\infty^2}{2}\delta_{ij} + \frac{1}{2}\begin{pmatrix} (\partial_2 h_\infty)^2 - (\partial_1 h_\infty)^2 & -2\partial_1 h_\infty \partial_2 h_\infty \\ -2\partial_1 h_\infty \partial_2 h_\infty & (\partial_1 h_\infty)^2 - (\partial_2 h_\infty)^2 \end{pmatrix}, \tag{I.39}$$

which happens to be the stress-energy tensor associated to the Lagrangian $\mathcal{L}$.

Thus, if we pass to the limit in the identities div $\frac{T_{ij}^n}{h_{\rm ex}^2} = 0$, we get a limiting identity div $L_{ij} = 0$ which provides some new information on h_∞, enough to get the result of the theorem. Indeed, formally again,

$$\text{div } L_{ij} = 0 \Leftrightarrow (-\Delta h_\infty + h_\infty)\nabla h_\infty = 0. \tag{I.40}$$

The main difficulty of the proof is to make these limits rigorous. Passing to the limit in $T_{ij}/h_{\rm ex}^2$ seems to require strong convergence in $H^1(\Omega)$ of $\frac{h_n}{h_{\rm ex}}$, but this convergence is false in general. Indeed, for $h_{\rm ex} \leq C|\log \varepsilon|$, there is a loss of compactness in the vortices, as seen in [SS3] for minimizers.

The problematic terms in T_{ij} behave like $(\partial_1 h_n)^2 - (\partial_2 h_n)^2$ and $\partial_1 h_n \partial_2 h_n$. One could think of passing to the limit in these terms by Delort's theorem, used to pass to the limit in Euler equations (see [De] or the presentation in [Ch]), which applies exactly to such terms. But it would require that $\Delta h_n/h_{\rm ex}$ be bounded in L^1 with an additional condition, essentially that the vorticity measure has a sign, or that its negative (or positive) part goes to 0. But here, such a condition is not necessarily fulfilled, and there would also be problems in controlling the other terms. Yet, this indicates that morally, in spite of the lack of strong convergence of $\frac{h_n}{h_{\rm ex}}$ in H^1, the terms in $T_{ij}/h_{\rm ex}^2$ should converge weakly thanks to a compensation phenomenon.

Finally, in order to overcome this convergence difficulty, we prove in Section II another result: there is strong convergence of $\frac{h_n}{h_{\rm ex}}$ to h_∞ in H^1 outside of a set of arbitrarily small perimeter. This set essentially is the union over n of the vortex-cores, i.e. $\cup_{n\geq N} \cup_i B_i$. If we think as the vortices as being of characteristic size ε, and the typical case when there are $O(h_{\rm ex})$ vortices, the total perimeter of the "bad set" is of order $\varepsilon h_{\rm ex}$.
Once we have it, we show that this result is still sufficient to get div $L = 0$, through the use of the co-area formula.

Remark I.2: What we prove amounts to the following lemma (see Lemma III.2): if f_n is a bounded family in $L^1(\Omega)$ and for all $\delta > 0$ there exists a set E_δ of perimeter $< \delta$ such that $\int_{\Omega \setminus E_\delta} |f_n - f| \to 0$, $f \in L^1(\Omega)$; and if Df_n is bounded in $L^1(\Omega)$, with D a linear combination of first order derivatives, then $Df_n \rightharpoonup Df$ in the weak sense of measures.

The conclusion that div $L = 0$ will yield some regularity on h_∞. Yet, the formal result (I.40) will only be true if we can have some additional regularity: $\nabla h_\infty \in C^0(\Omega)$.

For Theorem 2, the key point is to substract $h_{\rm ex} h_0$ to h_n and study the remainder f_n.

Such a splitting was already used in [S2]. Formally

$$\begin{cases} -\Delta f_n + f_n = 2\pi \sum_i d_i \delta_{a_i} & \text{in } \Omega \\ f_n = 0 & \text{on } \partial\Omega. \end{cases}$$

When $N_\varepsilon \ll h_{\text{ex}}$, f_n is negligible compared to $h_{\text{ex}} h_0$, thus, when we expand the terms in T_{ij} using $h_n = f_n + h_{\text{ex}} h_0$, there only remain the cross-terms, which yield $\nu_\infty \nabla h_0 = 0$.

Remark on notations : $\cdot$ denotes the scalar product in $\mathbb{R}^2$, $(.,.)$ the scalar product in $\mathbb{C}$ identified with $\mathbb{R}^2$, and C a positive constant.

Acknowledgments: We thank Professors A. Ancona and G. David for their helpful suggestions in the proof of Proposition II.1.

II $h_n/h_{\text{ex}} - h_\infty$ converges to 0 strongly except on a set of small perimeter

Let us recall the definition of the p-capacity (for further reference, see for example [EG, Fr, Z]). The p-capacity ($p \geq 1$) is first defined for compact sets by

$$\text{cap}_p(E) = \inf \left\{ \int_\Omega |\nabla \varphi|^p, \phi \in C_0^\infty(\Omega), \phi \geq 1 \text{ in } E \right\}.$$

Then, it can be extended to all Borel sets. We also recall that the 1-capacity is a definition of the perimeter (see [Fr]). As already mentioned, up to extraction, we can assume that $k_n = \frac{h_n}{h_{\text{ex}}}$ converges weakly in $H_1^1(\Omega)$ to some h_∞, with $-\Delta h_\infty + h_\infty \in \mathcal{M}(\Omega)$ hence $\Delta h_\infty \in \mathcal{M}(\Omega) \cap H^{-1}(\Omega)$. This section is devoted to proving the following proposition :

Proposition II.1 *For all $\delta > 0$, there exists $E_\delta \subset \Omega$ with $cap_1(E_\delta) < \delta$ and*

$$\int_{\Omega \setminus E_\delta} |\nabla(k_n - h_\infty)|^2 \longrightarrow 0 \qquad \textit{as } n \to \infty.$$

The proof of this proposition relies on the following idea: we have approximately

$$-\Delta h_n + h_n = 2\pi \sum_i d_i \delta_{a_i}.$$

This is of course not really true, but let us assume it for a moment for the sake of simplicity. Then, with our assumptions, we could say that $-\Delta k_n + k_n$ is bounded in the sense of measures. In addition, with a similar argument as in [SS3], we can say that k_n converges to h_∞ strongly in $W^{1,p}(\Omega)$, for all $p < 2$. Thanks to this fact, we can apply a standard result in capacity theory (see [Fr, Z]), which asserts that we can find $\delta_n \to 0$ such that

$\mathrm{cap}_1\,(\mathcal{A}_n := \{x \in \Omega, |k_n - h_\infty| > \delta_n\}) \to 0$ as $n \to \infty$.
Then, we denote $\overline{k_n - h_\infty}$ the function $k_n - h_\infty$ truncated at the level δ_n, i.e.

$$\overline{k_n - h_\infty} = \begin{cases} k_n - h_\infty \text{ where } |k_n - h_\infty| \le \delta_n \\ \delta_n \text{ where } k_n - h_\infty > \delta_n \\ -\delta_n \text{ where } k_n - h_\infty < -\delta_n \end{cases}$$

We have $\|\overline{k_n - h_\infty}\|_{L^\infty(\Omega)} \to 0$, and

$$\begin{aligned} \int_{\Omega\setminus\mathcal{A}_n} |\nabla(k_n - h_\infty)|^2 &= \int_\Omega |\nabla\overline{(k_n - h_\infty)}|^2 = \int_\Omega \nabla(k_n - h_\infty)\cdot\nabla\overline{(k_n - h_\infty)} \\ &= -\int_\Omega \Delta(k_n - h_\infty)\overline{(k_n - h_\infty)}, \end{aligned} \tag{II.1}$$

after integration by parts. But $\Delta(k_n - h_\infty)$ was supposed to be bounded in measures, and $\overline{k_n - h_\infty}$ converges uniformly to 0, hence the last integral tends to 0. We would thus get that $\int_{\Omega\setminus\mathcal{A}_n} |\nabla(k_n - h_\infty)|^2$ converges to 0, where $\mathrm{cap}_1(\mathcal{A}_n)$ tends to 0, which is the desired conclusion. Here is now the complete proof without simplification. Instead of comparing directly k_n to h_∞, we need to introduce auxiliary functions and to evaluate the difference between k_n and h_∞ as the sum of three differences.

We start with the following lemma, which allows to replace u by a unit-valued map, except on a small exceptional set, which consists of the balls B_i given by Proposition I.1.

Lemma II.1 *Let (u, A) satisfy the second Ginzburg-Landau equation* (I.2)-(I.3) *and the energy bound $J(u, A) \le C|\log\varepsilon|^2$. Let $(B_i)_{i\in I}$ be an associated family of vortex balls. Then there exists $(\tilde u, \tilde A) \in H^1(\Omega) \times H^1(\Omega, \mathbb{R}^2)$ such that, letting $\tilde h = \mathrm{curl}\,\tilde A$,*

1. *$\||\tilde u| - |u|\|_{L^\infty(\Omega)} \le |\log\varepsilon|^{-4}$ and $|\tilde u| \equiv 1$ in $\Omega \setminus \cup_{i\in I} B_i$.*
2. *$-\nabla^\perp \tilde h = (i\tilde u, \nabla_{\tilde A}\tilde u)$ in Ω and $\tilde h = h_{\mathrm{ex}}$ on $\partial\Omega$.*
3. *$\|(i\tilde u, \nabla_{\tilde A}\tilde u) - (iu, \nabla_A u)\|_{L^2(\Omega)} \le C|\log\varepsilon|^{-2}$ and $\|\tilde h - h\|_{H^1(\Omega)} \le C|\log\varepsilon|^{-2}$.*
4. *For any $i \in I$, $\deg(\tilde u, \partial B_i) = \deg(u, \partial B_i)$.*
5. *If (u, A) and $(\tilde u, \tilde A)$ satisfy the Coulomb gauge condition (I.7), then $\|(iu, \nabla u) - (i\tilde u, \nabla\tilde u)\|_{L^2(\Omega)} \le C|\log\varepsilon|^{-2}$.*

Proof : The strategy of the proof is to modify $|u|$ in order for 1) to be satisfied by the resulting map v. Then we minimize $J(v, B)$ with respect to B, to obtain a configuration $(v, \underline{B})$ that will satisfy 2). Properties 3) and 4) will be byproducts of this construction.

We define v. Let $\chi : [0,1] \to [0,1]$ be such that

$$\begin{cases} \chi(x) = x & \text{if } 0 \le x \le \frac{1}{2} \\ \chi(x) = 1 & \text{if } x \ge 1 - \frac{1}{|\log\varepsilon_n|^4} \\ \chi & \text{is affine between } \frac{1}{2} \text{ and } 1 - \frac{1}{|\log\varepsilon_n|^4} \end{cases} \tag{II.2}$$

We let

$$v := \chi(|u|)\frac{u}{|u|}.$$

Clearly, v satisfies property 1). We define

$$E(u,A) = \frac{1}{2}\int_\Omega \left|\frac{(iu,\nabla_A u)}{|u|}\right|^2 + |\mathrm{curl}\, A - h_{\mathrm{ex}}|^2,$$

which is the part of $J(u,A)$ that depends on A. To be precise,

$$J(u,A) = E(u,A) + \frac{1}{2}\int_\Omega |\nabla|u||^2 + \frac{1}{2\varepsilon^2}\left(1-|u|^2\right)^2.$$

Consider a minimizing sequence $(B_n)_n$ for $E(v,B)$ with v fixed. Although this sequence may not be bounded in a good function space, the configuration (v,B_n) is gauge equivalent to (u_n,A_n) with div $A_n = 0$ in Ω and $A_n.\nu = 0$ on $\partial\Omega$ and (u_n,A_n) is bounded in $H^1\times H^1$ (see [BR]). Thus a subsequence (u_n,A_n) weakly converges in $H^1\times H^1$ to a configuration $(\tilde u,\tilde A)$, we may also assume it converges a.e. and strongly in L^p, for any $1\le p<+\infty$.

Since $|u_n| = |v|$ for any n we have $|\tilde u| = |v|$, and since v satisfies 1) then so does $\tilde u$.

To prove 2), note that this is the Euler-Lagrange equation (with boundary condition) which expresses that $\tilde A$ is a critical point of the functional $B\mapsto E(\tilde u,B)$, which happens to be convex. Thus proving 2) is equivalent to proving that $\tilde A$ minimizes this functional or that for any B, $E(\tilde u,\tilde A+B)\ge E(\tilde u,\tilde A)$. But, using the weak H^1 convergence of (u_n,A_n) to (u,A), and the fact that $|u_n|$ is fixed,

$$\begin{aligned}
E(\tilde u,\tilde A+B)-E(\tilde u,\tilde A) &= \int_\Omega B\cdot(i\tilde u,\nabla_{\tilde A}\tilde u) + \frac{1}{2}|\tilde u|^2|B|^2 + (\tilde h - h_{\mathrm{ex}})\mathrm{curl}\, B + \frac{1}{2}(\mathrm{curl}\, B)^2\\
&= \lim_{n\to+\infty}\int_\Omega B\cdot(iu_n,\nabla_{A_n}u_n) + \frac{1}{2}|v|^2|B|^2 + (\mathrm{curl}\, A_n - h_{\mathrm{ex}})\mathrm{curl}\, B + \frac{1}{2}(\mathrm{curl}\, B)^2\\
&= \lim_{n\to+\infty}\int_\Omega B\cdot(iv,\nabla_{B_n}v) + \frac{1}{2}|v|^2|B|^2 + (\mathrm{curl}\, B_n - h_{\mathrm{ex}})\mathrm{curl}\, B + \frac{1}{2}(\mathrm{curl}\, B)^2\\
&= \lim_{n\to+\infty} E(v,B_n+B)-E(v,B_n),
\end{aligned}$$

which is nonnegative since $(B_n)_n$ is a minimizing sequence for $E(v,.)$. Thus 2) is proved.

We turn to 3). We may assume that $E(v,B_n)\le E(v,A)$ for any n. We claim that in this case

$$\text{(II.3)}\qquad \|(iu,\nabla_A u)-(iv,\nabla_{B_n}v)\|^2_{L^2(\Omega)}\le C|\log\varepsilon|^{-2}.$$

The first bound in 3) follows by passing to the limit in (II.3), the second bound being a consequence of the first one and the fact that $-\nabla^\perp\tilde h = (i\tilde u,\nabla_{\tilde A}\tilde u)$ and $-\nabla^\perp h = (iu,\nabla_A u)$ in Ω, while $h=\tilde h = h_{\mathrm{ex}}$ on $\partial\Omega$. We now prove (II.3).

Writing $u=\rho e^{i\varphi}$ and $v=\tilde\rho e^{i\varphi}$ (where $\tilde\rho = \chi(\rho)$) we have

$$\text{(II.4)}\qquad \begin{aligned}
|\rho(\nabla\varphi - A)-\tilde\rho(\nabla\varphi-A)| &= \left|\frac{\rho-\tilde\rho}{\rho}\right||\rho(\nabla\varphi-A)|\\
&\le C|\log\varepsilon|^{-4}|\rho(\nabla\varphi-A)|,
\end{aligned}$$

from the definition of χ. It follows, using the equality $(iu, \nabla_A u) = \rho^2(\nabla\varphi - A)$ and the bound $J(u,A) \le C|\log\varepsilon|^2$, that

$$|E(u,A) - E(v,A)| \le C|\log\varepsilon|^{-2} \tag{II.5}$$

and similarly that

$$|E(u,B_n) - E(v,B_n)| \le C|\log\varepsilon|^{-2}. \tag{II.6}$$

As already noted, the second Ginzburg-Landau equation (I.2)-(I.3) is equivalent to the fact that A minimizes $E(u,.)$. Therefore $E(u,A) \le E(u,B_n)$ while we assumed $E(v,B_n) \le E(v,A)$. Together with (II.5), (II.6), this yields

$$|E(u,A) - E(u,B_n)| \le C|\log\varepsilon|^{-2}.$$

Now (I.2)-(I.3) imply that

$$\begin{aligned}
E(u,A) - E(u,B_n) &= \frac{1}{2}\int_\Omega \rho(2\nabla\varphi - A - B_n)\cdot\rho(B_n - A) \\
&\qquad + (\operatorname{curl} A - \operatorname{curl} B_n)\cdot(\operatorname{curl} A + \operatorname{curl} B_n - 2h_{\mathrm{ex}}) \\
&= \frac{1}{2}\int_\Omega 2(B_n - A)\cdot\rho^2(\nabla\varphi - A) - \rho^2|A - B_n|^2 \\
&\qquad + 2(\operatorname{curl} A - \operatorname{curl} B_n)(h - h_{\mathrm{ex}}) - |h - \operatorname{curl} B_n|^2 \\
&= -\frac{1}{2}\int_\Omega |\rho(\nabla\varphi - A) - \rho(\nabla\varphi - B_n)|^2 + |h - \operatorname{curl} B_n|^2,
\end{aligned}$$

which combined with (II.4) yields

$$\left\| \frac{(iu, \nabla_A u)}{|u|} - \frac{(iv, \nabla_{B_n} v)}{|v|} \right\|^2_{L^2(\Omega)} \le C|\log\varepsilon|^{-2}.$$

Since $|1/|u| - 1/|v|| \le |\log\varepsilon|^{-4}$, (II.3) follows and 3) is proved.

We then prove 4). First it is clear that $\deg(u_n, \partial B_i) = \deg(v, \partial B_i) = \deg(u, \partial B_i)$ for every n. Moreover since $|u_n| = |v| = 1$ on ∂B_i

$$\deg(u_n, \partial B_i) = \int_{\partial B_i} \rho_n^2 \frac{\partial\varphi_n}{\partial\tau} = \int_{B_i} \nabla^\perp \rho_n^2 \cdot \nabla\varphi_n,$$

where $u_n = \rho_n e^{i\varphi_n}$. Using the fact that $\rho_n = |v|$ and the weak H^1 convergence of u_n to $\tilde{u}$ we find

$$\lim_{n\to+\infty} \deg(u_n, \partial B_i) = \deg(\tilde{u}, \partial B_i),$$

which proves 4).

Finally, we prove 5). From the upper bound $J(u,A) \le Ch_{\mathrm{ex}}^2$, if (I.7) is satisfied, we deduce the following a priori estimates

$$\|A\|_{H^2(\Omega)} \le Ch_{\mathrm{ex}} \qquad \|\nabla u\|_{L^2(\Omega)} \le Ch_{\mathrm{ex}}. \tag{II.7}$$

Indeed, A can be written $\nabla^\perp \xi$ for some $\xi \in H^2_0(\Omega)$, and $\frac{h}{h_{\rm ex}} = \frac{\Delta\xi}{h_{\rm ex}}$ is bounded in $H^1(\Omega)$ (see (I.37)), hence $\|\Delta\xi\|_{H^1(\Omega)} \le C h_{\rm ex}$ and $\|A\|_{H^2(\Omega)} \le C h_{\rm ex}$ follows. Then, $\|A\|_{L^\infty} \le C h_{\rm ex}$ which, combined with $\|\nabla_A u\|_{L^2(\Omega)} \le C h_{\rm ex}$ yields $\|\nabla u\|_{L^2(\Omega)} \le C h_{\rm ex}$. Similarly, since $(\tilde u, \tilde A)$ is also constructed to satisfy the Coulomb gauge, we have

$$\|A - \tilde A\|_{H^2(\Omega)} \le C\|h - \tilde h\|_{H^1(\Omega)} \le C|\log \varepsilon|^{-2}. \tag{II.8}$$

Also, from the energy bound,

$$\|(\rho^2 - 1)A\|_{L^2(\Omega)} \le \|A\|_{L^\infty}\|\rho^2 - 1\|_{L^2} \le C\varepsilon h_{\rm ex}^2, \tag{II.9}$$

and the same for $(\tilde u, \tilde A)$. Therefore,

$$\begin{aligned} \|\rho^2 A - \tilde\rho^2 \tilde A\|_{L^2(\Omega)} &\le \|(\rho^2-1)A - (\tilde\rho^2 - 1)\tilde A + A - \tilde A\|_{L^2} \\ &\le C|\log \varepsilon|^{-2}. \end{aligned}$$

Then, since $(iu, \nabla u) - (i\tilde u, \nabla \tilde u) = (iu, \nabla_A u) - (i\tilde u, \nabla_{\tilde A}\tilde u) + \rho^2 A - \tilde\rho^2 \tilde A$, the result follows with assertion 3). □

We can then deduce easily that

Lemma II.2 *In the Coulomb gauge, for all $\gamma > 0$,*

$$\left\| \mathrm{curl}\,(i\tilde{u_n}, \nabla \tilde{u_n}) - 2\pi \sum_{i\in I} d_i \delta_{a_i} \right\|_{(C^{0,\gamma}(\Omega))'} \le \varepsilon^{\alpha(\gamma)}, \tag{II.10}$$

where $(C^{0,\gamma}(\Omega))'$ is the dual of $C^{0,\gamma}$, and $\alpha(\gamma)$ some positive exponent depending on γ, and the (a_i, d_i) satisfy (I.12)—(I.15). Moreover,

$$\|\mathrm{curl}\,(iu_n, \nabla u_n) - \mathrm{curl}\,(i\tilde{u_n}, \nabla\tilde{u_n})\|_{H^{-1}(\Omega)} \to 0, \tag{II.11}$$

hence

$$\mathrm{curl}\,(iu_n, \nabla u_n) - 2\pi \sum_{i\in I} d_i \delta_{a_i} \to 0 \quad \text{in } (H^1_0(\Omega) \cap C^{0,\gamma}(\Omega))'$$

and thus in $W^{-1,p}(\Omega)$ for all $p < 2$.

As we already mentioned, this result was already stated in [SS1], Lemma II.3, [SS3], [ASS], Lemma II.2, except that here we give a finer estimate, and the trick of Lemma II.1 allows us to give a much shorter proof.

Proof : Using the fact that $\mathrm{curl}\,(i\tilde{u_n}, \nabla\tilde{u_n}) \equiv 0$ in $\Omega \setminus \cup_i B_i$, we have for any $\xi \in C^{0,\gamma}(\Omega)$,

$$\int_\Omega \xi \,\mathrm{curl}\,(i\tilde{u_n}, \nabla\tilde{u_n}) = \sum_i \int_{B_i} \xi\, \mathrm{curl}\,(i\tilde{u_n}, \nabla\tilde{u_n}). \tag{II.12}$$

On the other hand, using $|\text{curl}\,(i\tilde{u_n},\nabla\tilde{u_n})| \le |\nabla\tilde{u_n}|^2$, wehave

$$\begin{aligned}
\sum_i \int_{B_i} |(\xi-\xi(a_i))\text{curl}\,(i\tilde{u_n},\nabla\tilde{u_n})| &\le (\max_i r_i)^\gamma \|\xi\|_{C^{0,\gamma}(\Omega)} \sum_i \int_{B_i} |\nabla\tilde{u_n}|^2 \\
&\le \|\xi\|_{C^{0,\gamma}(\Omega)} (\max_i r_i)^\gamma \|\nabla\tilde{u_n}\|^2_{L^2(\Omega)} \\
&\le \|\xi\|_{C^{0,\gamma}(\Omega)} h_{\text{ex}}^2 (\max_i r_i)^\gamma \\
&\le \varepsilon^{\alpha(\gamma)} \|\xi\|_{C^{0,\gamma}(\Omega)}. \qquad \text{(II.13)}
\end{aligned}$$

Using Stokes formula,

$$\int_{B_i} \xi(a_i)\text{curl}\,(i\tilde{u_n},\nabla\tilde{u_n}) = \xi(a_i)\int_{\partial B_i}\left(i\tilde{u_n},\frac{\partial\tilde{u_n}}{\partial\tau}\right) = 2\pi d_i\xi(a_i), \qquad \text{(II.14)}$$

where we have used the fact that $|\tilde{u}| = 1$ on ∂B_i. Combining (II.12)—(II.14), we get (II.10). (II.11) follows from assertion 5) in Lemma II.1. □

We now introduce g_n as the solution of

$$\begin{cases} -\Delta g_n + g_n = \frac{2\pi}{h_{\text{ex}}}\sum_i d_i\delta_{a_i} & \text{in } \Omega \\ g_n = 1 & \text{on } \partial\Omega, \end{cases} \qquad \text{(II.15)}$$

where the (a_i, d_i) are the "vortices" as defined in (H2). For simplicity, we will also denote

$$\tilde{k_n} = \frac{\tilde{h_n}}{h_{\text{ex}}}.$$

Lemma II.3 *For all $p<2$, there exists $\alpha(p)>0$ such that*

$$\left\|\frac{\tilde{h_n}}{h_{\text{ex}}} - g_n\right\|_{W^{1,p}(\Omega)} \le C\varepsilon^{\alpha(p)}.$$

Moreover,

$$\|g_n - h_\infty\|_{W^{1,p}(\Omega)} \longrightarrow 0,$$

and thus

$$\left\|\frac{h_n}{h_{\text{ex}}} - h_\infty\right\|_{W^{1,p}(\Omega)} \longrightarrow 0.$$

Proof : Let us write $\tilde{k_n} = \frac{\tilde{h_n}}{h_{\text{ex}}}$. Recall that $(\tilde{u},\tilde{A})$ satisfies the second Ginzburg-Landau equation (cf Lemma II.1), i.e. $-\nabla^\perp\tilde{h} = (i\tilde{u},\nabla_{\tilde{A}}\tilde{u})$. Taking the curl of this equation, we have

$$-\Delta\tilde{h_n} + \tilde{h_n} = \text{curl}\,(i\tilde{u_n},\nabla\tilde{u_n}) + \text{curl}\,\left((1-|\tilde{u_n}|^2)\tilde{A_n}\right). \qquad \text{(II.16)}$$

Combining (II.16) with (II.15), we have
(II.17)

$$-\Delta(\tilde{k_n} - g_n) + \tilde{k_n} - g_n = \frac{1}{h_{\text{ex}}}\left(\text{curl}\,(i\tilde{u_n}, \nabla\tilde{u_n}) - 2\pi\sum_i d_i\delta_{a_i}\right) + \frac{1}{h_{\text{ex}}}\text{curl}\,\left((1-|\tilde{u_n}|^2)\tilde{A}_n\right).$$

Hence, in view of (II.9) and (II.10), we deduce

$$\left\|-\Delta(\tilde{k_n} - g_n) + \tilde{k_n} - g_n\right\|_{W^{-1,p}(\Omega)} \le C\varepsilon^{\alpha(p)}.$$

This provides the desired result, by elliptic regularity.
We treat the case of $h_\infty - g_n$.
(II.18)

$$\| - \Delta(h_\infty - g_n) + (h_\infty - g_n)\|_{W^{-1,p}(\Omega)} = \left\|\mu_\infty - \frac{2\pi\sum_i d_i\delta_{a_i}}{h_{\text{ex}}}\right\|_{W^{-1,p}(\Omega)} = \|\mu_\infty - \mu_\varepsilon\|_{W^{-1,p}(\Omega)}.$$

In view of (II.10), μ_ε is bounded in $W^{-1,p}(\Omega)$ for all $p < 2$. It is also bounded in measures, hence by Murat's theorem [Mu] (see [B] for a much simpler proof), it is compact in $W^{-1,p}(\Omega)$ for all $p < 2$, thus $\mu_\varepsilon \to \mu_\infty$ strongly in $W^{-1,p}$ for $p < 2$. Therefore, $\| - \Delta(h_\infty - g_n) + (h_\infty - g_n)\|_{W^{-1,p}(\Omega)} \to 0$ and, similarly, we get the result for $h_\infty - g_n$. □

Lemma II.4 *We can find a sequence of sets $\mathcal{A}_n$ such that $cap_1(\mathcal{A}_n) \to 0$ and*

$$\int_{\Omega\setminus\mathcal{A}_n} |\nabla(\tilde{k_n} - g_n)|^2 \longrightarrow 0 \quad \text{as } n \to \infty.$$

Proof :
- *Step 1 :* We apply again the standard result on capacities (see [Fr, Z]) stating that

$$\text{(II.19)} \qquad \forall p < 2 \qquad \text{cap}_p\left(\left\{x \in \Omega, |\tilde{k_n} - g_n| > \varepsilon^{\frac{\alpha(p)}{2}}\right\}\right) \le C\varepsilon^{-\frac{p\alpha(p)}{2}}\|\tilde{k_n} - g_n\|^p_{W^{1,p}(\Omega)}.$$

Hence, in view of Lemma II.3,

$$\text{(II.20)} \qquad \text{cap}_p\left(\left\{x \in \Omega, |\tilde{k_n} - g_n| > \varepsilon^{\frac{\alpha(p)}{2}}\right\}\right) \le C\varepsilon^{\frac{p\alpha(p)}{2}}.$$

Let us denote by $\overline{\tilde{k_n} - g_n}$ the function $\tilde{k_n} - g_n$ truncated at the level $\varepsilon^{\frac{\alpha(p)}{2}}$, and $C_n = \{x \in \Omega, |\tilde{k_n} - g_n| > \varepsilon^{\frac{\alpha(p)}{2}}\}$. (II.20) means that

$$\text{(II.21)} \qquad \text{cap}_p(C_n) \le C\varepsilon^{\frac{p\alpha(p)}{2}}.$$

- *Step 2 :* We prove that

$$\text{(II.22)} \qquad \int_{\Omega\setminus\cup_i B_i} |\nabla g_n|^2 \le C|\log\varepsilon|.$$

Indeed, g_n (see the definition in (II.15)) can be written as

$$g_n = \xi_0 + \frac{1}{h_{\rm ex}} \sum_i g_i,$$

where g_i is the solution of

$$\begin{cases} -\Delta g_i + g_i = 2\pi d_i \delta_{a_i} & \text{in } \Omega \\ g_i = 0 & \text{on } \partial\Omega, \end{cases}$$

and ξ_0 is the solution of

$$\begin{cases} -\Delta \xi_0 + \xi_0 = 0 & \text{in } \Omega \\ \xi_0 = 1 & \text{on } \partial\Omega. \end{cases}$$

It is easy to see that

$$\int_{\Omega\backslash B_i} |\nabla g_i|^2 \le C d_i^2 \log \frac{1}{r_i},$$

thus

$$\|\nabla g_n\|_{L^2(\Omega\backslash \cup_i B_i)} \le C + \frac{\sum_i \|\nabla g_i\|_{L^2(\Omega\backslash B_i)}}{h_{\rm ex}} \le C + C\frac{\sum_i |d_i|}{h_{\rm ex}} |\log r_i|^{\frac{1}{2}}.$$

But, by hypothesis (I.14), $r_i \ge C\varepsilon$ and $\sum_i |d_i| \le C h_{\rm ex}$, thus

$$\text{(II.23)} \qquad \int_{\Omega\backslash \cup_i B_i} |\nabla g_n|^2 \le C|\log \varepsilon|.$$

- *Step 3 :* We can find a ζ that satisfies

$$\text{(II.24)} \qquad \begin{cases} \zeta = 0 \ \ \text{in } \cup_i B(a_i, r_i) \\ \zeta = 1 \ \ \text{in } \Omega\backslash \cup_i B(a_i, 2r_i) \\ 0 \le \zeta \le 1 \\ \|\zeta\|_{H^1}^2 \le C|\log \varepsilon|^{20} \\ \|\nabla \zeta\|_{L^\infty} \le \frac{1}{\min_i r_i} \ll \frac{1}{\varepsilon|\log \varepsilon|^4} \end{cases}$$

Indeed, we need only choose ζ_i such that

$$\begin{cases} \zeta_i = 0 & \text{in } B(a_i, r_i) \\ \zeta_i = 1 & \text{in } \Omega\backslash B(a_i, 2r_i) \\ 0 \le \zeta_i \le 1 \\ \int_\Omega |\nabla \zeta_i|^2 \le C, \end{cases}$$

and then set $\zeta = \min_i \zeta_i$. Then,

$$\int_\Omega |\nabla \zeta|^2 \le \sum_i \int_\Omega |\nabla \zeta_i|^2 \le C\text{Card } I \le C|\log \varepsilon|^{20},$$

using (I.13).

- *Step 4 :* Let us then set $\mathcal{A}_n = C_n \cup (\cup_{i\in I} B(a_i, 2r_i))$. We have

$$\begin{aligned} \mathrm{cap}_1(\mathcal{A}_n) &\le \mathrm{cap}_1(C_n) + 2\sum_i r_i \\ &\le \mathrm{cap}_p(C_n) + o(1) \end{aligned}$$

because the cap_1 is dominated by the cap_p ($p \ge 1$). Thus, using (II.21), we are led to

$$\mathrm{cap}_1(\mathcal{A}_n) \to 0.$$

On the other hand,

$$\begin{aligned} \int_{\Omega\setminus\mathcal{A}_n} |\nabla(\tilde{k_n} - g_n)|^2 &\le \left| \int_\Omega \zeta \nabla(\tilde{k_n} - g_n) \cdot \nabla\overline{(\tilde{k_n} - g_n)} \right| \\ &\le \left| \int_\Omega \zeta \Delta(\tilde{k_n} - g_n)\overline{(\tilde{k_n} - g_n)} + \int_\Omega \overline{(\tilde{k_n} - g_n)} \nabla\zeta \cdot \nabla(\tilde{k_n} - g_n) \right|. \end{aligned} \tag{II.25}$$

But, since $-\Delta(\tilde{k_n} - g_n) + \tilde{k_n} - g_n$ is supported in $\cup_i B(a_i, r_i)$, and ζ vanishes there, we have

$$\int_\Omega \zeta \Delta(\tilde{k_n} - g_n)\overline{(\tilde{k_n} - g_n)} = \int_\Omega \zeta(\tilde{k_n} - g_n)\overline{(\tilde{k_n} - g_n)} = o(1).$$

Therefore,

$$\begin{aligned} \int_{\Omega\setminus\mathcal{A}_n} |\nabla(\tilde{k_n} - g_n)|^2 &\le C\|\overline{\tilde{k_n} - g_n}\|_{L^\infty(\Omega)} \|\nabla\zeta\|_{L^2(\Omega)} \left(\|\nabla\tilde{k_n}\|_{L^2(\Omega\setminus\cup_i B_i)} + \|\nabla g_n\|_{L^2(\Omega\setminus\cup_i B_i)} \right) \\ &\le C\varepsilon^{\frac{\alpha(p)}{2}} |\log\varepsilon|^{11}, \end{aligned} \tag{II.26}$$

where we have used (II.20), (II.24), (II.22). Thus, we can conclude that

$$\int_{\Omega\setminus\mathcal{A}_n} |\nabla(\tilde{k_n} - g_n)|^2 \le o(1), \tag{II.27}$$

which is the desired result. □

Lemma II.5 *There exists a set $\mathcal{B}_n$ such that $\mathrm{cap}_1(\mathcal{B}_n) \to 0$ and*

$$\int_{\Omega\setminus\mathcal{B}_n} |\nabla(g_n - h_\infty)|^2 \longrightarrow 0.$$

Proof : As seen in Lemma II.3, we have $\|g_n - h_\infty\|_{W^{1,p}} \to 0$ for $p < 2$, hence from the same theorem on capacities, writing $\mathcal{B}_n = \{x \in \Omega, |g_n - h_\infty| > \delta_n\}$, we have

$$\mathrm{cap}_p(\mathcal{B}_n) \le \frac{\|g_n - h_\infty\|^p_{W^{1,p}}}{\delta_n^p}.$$

Therefore, we can choose a suitable $\delta_n \to 0$ such that

$$\text{(II.28)} \qquad \text{cap}_1(\mathcal{B}_n) \to 0.$$

As previously, we denote by $\overline{g_n - h_\infty}$ the function $g_n - h_\infty$ truncated at the level δ_n. We have $\|\overline{g_n - h_\infty}\|_{L^\infty(\Omega)} \leq C\delta_n \to 0$, and furthermore

$$\text{(II.29)} \quad \int_{\Omega\setminus\mathcal{B}_n} |\nabla(g_n - h_\infty)|^2 = \int_\Omega \nabla(g_n - h_\infty) \cdot \nabla\overline{(g_n - h_\infty)} = -\int_\Omega \Delta(g_n - h_\infty)\overline{(g_n - h_\infty)}.$$

By definition of g_n (see (II.15) and (I.16)), $\Delta(g_n - h_\infty)$ remains bounded in the sense of measures, thus (II.29) implies that

$$\int_{\Omega\setminus\mathcal{B}_n} |\nabla(g_n - h_\infty)|^2 \to 0.$$

$\square$

Now, for any $\delta > 0$, we can consider $\mathcal{B}_n \cup \mathcal{A}_n$ (given by Lemmas II.4 and II.5), and extract a further subsequence such that $\forall n, \mathcal{B}_n \cup \mathcal{A}_n \subset E_\delta$, where E_δ satisfies $\text{cap}_1(E_\delta) < \delta$. Then, combining Lemmas II.4 and II.5, 3) of Lemma II.1, and the triangle inequality, we are led to the conclusion of Proposition II.1.

III Passing to the limit in the stress-energy tensor

We recall that the "stress-energy tensor" associated to the solution (u, A) of the Ginzburg-Landau equations is (dropping the subscripts)

$$\text{(III.1)} \qquad T_{ij} = \left(\frac{h^2}{2} - \frac{1}{4\varepsilon^2}(1 - |u|^2)^2\right)\delta_{ij} + \frac{1}{2}\begin{pmatrix} |\partial_1^A u|^2 - |\partial_2^A u|^2 & 2(\partial_1^A u, \partial_2^A u) \\ 2(\partial_1^A u, \partial_2^A u) & |\partial_2^A u|^2 - |\partial_1^A u|^2 \end{pmatrix}.$$

Here, δ_{ij} is the Kronecker symbol, and $(.,.)$ the scalar product in $\mathbb{C}$ identified to $\mathbb{R}^2$.
A tedious but straightforward computation allows to check that

$$\text{(III.2)} \qquad (u, A) \text{ is solution of } (G.L) \Longrightarrow \forall i, \sum_j \partial_j T_{ij} = 0.$$

In other words,

$$\forall i, \ \text{div } T_{ij} = 0.$$

As already explained in the introduction, this is a general fact known as "the stress-energy tensor associated to a critical point is divergence-free".

We then wish to apply this result to (u_n, A_n) and pass to the limit $n \to \infty$ in the tensor, in order to obtain a limiting tensor L_{ij}.

Proposition III.1 *For all $\delta > 0$, there exists E_δ such that $cap_1(E_\delta) < \delta$ and*

$$\forall i, j \quad \int_{\Omega \setminus E_\delta} \left| \frac{T^n_{ij}}{h^2_{\rm ex}} - L_{ij} \right| \longrightarrow 0 \quad \text{as } n \to \infty, \tag{III.3}$$

where L_{ij} was defined in (I.39).

(The set E_δ here is the one constructed in Proposition II.1.) Let us start by rewriting the tensor a little. In $\Omega \setminus \cup_i B_i$, since $|u_n| \geq 1 - o(1)$, u_n does not vanish (for n large enough), hence, as previously, we can write formally $u_n = \rho_n e^{i\varphi_n}$ with $\rho_n = |u_n|$. From now on, we will drop the subscripts again. Then, we can give a meaning to

$$\partial_j u - iA_j u = (\partial_j \rho) \frac{u}{|u|} + iu\partial_j \varphi - iuA_j$$

and thus

$$\begin{cases} |\partial_j^A u|^2 = (\partial_j \rho)^2 + \rho^2 (\partial_j \varphi - A_j)^2 \\ (\partial_1^A u, \partial_2^A u) = \partial_1 \rho \partial_2 \rho + \rho^2 (\partial_1 \varphi - A_1)(\partial_2 \varphi - A_2). \end{cases} \tag{III.4}$$

But, since (u, A) verifies (I.2), we recall we have

$$-\nabla^\perp h = \rho^2 (\nabla \varphi - A). \tag{III.5}$$

Hence, at least in $\Omega \setminus \cup_i B_i$, we can write

$$\begin{cases} |\partial_1^A u|^2 = \partial_1 |u|^2 + \frac{|\partial_2 h|^2}{|u|^2} \\ |\partial_2^A u|^2 = \partial_2 |u|^2 + \frac{|\partial_1 h|^2}{|u|^2} \\ (\partial_1^A u, \partial_2^A u) = \partial_1 |u| \partial_2 |u| - \frac{\partial_2 h \partial_1 h}{|u|^2}. \end{cases} \tag{III.6}$$

Thus, T_{ij} can be written in the following form :

$$\begin{aligned} T_{ij} = & \left(\frac{h^2}{2} - \frac{1}{4\varepsilon^2} (1 - |u|^2)^2 \right) \delta_{ij} \\ & + \frac{1}{2} \begin{pmatrix} \partial_1 |u|^2 - \partial_2 |u|^2 & 2\partial_1 |u| \partial_2 |u| \\ 2\partial_1 |u| \partial_2 |u| & \partial_2 |u|^2 - \partial_1 |u|^2 \end{pmatrix} \\ & + \frac{1}{2|u|^2} \begin{pmatrix} (\partial_2 h)^2 - (\partial_1 h)^2 & -2\partial_1 h \partial_2 h \\ -2\partial_1 h \partial_2 h & (\partial_1 h)^2 - (\partial_2 h)^2 \end{pmatrix} \end{aligned} \tag{III.7}$$

We prove that the second term in the right-hand side tends to 0.

Lemma III.1 *Writing $\rho_n = |u_n|$, we have*

$$\int_{\Omega \setminus \cup_i B(a_i, 2r_i)} \frac{|\nabla \rho_n|^2}{2} + \frac{1}{4\varepsilon^2} (1 - \rho_n^2)^2 \longrightarrow 0 \quad \text{as } n \to \infty.$$

Proof : For ease of notations, we drop again the subscripts n. We follow the scheme of the proof of Proposition VI.1 of [BR]. Projecting the first equation (I.1) on u yields the following equation on ρ, valid in $\Omega\backslash\cup_i B_i$:

$$-\Delta\rho+\frac{|\nabla h|^2}{\rho^3}=\frac{1}{\varepsilon^2}\rho(1-\rho^2). \tag{III.8}$$

We use again the function ζ defined in (II.24), multiply (III.8) by $\zeta(1-\rho)$ and integrate :

$$\int_\Omega -\zeta\Delta\rho(1-\rho)+\frac{|\nabla h|^2}{\rho^3}\zeta(1-\rho)=\frac{1}{\varepsilon^2}\int_\Omega\zeta(1-\rho)^2\rho(1+\rho).$$

Integrating by parts, since $\frac{\partial\rho}{\partial n}=0$ on $\partial\Omega$ (see the boundary condition (I.4)), we are led to

$$\int_\Omega|\nabla\rho|^2\zeta+\frac{(1-\rho)^2\rho(1+\rho)}{\varepsilon^2}\zeta=\int_\Omega\frac{|\nabla h|^2}{\rho^3}\zeta(1-\rho)+\nabla\zeta\cdot\nabla\rho(1-\rho). \tag{III.9}$$

Knowing that $\zeta\equiv 0$ on $\cup_i B(a_i,r_i)$, and that $0\le 1-\rho_n\le O(\frac{1}{|\log\varepsilon|^4})$ in $\Omega\backslash\cup_i B_i$, we can bound the first term of the right-hand side :

$$\begin{aligned}\left|\int_\Omega\frac{|\nabla h|^2}{\rho^3}\zeta(1-\rho)\right| &\le \int_\Omega|\nabla h|^2\|1-\rho\|_{L^\infty(\Omega\backslash\cup_i B_i)}\\ &\le o(1),\end{aligned}$$

using the fact that $\int_\Omega|\nabla h|^2\le Ch_{\text{ex}}^2$. The second term of the right-hand side can be bounded as follows :

$$\left|\int_\Omega(\nabla\zeta\cdot\nabla\rho)(1-\rho)\right|\le\|\nabla\rho\|_{L^2}\|1-\rho\|_{L^2}\|\nabla\zeta\|_{L^\infty}. \tag{III.10}$$

In view of (II.24), $\|\nabla\zeta\|_{L^\infty}\le 1/(\varepsilon|\log\varepsilon|^4)$, while, from (I.36), we have $\|\nabla\rho\|_{L^2}\le O(h_{\text{ex}})$ and $\int_\Omega(1-\rho)^2\le C\varepsilon^2h_{\text{ex}}^2$. Combining all this, we get

$$\left|\int_\Omega(\nabla\zeta\cdot\nabla\rho)(1-\rho)\right|=o(1),$$

hence the right-hand side of (III.9) tends to 0.
On the other hand, the left-hand side can be bounded from below as follows

$$\int_\Omega|\nabla\rho|^2\zeta+\frac{(1-\rho)^2\rho(1+\rho)}{\varepsilon^2}\zeta\ge\frac{1}{C}\int_{\Omega\backslash\cup_i B(a_i,2r_i)}|\nabla\rho|^2+\frac{(1-\rho^2)^2}{\varepsilon^2},$$

from which we deduce the desired result. □

Since $\cup_i B(a_i,2r_i)$ is included in E_δ, this lemma yields the convergence to 0 in $L^1(\Omega\backslash E_\delta)$ of the second term of the right-hand side of (III.7), as well as that of the term $\frac{1}{\varepsilon^2h_{\text{ex}}^2}(1-|u|^2)^2\delta_{ij}$.

$\frac{h}{h_{\rm ex}}$ converges to h_∞ weakly in H^1, hence strongly in $L^2(\Omega)$, hence the term $\frac{h^2}{2h_{\rm ex}^2}\delta_{ij}$ converges to $\frac{h_\infty^2}{2}\delta_{ij}$ in $L^1(\Omega)$.
For the third term, we use $\|\frac{1}{|u|^2}-1\|_{L^\infty(E_\delta)}\to 0$, (since $|u|\geq 1-o(1)$ in $\Omega\backslash\cup_i B_i$), and combine it with the result of Proposition II.1. We are thus led to
(III.11)

$$\frac{1}{h_{\rm ex}^2|u_n|^2}\begin{pmatrix}(\partial_2 h_n)^2-(\partial_1 h_n)^2 & -2\partial_1 h_n\partial_2 h_n\\ -2\partial_1 h_n\partial_2 h_n & (\partial_1 h_n)^2-(\partial_2 h_n)^2\end{pmatrix}\longrightarrow\begin{pmatrix}(\partial_2 h_\infty)^2-(\partial_1 h_\infty)^2 & -2\partial_1 h_\infty\partial_2 h_\infty\\ -2\partial_1 h_\infty\partial_2 h_\infty & (\partial_1 h_\infty)^2-(\partial_2 h_\infty)^2\end{pmatrix}$$

strongly in $L^1(\Omega\backslash E_\delta)$. This completes the proof of Proposition III.1. □

Proposition III.2

$$\forall i,\quad \mathrm{div}\ L_{ij}=0\quad in\ \Omega.$$

(or $\forall i,\sum_j\partial_j L_{ij}=0$ *in* Ω.*)*

Proof : It relies on Proposition III.1 and the co-area formula. Let ξ be a $C_0^\infty(\Omega)$ test-function. Let us denote $\gamma_t=\{x\in\Omega,\xi(x)=t\}$ the level-sets of ξ. Since div $T_{ij}=0$, we have

$$\text{(III.12)}\qquad \forall i=1,2,\qquad \int_{\gamma_t}T_{ij}\cdot\nu=0.$$

On the other hand, using the co-area formula of Federer and Fleming, we can write

$$\text{(III.13)}\qquad \forall i\quad \int_\Omega L_{ij}\cdot\nabla\xi=\int_t\left(\int_{\gamma_t}L_{ij}\cdot\nu\right)dt.$$

We recall that $\mathrm{cap}_1(E_\delta)<\delta$, where E_δ is given in Proposition III.1, and that the cap_1 controls the perimeter, hence

$$\text{(III.14)}\qquad meas\{t,\gamma_t\cap E_\delta\neq\varnothing\}\leq Cper(E_\delta)<o_\delta(1).$$

Using the coarea formula again,

$$\begin{aligned}\forall i\quad \left|\int_{t/\gamma_t\cap E_\delta=\varnothing}\left(\int_{\gamma_t}\left(L_{ij}-\frac{T_{ij}}{h_{\rm ex}^2}\right)\cdot\nu\right)dt\right| &= \left|\int_{x\in\Omega/\xi(x)\notin\xi(E_\delta)}\left(L_{ij}-\frac{T_{ij}}{h_{\rm ex}^2}\right)\cdot\nabla\xi\right|\\ &\leq \int_{x\in\Omega/\xi(x)\notin\xi(E_\delta)}\left|L_{ij}-\frac{T_{ij}}{h_{\rm ex}^2}\right||\nabla\xi|\\ \text{(III.15)}\qquad &\leq \|\nabla\xi\|_{L^\infty(\Omega)}\int_{\Omega\backslash E_\delta}\left|L_{ij}-\frac{T_{ij}}{h_{\rm ex}^2}\right|\longrightarrow 0,\end{aligned}$$

from the result of Proposition III.1. Thus, with (III.12),

$$\text{(III.16)}\qquad \forall\delta>0,\quad \int_{t/\gamma_t\cap E_\delta=\varnothing}\left(\int_{\gamma_t}L_{ij}\cdot\nu\right)dt=0.$$

We write $f(t) = \int_{\gamma_t} L_{ij} \cdot \nu$. $f \in L^1$ because $L_{ij} \in L^1$:

$$\int_t |f(t)| dt = \int_t \left| \int_{\gamma_t} L_{ij} \cdot \nu \right| dt \leq \int_t \int_{\gamma_t} |L_{ij}| \, dt \leq \int_\Omega |L_{ij}| |\nabla \xi| < \infty.$$

From (III.16),

$$\forall \delta > 0, \quad \int_t f(t) \mathbf{1}_{t/\gamma_t \cap E_\delta = \varnothing} \, dt = 0,$$

but from (III.14), $\mathbf{1}_{t/\gamma_t \cap E_\delta = \varnothing} \to 1$ almost everywhere, as $\delta \to 0$. Hence, from Lebesgue's dominated convergence theorem, passing to the limit $\delta \to 0$, we get

$$\int_t f(t) \, dt = 0. \tag{III.17}$$

Inserting (III.17) into (III.13), we obtain

$$\forall i, \quad \int_\Omega L_{ij} \cdot \nabla \xi = 0. \tag{III.18}$$

This is true for any $\xi \in C_0^\infty(\Omega)$, thus div $L_{ij} = 0$. □

Extending this proof, we can in fact get the more general result stated in the introduction:

Lemma III.2 *If Df is some linear combination of first order derivatives of f, f_n bounded in $L^1(\Omega)$, $f \in L^1(\Omega)$, Df_n bounded in $L^1(\Omega)$, and $\forall \delta > 0$, $\exists E_\delta$ such that $cap_1(E_\delta) < \delta$ and $\int_{\Omega \setminus E_\delta} |f_n - f| \to 0$, then $Df_n \to Df$ in the sense of distributions, hence also in the weak sense of measures.*

IV Consequences for h_∞ and μ_∞

Let us recall that the expression of L_{ij} was given in (I.39), and that L is itself the stress-energy tensor associated to the Lagrangian :

$$\mathcal{L}(h) = \frac{1}{2} \int_\Omega |\nabla h|^2 + h^2 = \int_\Omega \ell, \tag{IV.1}$$

defined over $H_1^1(\Omega)$. Indeed,

$$L_{ij}(h) = \left(\ell \delta_{ij} - \partial_j h \frac{\partial \ell}{\partial(\partial_i h)} \right)(h), \tag{IV.2}$$

which is exactly the expression for the stress-energy tensor associated to $\mathcal{L}$ (see [He] for example).
Similarly as for J, the following general property holds again for $\mathcal{L}$: h_∞ is stationary for $\mathcal{L}$ with respect to inner variations (i.e. domain diffeomorphisms), if and only if

div $L_{ij}(h_\infty) = 0$. Hence, we deduce from Proposition III.2 that h_∞ is stationary for H with respect to inner variations.

We now use the complex-variables notations $\partial = \partial_1 - i\partial_2$ and $\overline{\partial} = \partial_1 + i\partial_2$, and introduce the Hopf differential of h_∞ :

$$U = \frac{1}{2}\left((\partial_1 h_\infty)^2 - (\partial_2 h_\infty)^2\right) - i\partial_1 h_\infty \partial_2 h_\infty. \tag{IV.3}$$

Obviously,

$$U = \frac{1}{2}(\partial h_\infty)^2. \tag{IV.4}$$

Now, div $L_{ij} = 0$ is equivalent to

$$\operatorname{div}\begin{pmatrix} \frac{1}{2}\left((\partial_1 h_\infty)^2 - (\partial_2 h_\infty)^2\right) - i\partial_1 h_\infty \partial_2 h_\infty \\ \partial_1 h_\infty \partial_2 h_\infty + \frac{i}{2}\left((\partial_1 h_\infty)^2 - (\partial_2 h_\infty)^2\right) \end{pmatrix} = \operatorname{div}\begin{pmatrix} \frac{h_\infty^2}{2} \\ -i\frac{h_\infty^2}{2} \end{pmatrix}, \tag{IV.5}$$

which we can also write

$$\partial_1 U + i\partial_2 U = \frac{1}{2}(\partial_1 h_\infty^2 - i\partial_2 h_\infty^2)$$

or

$$\overline{\partial} U = \partial \frac{h_\infty^2}{2}. \tag{IV.6}$$

We notice that this is an elliptic equation for U, which will provide some extra regularity of U since $\partial h_\infty^2 \in L^p(\Omega)$ for all $p < 2$ (since $h_\infty \in H_1^1(\Omega)$).

Lemma IV.1 div $L_{ij} = 0$ *implies that*

$$|\nabla h_\infty|^2 \in W^{1,p}(\Omega) \quad \forall 1 \le p < \infty.$$

Proof : The main problem is to get regularity of U up to the boundary.
Equation (IV.5) expresses that h_∞ is stationary with respect to inner variations for (IV.1). Since Ω is simply connected, up to a conformal transformation, we can reduce to the case of the unit ball B_1, with a h stationary with respect to inner variations for a new functional $\int_{B_1} |\nabla h|^2 + \phi h^2$ over $H_1^1(B_1)$, thus solution of another equation of the form

$$\operatorname{div}\begin{pmatrix} \frac{1}{2}\left((\partial_1 h)^2 - (\partial_2 h)^2\right) - i\partial_1 h \partial_2 h \\ \partial_1 h \partial_2 h + \frac{i}{2}\left((\partial_1 h)^2 - (\partial_2 h)^2\right) \end{pmatrix} = \operatorname{div} F, \tag{IV.7}$$

where F has the same regularity as h_∞^2, thus in $W^{1,p}(B_1)$ for all $p < 2$. Then, since $h = 1$ on ∂B_1, we can perform a Schwartz reflexion to extend it to $\mathbb{R}^2$, by setting

$$\tilde{h}(z) = h(\frac{1}{\overline{z}}) \quad \in \mathbb{R}^2 \backslash B_1.$$

Notice that $\frac{\partial h}{\partial n} = -\frac{\partial \tilde{h}}{\partial n}$ on ∂B_1. $\tilde{h}$ is also stationary with respect to inner variations for a functional of the form $\int_{\mathbb{R}^2\backslash B_1} |\nabla h|^2 + \psi h^2$, thus again is solution of an equation of the form (IV.7). For simplicity, let us denote by h the function equal to h in B_1 extended by $\tilde{h}$ in $\mathbb{R}^2\backslash B_1$. We claim that $\operatorname{div} \begin{pmatrix} \frac{1}{2}\left((\partial_1 h)^2 - (\partial_2 h)^2\right) - i\partial_1 h \partial_2 h \\ \partial_1 h \partial_2 h + \frac{i}{2}\left((\partial_1 h)^2 - (\partial_2 h)^2\right) \end{pmatrix}$ does not have any singular part on ∂B_1. Indeed, its value should not depend on the choice of coordinates, thus, near a point $x_0 \in \partial B_1$, we can assume that the orthonormal coordinate frame is the local frame (τ, ν). Or in other words, we can work near the point $(0,1)$. Since $h = 1$ on ∂B_1, at this point, we have $\partial_1 h = 0$. Then

$$\operatorname{div} \begin{pmatrix} \frac{1}{2}\left((\partial_1 h)^2 - (\partial_2 h)^2\right) - i\partial_1 h \partial_2 h \\ \partial_1 h \partial_2 h + \frac{i}{2}\left((\partial_1 h)^2 - (\partial_2 h)^2\right) \end{pmatrix} = \operatorname{div} \begin{pmatrix} -\frac{1}{2}(\partial_2 h)^2 \\ -\frac{i}{2}(\partial_2 h)^2. \end{pmatrix}.$$

But $(\partial_2 h)^2(0,1) = \left(\frac{\partial h}{\partial n}\right)^2(0,1)$ remains continuous, thus this divergence will not have any singular part on ∂B_1. Therefore, we can write

$$\operatorname{div} \begin{pmatrix} \frac{1}{2}\left((\partial_1 h)^2 - (\partial_2 h)^2\right) - i\partial_1 h \partial_2 h \\ \partial_1 h \partial_2 h + \frac{i}{2}\left((\partial_1 h)^2 - (\partial_2 h)^2\right) \end{pmatrix} = \operatorname{div} F, \tag{IV.8}$$

over all $\mathbb{R}^2$, with F some $W^{1,p}(\mathbb{R}^2)$ field ($\forall p < 2$). Going back to our original problem, we can assume that equation (IV.6) holds in a strictly bigger domain than Ω. Consequently, by elliptic regularity,

$$U \in W^{1,p}(\Omega) \qquad \forall p < 2. \tag{IV.9}$$

Thus, $U \in L^q(\Omega)$, $\forall q$. But, from (IV.4),

$$|U| = \frac{1}{2}|\nabla h_\infty|^2, \tag{IV.10}$$

hence we deduce that $|\nabla h_\infty|^2 \in \cap_q L^q(\Omega)$. Thus $h_\infty \in \cap_q W^{1,q}(\Omega)$, and going back to (IV.6) and using elliptic regularity again, we get that $U \in \cap_q W^{1,q}(\Omega)$, hence L_{ij} too. Using (IV.10) again, we are led to $|\nabla h_\infty|^2 \in \cap_q W^{1,q}(\Omega)$. □

Proposition IV.1 *h_∞ is critical with respect to inner variations for $\mathcal{L}$ over $H^1_1(\Omega)$, and, if $\nabla h_\infty \in C^0(\Omega)$,*

$$(\nabla h_\infty)\mu_\infty = 0.$$

Proof : The first assertion has already been justified.
Let us set $h_t(x) = h_\infty(x + tX(x))$ where $X(x) \in C_0^\infty(\Omega, \mathbb{R}^2)$. If ∇h_∞ is continuous, then, by definition of the derivative,

$$\lim_{t\to 0} \frac{h_\infty(x + tX(x)) - h_\infty(x)}{t} = \nabla h_\infty(x) \cdot X(x) \quad \text{uniformly in } x, \tag{IV.11}$$

the uniformity coming from the continuity of ∇h_∞. Therefore,

$$h_t(x) = h_\infty(x) + t\nabla h_\infty(x) \cdot X(x) + o(t). \tag{IV.12}$$

Now, we perform variations in $\mathcal{L}$ with this family of functions.

$$\mathcal{L}(h_t) - \mathcal{L}(h_\infty) = \frac{1}{2}\int_\Omega |\nabla h_t|^2 - |\nabla h_\infty|^2 + |h_t|^2 - |h_\infty|^2.$$

$$\begin{aligned} \frac{\partial}{\partial t}\Big|_{t=0} \mathcal{L}(h_t) &= \frac{\partial}{\partial t}\Big|_{t=0} \int_\Omega \nabla h_t \cdot \nabla h_\infty + h_t h_\infty \\ &= \frac{\partial}{\partial t}\Big|_{t=0} \int_\Omega (-\Delta h_\infty + h_\infty) h_t. \end{aligned} \tag{IV.13}$$

Indeed, $-\Delta h_\infty + h_\infty$ is a measure, h_t is C^0, hence this integral and the integration by parts have a meaning. From (IV.12),

$$\begin{aligned} \frac{\partial}{\partial t}\Big|_{t=0} \mathcal{L}(h_t) &= \frac{\partial}{\partial t}\Big|_{t=0} \int_\Omega (-\Delta h_\infty + h_\infty)(h_\infty + t\nabla h_\infty \cdot X + o(t)) \\ &= \int_\Omega (-\Delta h_\infty + h_\infty)\nabla h_\infty \cdot X. \end{aligned} \tag{IV.14}$$

Again, this integration is valid since $-\Delta h_\infty + h_\infty$ is a measure and $\nabla h_\infty \cdot X$ is continuous. Since h_∞ is stationary with respect to domain-diffeomorphisms, $\frac{\partial}{\partial t}\big|_{t=0} \mathcal{L}(h_t) = 0$, and thus in view of (IV.14)

$$\forall X \in C_0^\infty(\Omega, \mathbb{R}^2) \qquad \int_\Omega (-\Delta h_\infty + h_\infty)\nabla h_\infty \cdot X = 0.$$

Consequently,

$$(-\Delta h_\infty + h_\infty)\nabla h_\infty = 0.$$

This is the desired conclusion, and again, it has a meaning as the product of a measure μ_∞ with a continuous function ∇h_∞. □

Proposition IV.2 *If $\nabla h_\infty \in C^0(\Omega)$ and $|\nabla h_\infty| \in BV(\Omega)$, then h_∞ and μ_∞ satisfy the additional properties*

$$0 \leq h_\infty \leq 1$$
$$\mu_\infty = h_\infty \mathbf{1}_{|\nabla h_\infty|=0}.$$

Hence μ_∞ is a positive measure, absolutely continuous with respect to the Lebesgue measure.

Proof : We start by proving the following lemma. For BV functions and perimeters, we refer to [EG, Gi].

Lemma IV.2 *Let $f \in BV(\Omega)$, there exists a sequence $s_n \to 0$ such that $s_n|\{f = s_n\}| \to 0$, where $|.|$ denotes the perimeter.*

Proof : Let us denote by $U_t = \{x \in \Omega / f(x) > t\}$. By definition of BV, we have, for $f \in L^1(\Omega)$,

$$f \in BV(\Omega) \Leftrightarrow \int_\Omega |\nabla f| < \infty \Leftrightarrow \int_{s\in\mathbb{R}} |\partial U_s| ds < \infty,$$

where $|.|$ denotes the perimeter. Let us assume by contradiction that there exists $\eta > 0$ such that, for all s in a neighborhood of 0, $s|\partial U_s| > \eta$. Then, $|\partial U_s| > \frac{\eta}{s}$ and $\int_s |\partial U_s| ds$ would diverge. Hence the assertion of the lemma is true. □

We apply this lemma to $f = |\nabla h_\infty|$ assumed to be in $BV(\Omega)$. There exists $s_n \to 0$ such that $s_n|\partial\{|\nabla h_\infty| > s_n\}| \to 0$.
Let now h_- denote the negative part of h_∞. We may write

$$\text{(IV.15)} \quad \int_{|\nabla h_\infty|>s_n} (-\Delta h_\infty + h_\infty) h_- = \\ -\int_{\partial\{|\nabla h_\infty|>s_n\}} \frac{\partial h_\infty}{\partial n} h_- + \int_{|\nabla h_\infty|>s_n} \nabla h_\infty \cdot \nabla h_- + h_\infty h_-.$$

But, by choice of s_n,

$$\left| \int_{\partial\{|\nabla h_\infty|>s_n\}} \frac{\partial h_\infty}{\partial n} h_- \right| \le \|h_\infty\|_{L^\infty(\Omega)} \int_{\partial\{|\nabla h_\infty|>s_n\}} |\nabla h_\infty| = \|h_\infty\|_{L^\infty(\Omega)} s_n |\partial\{|\nabla h_\infty| > s_n\}| \to 0.$$

Therefore, passing to the limit $n \to \infty$ in (IV.15), we get

$$\lim_{n\to\infty} \int_{|\nabla h_\infty|>s_n} (-\Delta h_\infty + h_\infty) h_- = \int_{|\nabla h_\infty|>0} |\nabla h_-|^2 + h_-^2.$$

But, $\forall n$,

$$\int_{|\nabla h_\infty|>s_n} (-\Delta h_\infty + h_\infty) h_- = 0$$

since, in view of Proposition IV.1, $(-\Delta h_\infty + h_\infty)\mathbf{1}_{|\nabla h_\infty|>0} = 0$. Consequently,

$$\int_{|\nabla h_\infty|>0} |\nabla h_-|^2 + h_-^2 = 0.$$

We deduce that

$$\{x \in \Omega, h_\infty \le 0\} \subset \{x \in \Omega, |\nabla h_\infty| = 0\},$$

in addition $h_\infty \ge 0$ on $\partial\Omega$, hence $h_\infty \ge 0$ in Ω.
Arguing similarly, let $(h_\infty - 1)_+$ be the positive part of $h_\infty - 1$. Testing it against $(-\Delta h_\infty + h_\infty)\mathbf{1}_{|\nabla h_\infty|>s_n}$, we obtain

$$\int_{|\nabla h_\infty|>0} h_\infty (h_\infty - 1)_+ + |\nabla (h_\infty - 1)_+|^2 = 0,$$

thus

$$\{x\in\Omega, h_\infty \geq 1\} \subset \{x\in\Omega, |\nabla h_\infty| = 0\}$$

and $h_\infty \leq 1$ in Ω.

For the second assertion, it goes as follows. Let us consider this time a sequence s_n such that $s_n|\partial\{|\nabla h_\infty| < s_n\}| \to 0$, and let us consider a test-function $\xi \in C_0^\infty(\Omega)$. In view of the result of Proposition IV.1, $\mu_\infty \mathbf{1}_{|\nabla h_\infty|>0} = 0$, hence we have

$$\begin{aligned} \int_\Omega \mu_\infty \xi &= \int_{|\nabla h_\infty|<s_n} \mu_\infty \xi \\ &= \int_{\partial\{|\nabla h_\infty|<s_n\}} \frac{\partial h_\infty}{\partial n}\xi + \int_{|\nabla h_\infty|<s_n} \nabla h_\infty \cdot \nabla \xi + h_\infty \xi. \end{aligned} \tag{IV.16}$$

But, on the one hand

$$\left|\int_{\partial\{|\nabla h_\infty|<s_n\}} \frac{\partial h_\infty}{\partial n}\xi\right| \leq \|\xi\|_{L^\infty} s_n |\partial\{|\nabla h_\infty| < s_n\}| \to 0$$

by choice of s_n, and, on the other hand

$$\left|\int_{|\nabla h_\infty|<s_n} \nabla h_\infty \cdot \nabla \xi\right| \leq C s_n \|\nabla \xi\|_{L^\infty} \to 0.$$

Thus, passing to the limit in (IV.16) yields the relation

$$\forall \xi \in C_0^\infty(\Omega),\ \int_\Omega \mu_\infty \xi = \int_{|\nabla h_\infty|=0} h_\infty \xi. \tag{IV.17}$$

We conclude that $\mu_\infty = h_\infty \mathbf{1}_{|\nabla h_\infty|=0}$, hence μ_∞ is a positive (recall that $h_\infty \geq 0$) L^∞ function. □

This completes the proof of Theorem 1.

V Proof of Theorem 2

Let us first rule out the case of $N \gg h_{\text{ex}}$. In view of Lemma II.2 combined with (II.16), we have

$$\left\|\frac{-\Delta h_n + h_n}{N} - \nu_n\right\|_{W^{-1,p}} \to 0. \tag{V.1}$$

But $\frac{h_n}{h_{\text{ex}}}$ is bounded in $H^1(\Omega)$ hence $\frac{h_n}{N} \to 0$ in $H^1(\Omega)$. Then, $\frac{1}{N}(-\Delta h_n + h_n) \rightharpoonup 0$ in H^{-1}, and, passing to the limit in (V.1), we are led to $\nu_n \rightharpoonup 0$.

From now on, we assume that we are in the other case $N \ll h_{\text{ex}}$. The method is the same as for Theorem 1. We start with the following lemma.

Lemma V.1 *$\frac{f_n}{N}$ is compact in $W_0^{1,p}(\Omega)$ and, up to extraction, converges to f_∞. For all $\delta > 0$, there exists E_δ such that $cap_1(E_\delta) < \delta$ and*

$$\int_{\Omega \setminus E_\delta} \left| \nabla \left(\frac{f_n}{N} - f_\infty \right) \right|^2 \longrightarrow 0 \qquad \text{as } n \to \infty.$$

Proof :
- *Step 1 :* From the result of Lemma II.2 with (II.16), and the fact that $-\Delta h_0 + h_0 = 0$, we have

$$\| - \Delta f_n + f_n - 2\pi \sum_i d_i \delta_{a_i} \|_{W^{-1,p}(\Omega)} \leq \varepsilon^{\alpha(p)}.$$

Hence

$$\left\| -\Delta \frac{f_n}{N} + \frac{f_n}{N} - \nu_n \right\|_{W^{-1,p}(\Omega)} \to 0.$$

But ν_n is bounded in the sense of measures hence is bounded in $W^{-1,p}(\Omega)$, and it converges weakly in $\mathcal{M}(\Omega)$ to a limiting measure ν_∞ (after extraction). Applying Murat's theorem again, we deduce that $-\Delta \frac{f_n}{N} + \frac{f_n}{N}$ is compact in $W^{-1,p}(\Omega)$ and converges to ν_∞. Since $f_n = 0$ on $\partial\Omega$, we deduce that $\frac{f_n}{N}$ converges strongly to some f_∞ in $W^{1,p}(\Omega)$, for $p < 2$. It is clear that $\nu_\infty = -\Delta f_\infty + f_\infty$.

- *Step 2 :* If we set $\tilde{f}_n = \tilde{h}_n - h_{\text{ex}} h_0$, where $\tilde{h}_n$ was defined in Lemma II.1, we have $\|\tilde{f}_n - f_n\|_{H^1(\Omega)} \to 0$. Then, we introduce, as in Proposition II.1 g_n solution of

$$\text{(V.2)} \qquad \begin{cases} -\Delta g_n + g_n = \nu_n & \text{in } \Omega \\ g_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Again, $\int_\Omega |\nabla g_n|^2 \leq C |\log \varepsilon|$. Then, it follows exactly as in Lemma II.3 that for any $p < 2$, there exists $\alpha(p)$ such that

$$\left\| \frac{\tilde{f}_n}{N} - g_n \right\|_{W^{1,p}(\Omega)} \leq \varepsilon^{\alpha(p)}$$
$$\|g_n - f_\infty\|_{W^{1,p}(\Omega)} \to 0.$$

- *Step 3 :* Exactly as in the proof of Lemmas II.4 and II.5, we can find $\mathcal{A}_n$ and $\mathcal{B}_n$ with $\text{cap}_1(\mathcal{A}_n) \to 0$ and $\text{cap}_1(\mathcal{B}_n) \to 0$ such that

$$\text{(V.3)} \qquad \int_{\Omega \setminus \mathcal{B}_n} |\nabla (g_n - f_\infty)|^2 \to 0$$

$$\text{(V.4)} \qquad \int_{\Omega \setminus \mathcal{A}_n} \left| \nabla \left(\frac{\tilde{f}_n}{N} - g_n \right) \right|^2 \to 0.$$

We can then add the union of the balls to get E_δ as done in Proposition II.1. $\square$

We are then going to pass to the limit again in the relation div $T_{ij} = 0$. Let us define the following tensor

$$R_{ij} = \frac{h_0^2}{2}\delta_{ij} + \frac{1}{2}\begin{pmatrix} (\partial_2 h_0)^2 - (\partial_1 h_0)^2 & -2\partial_1 h_0 \partial_2 h_0 \\ -2\partial_1 h_0\partial_2 h_0 & (\partial_1 h_0)^2 - (\partial_2 h_0)^2 \end{pmatrix} \tag{V.5}$$

It is easy to see that, since $-\Delta h_0 + h_0 = 0$, we have div $R_{ij} = 0$ for $i = 1, 2$. Let us then write

$$M_{ij} = \frac{h_n^2}{2}\delta_{ij} + \frac{1}{2}\begin{pmatrix} (\partial_2 h_n)^2 - (\partial_1 h_n)^2 & -2\partial_1 h_n \partial_2 h_n \\ -2\partial_1 h_n\partial_2 h_n & (\partial_1 h_n)^2 - (\partial_2 h_n)^2 \end{pmatrix} \tag{V.6}$$

We start again from the expression of T_{ij} given in (III.7). Using Lemma III.1, we deduce that

$$\int_{\Omega\backslash E_\delta} |T_{ij} - M_{ij}| \to 0. \tag{V.7}$$

Then, prove the following

Lemma V.2

$$\int_{\Omega\backslash E_\delta} \left| \frac{M_{ij} - h_{\text{ex}}^2 R_{ij}}{N h_{\text{ex}}} - K_{ij} \right| \to 0$$

where

$$K_{ij} = h_0 f_\infty \delta_{ij} + \frac{1}{2}\begin{pmatrix} \partial_2 h_0 \partial_2 f_\infty - \partial_1 h_0 \partial_1 f_\infty & -2(\partial_1 f_\infty \partial_2 h_0 + \partial_2 f_\infty \partial_1 h_0) \\ -2(\partial_1 f_\infty \partial_2 h_0 + \partial_2 f_\infty \partial_1 h_0) & \partial_1 h_0 \partial_1 f_\infty - \partial_2 h_0 \partial_2 f_\infty \end{pmatrix}. \tag{V.8}$$

Proof : Just expand h_n as $f_n + h_{\text{ex}} h_0$ and observe that

$$\begin{aligned} \frac{M_{ij} - h_{\text{ex}}^2 R_{ij}}{N h_{\text{ex}}} &= \frac{f_n^2}{2N h_{\text{ex}}}\delta_{ij} + \frac{1}{2N h_{\text{ex}}}\begin{pmatrix} (\partial_2 f_n)^2 - (\partial_1 f_n)^2 & -2\partial_1 f_n \partial_2 f_n \\ -2\partial_1 f_n \partial_2 f_n & (\partial_1 f_n)^2 - (\partial_2 f_n)^2 \end{pmatrix} \\ &+ \frac{f_n h_0}{N}\delta_{ij} + \frac{1}{2N}\begin{pmatrix} \partial_2 h_0 \partial_2 f_n - \partial_1 h_0 \partial_1 f_n & -2(\partial_1 f_n \partial_2 h_0 + \partial_2 f_n \partial_1 h_0) \\ -2(\partial_1 f_n \partial_2 h_0 + \partial_2 f_n \partial_1 h_0) & \partial_1 h_0 \partial_1 f_n - \partial_2 h_0 \partial_2 f_n \end{pmatrix}. \end{aligned}$$

In view of Lemma V.1, and since $N \ll h_{\text{ex}}$, the first part tends to 0 in $L^1(\Omega\backslash E_\delta)$, while the second part tends to K_{ij} in $L^1(\Omega\backslash E_\delta)$. $\square$

Now div $(T_{ij} - h_{\text{ex}}^2 R_{ij}) = 0$ for all n. Using (V.7) and Lemma V.2, we may pass to the limit in this relation as we did in Proposition III.2. We deduce that

$$\text{div } K_{ij} = 0,$$

where K_{ij} was defined in (V.8). Since $h_0 \in C^\infty$, we can check, by using test-functions, that in the sense of distributions,

$$\text{div } K_{ij} = \nabla h_0(-\Delta f_\infty + f_\infty).$$

Observe that this product has a meaning since $-\Delta f_\infty + f_\infty$ is a measure and ∇h_0 a continuous function. Therefore, we have

$$\nabla h_0(-\Delta f_\infty + f_\infty) = 0.$$

This completes the proof of Theorem 2.

VI The case without magnetic field

Exactly as in the magnetic-field case, knowing that u_ε is solution of (I.27) implies that the stress-energy tensor is divergence-free:

$$\text{(VI.1)} \qquad \operatorname{div} \left[-\frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\delta_{ij} + \begin{pmatrix} |\partial_1 u_\varepsilon|^2 - |\partial_2 u_\varepsilon|^2 & 2(\partial_1 u_\varepsilon, \partial_2 u_\varepsilon) \\ 2(\partial_1 u_\varepsilon, \partial_2 u_\varepsilon) & |\partial_2 u_\varepsilon|^2 - |\partial_1 u_\varepsilon|^2 \end{pmatrix} \right] = 0.$$

(This is a fact that was used in [BBH].) The method consists again in passing to the limit in this equation. But we saw in the introduction that we can define a U_ε by

$$\text{(VI.2)} \qquad \begin{cases} \nabla^\perp U_\varepsilon = \rho_\varepsilon^2 \nabla\varphi_\varepsilon = (iu_\varepsilon, \nabla u_\varepsilon) \\ \int_\Omega U_\varepsilon = 0 \end{cases}$$

First of all

$$\text{(VI.3)} \qquad |\nabla U_\varepsilon|^2 = \rho_\varepsilon^4 |\nabla\varphi_\varepsilon|^2 \le \rho_\varepsilon^2 |\nabla\varphi_\varepsilon|^2 \le |\nabla u_\varepsilon|^2,$$

where we have used $\rho_\varepsilon \le 1$, hence

$$\text{(VI.4)} \qquad \frac{1}{2}\int_\Omega |\nabla U_\varepsilon|^2 \le E_\varepsilon(u_\varepsilon)$$

and $\frac{U_\varepsilon}{\sqrt{E_\varepsilon(u_\varepsilon)}}$ is bounded in $H^1(\Omega)$ and has a weak limit U, up to extraction.

Formally, since $|u_\varepsilon| \simeq 1$,

$$\text{(VI.5)} \qquad \begin{cases} |\partial_1 u_\varepsilon|^2 - |\partial_2 u_\varepsilon|^2 \simeq (\partial_2 U_\varepsilon)^2 - (\partial_1 U_\varepsilon)^2 \\ (\partial_1 u_\varepsilon, \partial_2 u_\varepsilon) \simeq \partial_1 U_\varepsilon \partial_2 U_\varepsilon. \end{cases}$$

and the same method as in Sections II—IV, will yield

$$\text{(VI.6)} \qquad \operatorname{div} \begin{pmatrix} (\partial_1 U)^2 - (\partial_2 U)^2 & 2\partial_1 U \partial_2 U \\ 2\partial_1 U \partial_2 U & (\partial_2 U)^2 - (\partial_1 U)^2 \end{pmatrix} = 0,$$

which is the same as saying that the Hopf differential of U is holomorphic.

VI.1 Step 1

From u_ε we can construct vortex-balls as in Proposition I.1 and define $\tilde{u_\varepsilon}$ and $\tilde{\rho_\varepsilon} = |\tilde{u_\varepsilon}|$ as in (II.2), and then solve for

$$\text{(VI.7)} \qquad \begin{cases} \Delta\xi_\varepsilon = \operatorname{div}\,(\tilde{\rho_\varepsilon}^2 \nabla\varphi_\varepsilon) = \operatorname{div}\,((\tilde{\rho_\varepsilon}^2 - \rho_\varepsilon^2)\nabla\varphi_\varepsilon) & \text{in } \Omega \\ \xi_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

Using the fact that $\|(\tilde{\rho_\varepsilon}^2 - \rho_\varepsilon^2)\nabla\varphi_\varepsilon\|_{L^2(\Omega)} \to 0$, we have $\|\nabla\xi_\varepsilon\|_{L^2(\Omega)} \to 0$. Then, using Poincaré's lemma again, there exists $\tilde{U_\varepsilon}$ such that

$$\text{(VI.8)} \qquad \begin{cases} \tilde{\rho_\varepsilon}^2 \nabla\varphi_\varepsilon - \nabla\xi_\varepsilon = \nabla^\perp \tilde{U_\varepsilon} \\ \int_\Omega \tilde{U_\varepsilon} = 0. \end{cases}$$

Then again, substracting this relation from (VI.2)

$$\|U_\varepsilon - \tilde{U}_\varepsilon\|_{H^1(\Omega)} \to 0.$$

Like for the magnetic field case, we can replace ρ_ε by $\tilde{\rho}_\varepsilon$ and U_ε by $\tilde{U}_\varepsilon$, and we will only make an $o(1)$ error in the quadratic terms in ∇U_ε. Hence, without loss of generality, we may assume that $\rho_\varepsilon = 1$ outside of the balls B_i, and thus that $\Delta U_\varepsilon = \operatorname{curl}(\rho_\varepsilon^2 \nabla \varphi_\varepsilon) = 0$ outside of the balls too.

VI.2 Step 2

As already mentioned, we define $\mu_\varepsilon = 2\pi \sum_i d_i \delta_{a_i}$ where the (a_i, d_i) are any vortices satisfying the results of Proposition I.1. We have as in Lemma II.2,

$$\|\Delta U_\varepsilon - \mu_\varepsilon\|_{W^{-1,p}(\Omega)} \le \varepsilon^{\alpha(p)} \quad \forall p < 2. \tag{VI.9}$$

If $N_\varepsilon = \int_\Omega |\mu_\varepsilon| \ll \sqrt{E_\varepsilon(u_\varepsilon)}$ then $\frac{\mu_\varepsilon}{\sqrt{E_\varepsilon(u_\varepsilon)}} \to 0$ in the strong sense of measures and thus strongly in $W^{-1,p}(\Omega)$ for $p < 2$. From (VI.9), we get that ΔU_ε is compact in $W^{-1,p}$ $(p < 2)$ and its limit is 0.

If $N_\varepsilon \gg \sqrt{E_\varepsilon(u_\varepsilon)}$ then $\frac{\nabla U_\varepsilon}{N_\varepsilon} \to 0$ in $L^2(\Omega)$ from (VI.4). Thus, from (VI.9), $\frac{\mu_\varepsilon}{N_\varepsilon} \to 0$ in $W^{-1,p}$, $(p < 2)$. But, it is also bounded in the sense of measures, thus converges weakly to 0 in that sense.

We may now restrict to the third case where $N_\varepsilon \sim C\sqrt{E_\varepsilon(u_\varepsilon)}$ as $\varepsilon \to 0$. Then $\frac{\mu_\varepsilon}{\sqrt{E_\varepsilon(u_\varepsilon)}}$ remains bounded in the sense of measures, thus up to extraction, converges weakly to the limiting vorticity measure μ, which in view of (VI.9), is equal to ΔU.

Defining this time g_ε by

$$\begin{cases} \Delta g_\varepsilon = \frac{\mu_\varepsilon}{\sqrt{E_\varepsilon(u_\varepsilon)}} & \text{in } \Omega \\ g_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{VI.10}$$

One can check that $\int_{\Omega \setminus \cup_i B_i} |\nabla g_\varepsilon|^2 \le C|\log \varepsilon|$.

From now on we work on any compact $K \Subset \Omega$, and the constants will depend on K. We have the analogue results of Lemma II.2 and II.3, i.e.

$$\left\| \frac{U_\varepsilon}{\sqrt{E_\varepsilon(u_\varepsilon)}} - g_\varepsilon \right\|_{W^{1,p}(K)} \le C\varepsilon^{\alpha(p)} \tag{VI.11}$$

$$\|g_\varepsilon - U\|_{W^{1,p}_{loc}(\Omega)} \to 0 \tag{VI.12}$$

$$\left\| \frac{U_\varepsilon}{\sqrt{E_\varepsilon(u_\varepsilon)}} - U \right\|_{W^{1,p}_{loc}(\Omega)} \to 0. \tag{VI.13}$$

VI.3 Step 3

Using a truncation function $\chi \in C_0^\infty(\Omega)$ which is 1 in K, we can prove exactly as in Lemmas II.4 and II.5 that for all $\delta > 0$ there exists $E_\delta \subset \Omega$ with $\mathrm{cap}_1(E_\delta) < \delta$ and

$$\int_\Omega \chi \left| \nabla \left(\frac{U_\varepsilon}{\sqrt{E_\varepsilon(u_\varepsilon)}} - U \right) \right|^2 \to 0. \tag{VI.14}$$

Also, like in Lemma III.1, we can prove that

$$\frac{1}{E_\varepsilon(u_\varepsilon)} \int_{\Omega \setminus \cup_i B_i} \chi \left(|\nabla \rho_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-\rho_\varepsilon^2)^2 \right) \to 0. \tag{VI.15}$$

If we denote by T_{ij} the elements of the tensor

$$-\frac{1}{2\varepsilon^2}(1-\rho_\varepsilon^2)^2 \delta_{ij} + \begin{pmatrix} |\partial_1 u_\varepsilon|^2 - |\partial_2 u_\varepsilon|^2 & 2(\partial_1 u_\varepsilon, \partial_2 u_\varepsilon) \\ 2(\partial_1 u_\varepsilon, \partial_2 u_\varepsilon) & |\partial_2 u_\varepsilon|^2 - |\partial_1 u_\varepsilon|^2 \end{pmatrix}$$

and

$$L_{ij} = \begin{pmatrix} (\partial_2 U)^2 - (\partial_1 U)^2 & 2\partial_1 U \partial_2 U \\ 2\partial_1 U \partial_2 U & (\partial_1 U)^2 - (\partial_2 U)^2 \end{pmatrix}$$

then, we deduce that for all $\delta > 0$, there exists $E_\delta \subset \Omega$ with $\mathrm{cap}_1(E_\delta) < \delta$ and

$$\int_{\Omega \setminus E_\delta} \chi \left| \frac{T_{ij}}{E_\varepsilon(u_\varepsilon)} - L_{ij} \right| \to 0. \tag{VI.16}$$

Like in Proposition III.2, this implies that

$$\mathrm{div}\ L_{ij} = 0 \quad \text{in } K.$$

This being true for every $K \Subset \Omega$, we have div $L_{ij} = 0$ in Ω which is exactly like writing

$$\overline{\partial} \omega = 0 \quad \text{in} \quad \Omega$$

where $\omega = (\partial U)^2 = (\partial_1 U)^2 - (\partial_2 U)^2 - 2i \partial_1 U \partial_2 U$.

VI.4 Step 4

As a consequence, we get that $|\nabla U|^2 \in C^\infty(\Omega)$ but not necessarily ∇U. If we assume in addition that $\nabla U \in C^0$, (which will be the case in particular if $\mu = \Delta U \in L^p, p > 2$), then, we may prove as in Proposition IV.2 that $\mu \nabla U = 0$. Then if also $|\nabla U| \in BV$ (which will also be implied by $\mu \in L^p, p > 2$), we will deduce as in Proposition IV.2 that $\Delta U = \mu = 0$.

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