A Rigorous Derivation of a Free-Boundary Problem Arising in Superconductivity

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Résumé

On étudie l'énergie de Ginzburg-Landau des supraconducteurs soumis à un champ magnétique éventuellement non-uniforme, dans la limite d'un grand paramètre de Ginzburg-Landau κ. On montre que le champ magnétique induit associé aux minimiseurs de l'énergie converge lorsque $\kappa \to +\infty$ vers la solution d'un problème à frontière libre. Ce problème à frontière libre n'admet de solution non triviale que lorsque le champ magnétique appliqué est de l'ordre du "premier champ critique", c'est-à-dire de l'ordre de $\log \kappa$. Dans les autres cas, nos résultats sont inclus dans ceux que nous avions précédemment obtenus ([SS2, S1, SS1]). On obtient aussi un résultat de convergence de la densité de vortex.

Abstract

We study the Ginzburg-Landau energy of superconductors submitted to a possibly non-uniform magnetic field, in the limit of a large Ginzburg-Landau parameter κ. We prove that the induced magnetic fields associated to minimizers of the energy-functional converge as $\kappa \to +\infty$ to the solution of a free-boundary problem. This free-boundary problem has a nontrivial solution only when the applied magnetic field is of the order of the "first critical field", i.e. of the order of $\log \kappa$. In other cases, our results are contained in those we had previously obtained ([SS2, S1, SS1]). We also derive a convergence result for the density of vortices.

I Introduction

I.1 The Ginzburg-Landau model of superconductivity

The Ginzburg-Landau model was introduced in the fifties by Ginzburg and Landau as a phenomenological model of superconductivity. In this model, the Gibbs energy of a superconducting material, submitted to an external magnetic field is, in a suitable normalization,

$$J(u,A) = \frac{1}{2}\int_{\Omega} |\nabla_A u|^2 + \frac{\kappa^2}{2}\left(1-|u|^2\right)^2 + \int_{\mathbb{R}^3} |h-h_{\text{ex}}|^2. \tag{I.1}$$

Here, Ω is the domain occupied by the superconductor, κ is a dimensionless constant (the Ginzburg-Landau parameter) depending only on characteristic lengths of the material and of temperature. h_{ex} is the applied magnetic field, $A : \Omega \mapsto \mathbb{R}^3$ is the vector-potential, and the induced magnetic field in the material is $h = \operatorname{curl} A$. $\nabla_A = \nabla - iA$ is the associated covariant derivative. The complex-valued function u is called the "order-parameter". It is a pseudo-wave function that indicates the local state of the material. There can be essentially two phases in a superconductor : $|u(x)| \simeq 0$ is the normal phase, $|u(x)| \simeq 1$, the superconducting phase. The Ginzburg-Landau model was based on Landau's theory of phase-transitions. Since then, the model has been justified by the microscopic theory of Bardeen-Cooper-Schrieffer (BCS theory). $|u(x)|$ is then understood as the local density of superconducting electron pairs, called "Cooper pairs", responsible for the superconductivity phenomenon.

A common simplification, that we make, is to restrict to the two-dimensional model corresponding to a infinite cylindrical domain of section $\Omega \subset \mathbb{R}^2$ (smooth and simply connected), when the applied field is parallel to the axis of the cylinder, and all the quantities are translation-invariant. The energy-functional then reduces to

$$J(u,A) = \frac{1}{2}\int_{\Omega} |\nabla_A u|^2 + |h-h_{\text{ex}}|^2 + \frac{\kappa^2}{2}\left(1-|u|^2\right)^2. \tag{I.2}$$

Then $A : \Omega \mapsto \mathbb{R}^2$, h is real-valued, and h_{ex} is just a real parameter. The Ginzburg-Landau equations associated to this functional are

$$\text{(G.L.)}\begin{cases} -\nabla_A^2 u = \kappa^2 u(1-|u|^2) \\ -\nabla^{\perp} h = < iu, \nabla_A u > . \end{cases}$$

with the boundary conditions

$$\begin{cases} h = h_{\text{ex}} & \text{on } \partial\Omega \\ < \nabla u - iAu, n > = 0 & \text{on } \partial\Omega. \end{cases}$$

(Here $\nabla^{\perp}$ denotes $(-\partial_{x_2}, \partial_{x_1})$, and $< .,. >$ denotes the scalar-product in $\mathbb{R}^2$.) One can also notice that the problem is invariant under the gauge-transformations :

$$\begin{cases} u \to ue^{i\Phi} \\ A \to A + \nabla\Phi, \end{cases}$$

where $\Phi \in H^2(\Omega, \mathbb{R})$. Thus, the only quantities that are physically relevant are those that are gauge invariant, such as the energy J, the magnetic field h, the current $j = < iu, \nabla_A u >$, the zeros of u. We saw in [S1, SS1] that, up to a gauge-transformation, the natural space over which to minimize J is $\{(u, A) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)\}$.

I.2 Critical fields and vortices

When $\kappa > \frac{1}{\sqrt{2}}$ the superconductor is said to be "of type-II". As in our previous studies [SS1, SS2, S1, S2], we shall only consider the case in which κ is large (i.e. we study the "London limit" $\kappa \to \infty$ of "extreme type-II superconductors"), and for simplicity we set $\varepsilon = \frac{1}{\kappa}$. Superconductors in general, when cooled down below a critical temperature, become "superconducting", which means in particular that there can be permanent currents without energy dissipation. But they also have a particular behaviour when a magnetic field is applied, which we now describe for type-II superconductors (for further physics reference, see for example [T]).

When the applied field $h_{\rm ex}$ is small enough, the material is superconducting : $|u| \sim 1$, and the magnetic field is "expelled" (this is called the Meissner effect) :

$$\begin{cases} -\Delta h + h = 0 & \text{in } \Omega \\ h = h_{\rm ex} & \text{on } \partial\Omega. \end{cases}$$

When $h_{\rm ex}$ is raised, one observes two main phase-transitions, for two critical fields H_{c_1} and H_{c_2} : for $h_{\rm ex} = H_{c_1} = O(|\log \varepsilon|)$, there is a phase-transition from the superconducting state described above to the *"mixed-phase"* or *"mixed-state"*; for $h_{\rm ex} = H_{c_2} = O(\frac{1}{\varepsilon^2})$, from the "mixed-phase" to the normal state ($u \equiv 0, h \equiv h_{\rm ex}$). The mixed-phase is defined by the coexistence of the normal and superconducting phases in the sample. The normal phase is localized in small regions of characteristic size $\varepsilon = \frac{1}{\kappa}$ called *"vortices"*, surrounded by a superconducting region in which $|u| \sim 1$. u vanishes at the center of a vortex, and if C is a small circle centered at this point and φ the phase of u on C, the *degree* of the vortex is defined as

$$\frac{1}{2\pi} \int_C \frac{\partial \varphi}{\partial \tau} = d \in \mathbb{Z}$$

or as the topological-degree of the map $\frac{u}{|u|} : C \mapsto S^1$. In stable stationary situations, the vortices are generally all of degree 1, and they repell one another according to a coulombian interaction. When $h_{\rm ex}$ is close to H_{c_1}, there are only a few vortices, and when $h_{\rm ex}$ is increased, their number increases, and, in order to minimize their repulsion, they tend to arrange in a triangular lattice, called the "Abrikosov lattice". The induced magnetic field approximately satisfies

$$\begin{cases} -\Delta h + h = 2\pi \sum_i d_i \delta_{a_i} & \text{in } \Omega \\ h = h_{\rm ex} & \text{on } \partial\Omega, \end{cases}$$

where the a_i's are the centers of the vortices, and the d_i's their degrees.

We are interested in describing rigorously the mixed-phase (which we have started to do in [SS2]). Several numerical or formal studies have been carried out to describe this mixed-phase, for example those of Chapman, Rubinstein and Chapman [CRS], and Berestycki, Bonnet and Chapman [BBC].

I.3 Previous studies of vortices

On a mathematical viewpoint, many papers have made clear the mathematical mechanisms of the apparition of vortices, and the definitions of what can be called a "vortex-structure".

The first and main work was the book of Bethuel, Brezis and Hélein [BBH], where the authors studied the functional

$$F_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2, \tag{I.3}$$

in the limit $\varepsilon \to 0$. This corresponds to setting A and $h_{\rm ex}$ equal to zero. Then, the influence of the fields has to be replaced by a Dirichlet boundary condition $u = g$ on $\partial\Omega$, where g is an S^1-valued map of winding-degree $d > 0$. This boundary condition triggers the apparition of vortices. They proved that minimizers of F_ε have d vortices of degree one, and that the following expansion of the energy holds :

$$F_\varepsilon(u) \sim \pi d|\log\varepsilon| + W(a_1, \cdots, a_d) \quad \text{as } \varepsilon \to 0, \tag{I.4}$$

W, the "renormalized-energy" being a function depending only on the vortex-centers a_i.

Afterwards, Almeida-Bethuel [AB], Sandier [Sa], Jerrard [J], independently developed methods to construct vortices (or define a vortex-structure) of energetical cost $\pi|d||\log\varepsilon|$ for arbitrary maps u (and not only for critical points of F).

There has also been a lot of research on the full Ginzburg-Landau functional J itself, for example the study of radial solutions by Berger and Chen. Bethuel and Rivière have proved in [BR] results in the spirit of (I.4), replacing again the boundary conditions by Dirichlet boundary conditions (with $h_{\rm ex} = 0$).

In [S1, S2, S3], the full functional was studied for $h_{\rm ex}$ not too high above H_{c_1}, and without Dirichlet boundary conditions. More precisely, local minimizers of J were found by minimizing it over the set of configurations such that

$$F_\varepsilon(u) < M|\log\varepsilon|, \tag{I.5}$$

corresponding to configurations with a bounded number of vortices (as $\varepsilon \to 0$). This led to the following asymptotic expansion of H_{c_1} :

$$H_{c_1} \sim k_1(\Omega)|\log\varepsilon| + O(1) \quad \text{as } \varepsilon \to 0, \tag{I.6}$$

where

$$k_1(\Omega) = (2 \max |\xi_0|)^{-1},$$

ξ_0 being defined by

$$\text{(I.7)} \qquad \begin{cases} -\Delta \xi_0 + \xi_0 = -1 & \text{in } \Omega \\ \xi_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

It also led to the existence of branches of stable n-vortices solutions, whose vortices were located. The estimate (I.5) allowed to use the method of construction of vortices of Almeida and Bethuel, with an a priori uniform bound on the number of vortices. Then, thanks to this construction and uniform bound, an expansion of J of the type (I.4), involving also a renormalized energy, was derived.

The study of global minimizers of the energy (thus without the bound (I.5)) is more delicate in the sense that the number of vortices does not remain bounded as $\varepsilon \to 0$, and thus the type of analysis of [BBH] can no longer be reproduced. We have already studied two situations for the repartition of vortices. In the case $h_{\text{ex}} < H_{c_1}$, we proved in [SS1] that there are no vortices in the minimizing configurations. In the case $H_{c_1} \ll h_{\text{ex}} \ll \frac{1}{\varepsilon^2}$, we proved in [SS2] that there is a uniform repartition of vortices, of density $\frac{h_{\text{ex}}}{2\pi}$. (Here, notice that h_{ex} has to be considered as a function of ε, and recall that $H_{c_2} = O(\frac{1}{\varepsilon^2})$.) In both situations, vortices are defined through the methods of Sandier and Jerrard, i.e. in a weaker sense than as in [BBH] or [AB], where their number remains bounded.

I.4 Purpose of this paper

Our aim here is to describe the repartition of vortices in the minimizers for arbitrary applied fields, in particular for fields of the order of $|\log \varepsilon|$, case which was left open. We show that in the limit $\varepsilon \to 0$, minimizers of J have a uniform vortex-distribution in a sub-region $\omega_\lambda \subset \Omega$ which is the solution of a free-boundary problem resembling the model of [CRS, BBC]. In [CRS], the authors formally derive the equation for the limiting magnetic field without however computing the number of vortices for a minimizing configuration. On the contrary, our approach, which consists in expanding $\min J$ as in (I.4) (except that the positions of vortices are replaced by a measure), allows it.

For the sake of more generality, we consider in the sequel the functional

$$\text{(I.8)} \qquad J(u, A) = \frac{1}{2} \int_\Omega |\nabla_A u|^2 + |h - p h_{\text{ex}}|^2 + \frac{\kappa^2}{2} \left(1 - |u|^2\right)^2,$$

where $p(x)$ is a smooth (C^2 is fine) positive weight. When $p \equiv 1$, it is the standard Ginzburg-Landau energy. Otherwise, we consider applied fields of intensity $p(x)h_{\text{ex}}$, which may be able to account for non-uniform applied fields. Note that in [SS1, SS2], only uniform applied fields were considered, but the analysis there could probably be adapted to the case of a weighted field.

I.5 The main result

We consider $h_{\rm ex}$ as a function of ε, and assume throughout this paper that the limit

$$\lambda = \lim_{\varepsilon\to 0} \frac{|\log \varepsilon|}{h_{\rm ex}(\varepsilon)}$$

exists and is finite (this implies in particular that $h_{\rm ex} \to \infty$ as $\varepsilon \to 0$), and we also assume that (in the case $\lambda = 0$)

$$h_{\rm ex}(\varepsilon) \ll \frac{1}{\varepsilon^2}.$$

From now on we will write $h_{\rm ex}$ instead of $h_{\rm ex}(\varepsilon)$ and J for the corresponding functional, the ε-dependence being implicit.

By testing J with $(u \equiv 1, A \equiv 0)$ it is clear that, for minimizers, $\frac{J(u,A)}{h_{\rm ex}^2}$ remains bounded as $\varepsilon \to 0$. Here, we prove that $\frac{J}{h_{\rm ex}^2}$ converges in a sense similar to Γ-convergence to the limiting functional

$$E(f) = \frac{\lambda}{2}\int_\Omega |-\Delta(f-p)+f| + \frac{1}{2}\int_\Omega |\nabla(f-p)|^2 + |f-p|^2\,, \tag{I.9}$$

defined over

$$V = \{f \in H^1_p(\Omega) / -\Delta(f-p)+f \text{ is a Radon measure}\},$$

where H^1_p denotes the functions f in $H^1(\Omega)$ such that $f(x) = p(x)$ on the boundary. More precisely, we prove that the induced magnetic fields of minimizers of J converge, after a renormalization, to the minimizer of E.

The minimizer h_* of E will be proved to be unique and to be the solution of the following variational inequality, usually called an "obstacle problem" :

$$(\mathrm{P})\begin{cases} h_* \in H^1_p(\Omega) \\ h_* \geq p - \dfrac{\lambda}{2} \text{ in } \Omega \\ \forall v \in H^1_p(\Omega) \text{ such that } v \geq p - \dfrac{\lambda}{2}, \quad \displaystyle\int_\Omega (-\Delta(h_*-p)+h_*)(v-h_*) \geq 0 \end{cases}$$

This obstacle problem is quite standard (one can refer to [R] for example). It is a *free boundary problem* in the sense that h_* is determined by its coincidence set, defined by

$$\omega_\lambda := \left\{ x \in \Omega \mid h_*(x) = p(x) - \frac{\lambda}{2} \right\}.$$

It is also a classical result that the free boundary $\partial\omega_\lambda$ is not always smooth (there have been counterexamples by Schaeffer), yet, A. Bonnet and R. Monneau proved recently in

[BM] that it is smooth for almost every value of λ. When this is the case, ω_λ is determined by the fact that there exists a solution h_* to the over-determined system

$$\begin{cases} -\Delta(h_* - p) + h_* = 0 & \text{in } \Omega\backslash\omega_\lambda \\ h_* = p - \dfrac{\lambda}{2} & \text{in } \omega_\lambda \\ \dfrac{\partial(h_* - p)}{\partial n} = 0 & \text{on } \partial\omega_\lambda \\ h_* = p & \text{on } \partial\Omega, \end{cases}$$

In addition, $h_* \in C^{1,\alpha}(\Omega), \forall\alpha < 1$.

The first theorem we prove is

Theorem 1 *Assume $\lambda = \lim_{\varepsilon\to 0} \frac{|\log \varepsilon|}{h_{\text{ex}}}$ exists and is finite and, if $\lambda = 0$, $h_{\text{ex}} \ll 1/\varepsilon^2$. Consider, for every ε, $(u_\varepsilon, A_\varepsilon)$ minimizing J, and $h_\varepsilon = \operatorname{curl} A_\varepsilon$ the associated magnetic field. Then, as $\varepsilon \to 0$,*

$$\frac{h_\varepsilon}{h_{\text{ex}}} \underset{\varepsilon\to 0}{\rightharpoonup} h_* \quad \textit{weakly in } H^1(\Omega), \qquad \frac{h_\varepsilon}{h_{\text{ex}}} \underset{\varepsilon\to 0}{\longrightarrow} h_* \quad \textit{strongly in } W^{1,q}(\Omega), \forall q < 2,$$

where h_* is the unique minimizer of E, and the solution of the free-boundary problem (P). The lack of compactness in the above convergence is described by a defect measure :

$$\left|\nabla\left(\frac{h_\varepsilon}{h_{\text{ex}}} - h_*\right)\right|^2 \underset{\varepsilon\to 0}{\longrightarrow} \lambda\mu,$$

in the sense of measures, where $\mu = -\Delta(h_* - p) + h_*$.
In addition,

$$\lim_{\varepsilon\to 0} \frac{J(u_\varepsilon, A_\varepsilon)}{h_{\text{ex}}^2} = E(h_*) = \frac{\lambda}{2}\int_\Omega |\mu| + \frac{1}{2}\int_\Omega |\nabla(h_* - p)|^2 + |h_* - p|^2,$$

where E is defined by (I.9).

One can easily notice that if $\lambda = 0$ (i.e. if $h_{\text{ex}} \gg |\log \varepsilon|$), the solution of (P) is $h_* = p$, and $E(h_*) = 0$. In this case, the theorem asserts that

$$\frac{h}{h_{\text{ex}}} \to p \quad \text{strongly in } H^1, \quad \text{and} \quad \frac{\min J}{h_{\text{ex}}^2} \to 0.$$

This was already proved in [SS2] in the case $p \equiv 1$, and will be checked in other cases. In [SS2], the stronger result

$$\min J \sim \frac{1}{2}\text{vol}(\Omega) h_{\text{ex}} \log \frac{1}{\varepsilon\sqrt{h_{\text{ex}}}}$$

was proved (still for $\lambda = 0$). We conjecture that this stronger result holds when the applied field is $ph_{\rm ex}$, in the modified form:

$$\min J \sim \frac{1}{2}\left(\int_\Omega p(x)\,dx\right) h_{\rm ex}\log\frac{1}{\varepsilon\sqrt{h_{\rm ex}}}.$$

We will thus treat separately the simple case $\lambda = 0$ in Corollary II.2, and we now describe some stronger results for $\lambda > 0$.

I.6 Results for the case $\lambda > 0$

In this case, $h_{\rm ex}$ satisfies the a priori bound $h_{\rm ex} \le C|\log \varepsilon|$. Once we restrict ourselves to this case, the proof of Theorem 1 uses, as our previous papers, a construction of vortex-balls. Here, we use the following result, which is adjusted from [SS2] :

Proposition I.1 *If $h_{\rm ex} \le C|\log\,\varepsilon|$, there exists ε_0 such that if $\varepsilon < \varepsilon_0$ and $(u_\varepsilon, A_\varepsilon)$ is a minimizer of J, there exists a family of balls (depending on ε) $(B_i)_{i\in I_\varepsilon} = (B(a_i, r_i))_{i\in I_\varepsilon}$ satisfying :*

$$\left\{x/|u_\varepsilon| \le \frac{1}{2}\right\} \subset \bigcup_{i\in I_\varepsilon} B(a_i, r_i). \tag{I.10}$$

$$\sum_{i\in I_\varepsilon} r_i \le \frac{1}{|\log\,\varepsilon|^6} \tag{I.11}$$

$$\frac{1}{2}\int_{B_i} |\nabla(h_\varepsilon - ph_{\rm ex})|^2 \ge \pi|d_i||\log\,\varepsilon|(1-o(1)), \tag{I.12}$$

where $h_\varepsilon = \mathrm{curl}\,A_\varepsilon$, and $d_i = deg(\frac{u_\varepsilon}{|u_\varepsilon|}, \partial B_i)$ if $\overline{B_i} \subset \Omega$ and 0 otherwise.

This proposition will be proved at the beginning of Section III.

Using it, Theorem 1 can be made more precise :

Theorem 2 *Under the hypotheses of Theorem 1, assuming in addition that $\lambda > 0$, we have*

$$\frac{2\pi}{h_{\rm ex}}\sum_{i\in I_\varepsilon} d_i\delta_{a_i} \underset{\varepsilon\to 0}{\longrightarrow} \mu, \tag{I.13}$$

$$\frac{2\pi}{h_{\rm ex}}\sum_{i\in I_\varepsilon} |d_i|\delta_{a_i} \underset{\varepsilon\to 0}{\longrightarrow} \mu, \tag{I.14}$$

in the sense of measures, where

$$\mu = -\Delta(h_* - p) + h_* = (p - \frac{\lambda}{2})\mathbf{1}_{\omega_\lambda}$$

and the (a_i, d_i)'s are any "vortices" satisfying the results of Proposition I.1.

I.7 Interpretations and consequences

These results first indicate that $h_{\rm ex}h_*$ can be seen as a good approximation of h_ε as $\varepsilon \to 0$, and provide a new asymptotic expansion of the energy. Also, the vortex-density is approximately $h_{\rm ex}\mu$ so that, when there are vortices, the domain is split in two regions given by problem (P) : one central region ω_λ in which the vortex-density is roughly equal to $ph_{\rm ex} - \frac{1}{2}|\log \varepsilon|$ and vortices have positive degrees, and a peripheral region in which there are no vortices. When $h_{\rm ex}$ becomes much higher than $|\log \varepsilon|$, the central region tends to occupy the whole domain, and we are led to a vortex-density $ph_{\rm ex}$, which generalizes the results of [SS2].

Let us also point out that the defect measure for the H^1-convergence of $\frac{h}{h_{\rm ex}}$ to h_* in Theorem 1 exactly corresponds to the concentration of energy in the vortices, whereas $\frac{1}{2}\|h_* - p\|_{H^1}^2$ corresponds to the remaining energy outside of $\cup_i B_i$. This appears clearly in the proofs (see Section I.8 for a sketch). This phenomenon is of the same type as the one that was described by D. Cioranescu and F. Murat in [CM].

In this intermediate case $h_{\rm ex} \sim C|\log \varepsilon|$ (or $0 < \lambda < \infty$), the energy that concentrates in the vortices (tending to the first term in $E(h_*)$) is of the same order as the energy outside of the vortex-cores (which corresponds to the second term in $E(h_*)$); whereas when $h_{\rm ex} \gg |\log \varepsilon|$, the outside energy becomes negligible as can be seen from the construction of [SS2]. This also explains why the analysis in this intermediate case is more delicate, the two contributions of energy having to be precisely taken into account.

As already mentioned, these results, which describe a homogeneized behaviour of vortices, are reminiscent of existing free-boundary models for superconductivity of [CRS] and [BBC]. To our knowledge however, the range of $h_{\rm ex}$ for which our model is valid is somewhat clearer, and our approach, which consists in deriving the problem (P) rigorously from the minimization of J and *using the vortices*, seems to be new.
Let us also mention that the behaviour of solutions of problems like (P) has been studied (see [R] for instance). The similar Berestycki-Bonnet-Chapman model has also been studied by A. Bonnet and R. Monneau in [BM].

Now, problem (P) can be further analyzed to understand the apparition and location of vortices in link with the behaviour of the coincidence set. We proved in [S1] and [SS1] that — in the case $p(x) \equiv 1$ — below the first critical field H_{c_1}, the minimizer of the energy is vortex-less, where

$$H_{c_1} = \frac{1}{2\max|\xi_0|}|\log \varepsilon| + O(1), \tag{I.15}$$

and where ξ_0 was defined in (I.7). Then, we naturally wish to compare this former result with the ones we present here. This is the content of the following proposition, easily derived from the maximum principle. Let ψ be the solution of

$$\begin{cases} -\Delta\psi + \psi = -p & \text{in } \Omega \\ \psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{I.16}$$

ψ is the analogue of ξ_0 with the weight p.

Proposition I.2 *The function ψ being defined in (I.16), and ω_λ being the coincidence set corresponding to problem (P),*

1) $\Omega\backslash\omega_\lambda$ is connected.

2)

$$\omega_\lambda = \varnothing \Longleftrightarrow \lambda > 2\ \max|\psi| \Longleftrightarrow \lim_{\varepsilon\to 0}\frac{h_{\text{ex}}}{|\log\ \varepsilon|} < \frac{1}{2\ \max|\psi|}.$$

In this case $h_* = p + \psi = \Delta\psi$. In all cases, $h_* \geq p + \psi$.

3)

$$\exists C > 0, \quad \text{dist}(\omega_\lambda, \partial\Omega) \geq C\lambda.$$

For $p(x) \equiv 1$, our results match with [SS1, S1] since in this case $\psi = \xi_0$, giving the value (I.15) for H_{c_1} (actually, this only gives an equivalent as $\varepsilon \to 0$ of H_{c_1}, hence is less precise than our previous results). For $\lambda > 2\ \max|\xi_0|$, i.e. roughly for $h_{\text{ex}} < H_{c_1}$, Theorem 1 tells us that $\frac{h}{h_{\text{ex}}} \to \Delta\xi_0$, which was already proved in [S3] : this corresponds to the Meissner (i.e. vortex-less) solution, for which the magnetic field was known to be approximately solution of the London equation

$$\begin{cases} -\Delta h + h = 0 & \text{in } \Omega \\ h = h_{\text{ex}} & \text{on } \partial\Omega, \end{cases}$$

the solution of this equation being exactly $h_{\text{ex}}\Delta\xi_0$. Therefore, when $p \equiv 1$, the results we get here are really new only for $0 < \lambda < 2\ \max|\xi_0|$, which corresponds to the intermediate region $H_{c_1} \leq h_{\text{ex}} \leq O(|\log\ \varepsilon|)$. On the other hand, if $p(x)$ is not identically 1, they yield a new estimate of the corresponding "first critical field"

$$H_{c_1} \sim \frac{1}{2\ \max\ |\psi|}|\log\ \varepsilon|.$$

The last assertion in Proposition I.2 allows us to control the growth of the vortex-region ω_λ as it tends to occupy all of Ω.

Proof of Proposition I.2, assuming Theorem 2 : The proof of the first assertion follows [CRS]. If $\Omega \setminus \omega_\lambda = \varnothing$, there is nothing to prove. If not, then $\lambda \neq 0$, therefore $h_* - p = -\lambda/2 < 0$ on ω_λ while $h_* - p = 0$ on $\partial\Omega$, thus $\partial\omega_\lambda \cap \partial\Omega = \varnothing$. Then from the simple connectedness of Ω, it follows that if $\Omega\backslash\omega_\lambda$ was not connected, it would have a connected component Ω' such that $\partial\Omega' \subset \Omega$. Hence

$$\begin{cases} -\Delta(h_* - p) + h_* = 0 & \text{in } \Omega' \\ h_* = p - \frac{\lambda}{2} & \text{on } \partial\Omega'. \end{cases}$$

Thus, $h_* < p - \frac{\lambda}{2}$ in Ω' by the maximum principle, contradicting (P).

We turn to the second assertion. From Theorems 1 and 2, $-\Delta(h_* - p) + h_* \geq 0$ which can be written as $-\Delta(h_* - p) + h_* - p \geq -p$, while $h_* - p = 0$ on $\partial\Omega$. Subtracting (I.16) and using the maximum principle, we find that

$$h_* - p \geq \psi. \tag{I.17}$$

But $h_* - p = -\frac{\lambda}{2}$ on ω_λ. Thus, if $\omega_\lambda \neq \emptyset$, then $|\psi| \geq \lambda/2$ on ω_λ hence $\lambda \leq 2 \max|\psi|$. On the other hand, if $\omega_\lambda = \emptyset$, $h_* - p = \psi$. But from the second equation of (P), we have $h_* - p > -\lambda/2$, this implies $|\psi| < \lambda/2$, finishing the proof of the second assertion.

For the third assertion, we use (I.17) again, which yields $\omega_\lambda \subset \{x \in \Omega \mid \psi(x) \leq -\frac{\lambda}{2}\}$. As $\psi = 0$ on $\partial\Omega$, there exists a $C > 0$ such that $\psi \geq -C \operatorname{dist}(., \partial\Omega)$, hence the distance between $\partial\Omega$ and the set $\{x/\psi \leq -\frac{\lambda}{2}\}$ remains bounded from below by $C\lambda$ for some other $C > 0$, proving the third assertion and the proposition. □

I.8 Outline of the paper

In Section II, we prove the upper bound $\frac{\min J}{h_{\text{ex}}^2} \leq E(h) + o(1)$ — where E is defined in (I.9) — for any $h \in H_p^1(\Omega)$ such that $\mu = -\Delta(h - p) + h$ is a positive Radon measure, absolutely continuous with respect to the Lebesgue measure. It will be proved in Section III that the measure corresponding to h_*, the minimizer of E, indeed satisfies this condition. The upper bound is computed as follows : given h, and the corresponding measure μ, we construct a test-configuration having vortex-density close to μ. This construction is somewhat similar to that of the upper bound of [SS2] in that it starts from the idea of constructing first a magnetic field satisfying $-\Delta(h - p) + h = 2\pi \sum_i \delta_{a_i}$, where the a_i's denote the vortices. Again, this construction, and particularly that of [SS2], has some similarity with that of [CM]. In [CM] they constructed a periodic function on a domain with small holes, (corresponding to our vortex-cores) whose number diverges. The scale of the holes relatively to their distances were the same as in our construction. Yet, one of the main differences here is that the test-function is no longer periodic and thus we had to change the method of construction, as well as the method of evaluation of the energy of h.

Once h is set, we construct a suitable u and evaluate the energy of the configuration, expressed in terms of an energy for the vorticity measure, similar to the interaction energy in potential theory.

In Section III, the matching lower bound is derived. Let us sketch the main steps of the proof. Considering $(u_\varepsilon, A_\varepsilon)$ any minimizer of the energy J, as a critical point, it satisfies the following Ginzburg-Landau equations :

$$(G.L.) \begin{cases} -(\nabla - iA)^2 u = \frac{1}{\varepsilon^2} u(1 - |u|^2) & \text{in } \Omega \\ -\nabla^\perp (h - p h_{\text{ex}}) = < iu, \nabla u - iAu > = \rho^2(\nabla\varphi - A) & \text{in } \Omega \\ h = h_{\text{ex}} & \text{on } \partial\Omega, \end{cases}$$

where $h = \operatorname{curl} A$ is still the induced magnetic field, and u is written (where possible) $u = \rho e^{i\varphi}$, with $\rho \leq 1$ — a standard consequence of the maximum principle (see [S1] for instance). On the other hand

$$|\nabla_A u|^2 = |\nabla \rho|^2 + \rho^2 |\nabla \varphi - A|^2,$$

hence from the second (G.L.) equation, we deduce that

$$\int_\Omega |\nabla_A u|^2 \geq \int_\Omega |\nabla \rho|^2 + \frac{|\nabla (h - p h_{\text{ex}})|^2}{\rho^2} \geq \int_\Omega |\nabla (h - p h_{\text{ex}})|^2. \tag{I.18}$$

Next, we use the construction of vortices that we recalled in Proposition I.1. Once these vortices of u_ε are given, we can define the family of measures

$$\mu_\varepsilon = \frac{2\pi \sum_{i \in I_\varepsilon} d_i \delta_{a_i}}{h_{\text{ex}}}. \tag{I.19}$$

Moreover, as from (I.12) each vortex carries at least an energy $\pi |d_i| |\log \varepsilon|$, and $J(u, A) \leq C |\log \varepsilon|^2$; we have $\sum_i |d_i| \leq C |\log \varepsilon| \leq C h_{\text{ex}}$, thus μ_ε is a bounded family of measures. Therefore, up to extraction of a subsequence, we can find a measure μ_0 such that

$$\mu_\varepsilon \to \mu_0 \tag{I.20}$$

in the sense of measures. Similarly, from (I.18) and $J(u, A) \leq C h_{\text{ex}}^2$, we can find h_0 such that

$$\frac{h_\varepsilon}{h_{\text{ex}}} \rightharpoonup h_0 \quad \text{weakly in } H^1_p. \tag{I.21}$$

Formally, when $\varepsilon = 0$, the London-type equation holds :

$$-\Delta (h - p h_{\text{ex}}) + h = 2\pi \sum_i d_i \delta_{a_i}.$$

We can make this statement rigorous by proving the following identity :

$$-\Delta (h_0 - p) + h_0 = \mu_0. \tag{I.22}$$

We can then derive the lower bound $\min \frac{J}{h_{\text{ex}}^2} \geq E(h_0) - o(1)$. First, from (I.18), we have

$$J(u_\varepsilon, A_\varepsilon) \geq \frac{1}{2} \int_\Omega |\nabla (h - p h_{\text{ex}})|^2 + |h - h_{\text{ex}}|^2. \tag{I.23}$$

Then, we can split this energy between the vortex-energy contained in $\cup B_i$, and the outer energy :

$$\frac{J(u_\varepsilon, A_\varepsilon)}{h_{\text{ex}}^2} \geq \frac{1}{2} \sum_i \frac{1}{h_{\text{ex}}^2} \int_{B_i} |\nabla (h - p h_{\text{ex}})|^2 + \frac{1}{2 h_{\text{ex}}^2} \int_{\Omega \setminus \cup_i B_i} |\nabla (h - p h_{\text{ex}})|^2 + |h - p h_{\text{ex}}|^2. \tag{I.24}$$

Using assertion (I.12) in Proposition I.1, the first sum in (I.24) is larger than $\frac{|\log\varepsilon|}{2h_{\text{ex}}^2}\int_\Omega |\mu_\varepsilon|$, and then from (I.20) and lower semi-continuity, larger than $\frac{\lambda}{2}\int_\Omega |\mu_0|$. By (I.21) and lower semi-continuity again the second term in (I.24) is larger than $\frac{1}{2}\int_\Omega |\nabla(h_0-p)|^2+|h_0-p|^2$. Hence we obtain

$$\liminf_{\varepsilon\to 0}\frac{J(u_\varepsilon,A_\varepsilon)}{h_{\text{ex}}^2}\geq \frac{\lambda}{2}\int_\Omega |\mu_0|+\frac{1}{2}\int_\Omega |\nabla(h_0-p)|^2+|h_0-p|^2. \tag{I.25}$$

Using (I.22), the right-hand side in (I.25) is exactly $E(h_0)$, hence

$$\liminf_{\varepsilon\to 0}\frac{J(u_\varepsilon,A_\varepsilon)}{h_{\text{ex}}^2}\geq E(h_0)\geq \min E = E(h_*).$$

The rest of Section III is then devoted to minimizing E and proving properties of its unique minimizer h_*.

Finally, in Section IV, we are able to derive the convergence of $\frac{h_\varepsilon}{h_{\text{ex}}}$ and of the measures μ_ε from the fact that the upper and lower bounds of Sections II and III match.

Remark on notations : C always denotes a positive constant independent of ε.

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II Upper bound

In this section, $0\leq\lambda<+\infty$.
Recall the expression

$$E(f)=\frac{\lambda}{2}\int_\Omega |-\Delta(f-p)+f|+\frac{1}{2}\int_\Omega |\nabla(f-p)|^2+|f-p|^2, \tag{II.1}$$

defined over

$$V=\{f\in H^1_p(\Omega)/-\Delta(f-p)+f \text{ is a Radon measure}\}.$$

The minimum of E is achieved, as will be shown in Section III, by a function $h_*\in V$ for which $\mu=-\Delta(h_*-p)+h_*$ is in fact a positive absolutely continuous measure. Since trivially any $f\in V$ is the solution of

$$\begin{cases} -\Delta(f-p)+f-p=\mu-p & \text{in } \Omega\\ f-p=0 & \text{on } \partial\Omega, \end{cases} \tag{II.2}$$

where $\mu=-\Delta(f-p)+f$, we have

$$(f-p)(x)=\int_\Omega G(x,y)\,d(\mu-p)(y),$$

where $G(x,y)$ is the Green potential, solution of

$$\text{(II.3)} \qquad \begin{cases} -\Delta_x G + G = \delta_y & \text{in } \Omega \\ G(x,y) = 0 & \text{for } x \in \partial\Omega. \end{cases}$$

From (II.2), (II.3) it follows that

$$\text{(II.4)} \qquad E(f) = I(\mu) = \frac{\lambda}{2}\|\mu\| + \frac{1}{2}\int_{\Omega\times\Omega} G(x,y)\, d(\mu - p)(x)\, d(\mu - p)(y),$$

where $\|\mu\| = |\mu|(\Omega)$ is the total variation of μ. Note that $I(\mu)$ makes sense for any positive Radon measure if we admit the value $+\infty$.

We prove in this section

Proposition II.1 *Let $h_{\rm ex}$ be such that $\lim_{\varepsilon\to 0} \frac{|\log \varepsilon|}{h_{\rm ex}} = \lambda$, with the additional condition, if $\lambda = 0$, that $h_{\rm ex} \ll \frac{1}{\varepsilon^2}$; and μ be a positive Radon measure absolutely continuous with respect to the Lebesgue measure. Then, letting $(u_\varepsilon, A_\varepsilon)$ be a minimizer of J over $H^1(\Omega,\mathbb{C})\times H^1(\Omega,\mathbb{R}^2)$,*

$$\text{(II.5)} \qquad \limsup_{\varepsilon\to 0} \frac{1}{h_{\rm ex}^2} J(u_\varepsilon, A_\varepsilon) \le I(\mu).$$

As already mentioned, the measure μ corresponding to h_* will be proved to be indeed positive and absolutely continuous (see Proposition III.1), thus we have

Corollary II.1 *Under the hypothesis of the proposition,*

$$\text{(II.6)} \qquad \limsup_{\varepsilon\to 0} \frac{1}{h_{\rm ex}^2} J(u_\varepsilon, A_\varepsilon) \le I(\mu) = E(h_*) = \min_{f\in V} E(f).$$

Corollary II.2 *If $\lambda = 0$, and h_ε is the induced magnetic field of a minimizing configuration,*

$$\frac{h_\varepsilon}{h_{\rm ex}} \underset{\varepsilon\to 0}{\longrightarrow} p \quad \textit{strongly in } H^1,$$

proving Theorem 1 in this particular case.

Proof : If $\lambda = 0$ then it is clear that the minimum of E over V is zero, and is uniquely achieved by $h_* = p$. Hence from Corollary II.1

$$\limsup_{\varepsilon\to 0} \frac{1}{h_{\rm ex}^2} J(u_\varepsilon, A_\varepsilon) = 0.$$

Therefore $\lim_{\varepsilon\to 0} \frac{J}{h_{\rm ex}^2} = 0$. But as pointed out in (I.17),

$$J(u_\varepsilon, A_\varepsilon) \ge \frac{1}{2}\int_\Omega |\nabla(h - p h_{\rm ex})|^2 + |h - p h_{\rm ex}|^2,$$

thus $\frac{h}{h_{\text{ex}}} \to p$ in H^1. □

The proof of the above proposition involves constructing a test configuration $(u_\varepsilon, A_\varepsilon)$ for J_ε with normalized vortex density given by μ. But the size of a vortex, i.e. of the region where $|u| < 1$ is expected to be determined by the potential term in J. The factor $1/\varepsilon^2$ is scaled out from this term by letting $v(x) = u(\varepsilon x)$ and expressing $J(u, A)$ in terms of v. Thus we are led to believe that vortices have a size of the order of ε, and then that the vortex measure should be concentrated in balls of this size, each carrying a weight 2π. Our main task will therefore be to approximate μ by measures μ_ε having this property, and to control $I(\mu_\varepsilon)$ as $\varepsilon \to 0$. In fact $I(\mu_\varepsilon)$ does not converge to $I(\mu)$ because there is a loss of energy due to the concentration at vortices. However we have

Proposition II.2 *Let μ, h_{ex}, λ be as in Proposition II.1. Then for $\varepsilon > 0$ small enough, there exists points a_i^ε, $1 \le i \le n(\varepsilon)$ such that*

(1)

$$n(\varepsilon) \underset{\varepsilon\to 0}{\sim} \frac{h_{\text{ex}}\mu(\Omega)}{2\pi}, \qquad |a_i^\varepsilon - a_j^\varepsilon| > 4\varepsilon$$

and, letting μ_ε^i be the uniform measure on $\partial B(a_i^\varepsilon, \varepsilon)$ of mass 2π,

(2)

$$\mu_\varepsilon = \frac{1}{h_{\text{ex}}} \sum_i \mu_\varepsilon^i \to \mu$$

in the sense of measures, as $\varepsilon \to 0$. Finally

(3)

$$\limsup_{\varepsilon\to 0} \frac{1}{2} \int_{\Omega\times\Omega} G(x, y)\, d\mu_\varepsilon(x)\, d\mu_\varepsilon(y) \le \frac{\lambda}{2}\mu(\Omega) + \frac{1}{2} \int_{\Omega\times\Omega} G(x, y)\, d\mu(x)\, d\mu(y).$$

This proposition shall be proved at the end of this section. Proposition II.1 follows easily from the above result, once we know some easy properties of the function $G(x, y)$ that we now list without proof.

Lemma II.1 *The function $G(x, y)$, solution to (II.3) has the following properties (Δ is the diagonal of $\mathbb{R}^2 \times \mathbb{R}^2$).*

(1) $G(x, y)$ is symmetric and positive.

(2) $G(x, y) + \dfrac{1}{2\pi} \log |x - y|$ is continuous on $\Omega \times \Omega$.

(3) There exists $C > 0$ such that for all $x, y \in \Omega \times \Omega \setminus \Delta$

$$\frac{1}{2\pi} \log \frac{1}{|x - y|} - C \le G(x, y) \le C \left(\log \frac{1}{|x - y|} + 1 \right).$$

II.1 Proof of Proposition II.1

Step 1

Let μ_ε be the sequence of measures given by Proposition II.2, then

$$\limsup_{\varepsilon\to 0} \frac{1}{2} \int_{\Omega\times\Omega} G(x,y)\, d(\mu_\varepsilon - p)(x)\, d(\mu_\varepsilon - p)(y) \le I(\mu). \tag{II.7}$$

Indeed from Proposition II.2,

$$\limsup_{\varepsilon\to 0} \frac{1}{2} \int_{\Omega\times\Omega} G(x,y)\, d\mu_\varepsilon(x)\, d\mu_\varepsilon(y) \le \frac{\lambda}{2}\mu(\Omega) + \frac{1}{2} \int_{\Omega\times\Omega} G(x,y)\, d\mu(x)\, d\mu(y). \tag{II.8}$$

Moreover, from the weak convergence of μ_ε to μ,

$$\limsup_{\varepsilon\to 0} \frac{1}{2} \int_{\Omega} \left(\int_{\Omega} G(x,y) p(x)\, dx \right) d\mu_\varepsilon(y) = \frac{1}{2} \int_{\Omega} \left(\int_{\Omega} G(x,y) p(x)\, dx \right) d\mu(y), \tag{II.9}$$

the inner integral being a continuous function of y — an easy consequence of (II.3). Combining (II.8) and (II.9) yields (II.7).

Step 2

Let h_ε be the solution to

$$\begin{cases} -\Delta(h_\varepsilon - p h_{\text{ex}}) + h_\varepsilon = h_{\text{ex}}\mu_\varepsilon & \text{in } \Omega \\ h_\varepsilon = p h_{\text{ex}} & \text{on } \partial\Omega. \end{cases} \tag{II.10}$$

This function will serve as the induced magnetic field of our test-configuration. For future reference, note that since

$$\frac{1}{2} \int_{\Omega} |\nabla(h_\varepsilon - p h_{\text{ex}})|^2 + |h_\varepsilon - p h_{\text{ex}}|^2\, dx = h_{\text{ex}}^2 \int_{\Omega\times\Omega} G(x,y)\, d(\mu_\varepsilon - p)(x)\, d(\mu_\varepsilon - p)(y),$$

it follows from (II.7) that

$$\limsup_{\varepsilon\to 0} \frac{1}{h_{\text{ex}}^2} \left(\frac{1}{2} \int_{\Omega} |\nabla(h_\varepsilon - p h_{\text{ex}})|^2 + |h_\varepsilon - p h_{\text{ex}}|^2\, dx \right) \le I(\mu). \tag{II.11}$$

Step 3

We construct a test configuration $(u_\varepsilon, A_\varepsilon)$ such that

$$\text{(II.12)} \qquad \limsup_{\varepsilon\to 0} \frac{1}{h_{\text{ex}}^2} J(u_\varepsilon, A_\varepsilon) \le I(\mu).$$

We define A_ε to be such that $\operatorname{curl} A_\varepsilon = h_\varepsilon$, and define $u_\varepsilon = \rho_\varepsilon e^{i\varphi_\varepsilon}$ as follows. Recall from Proposition II.2 that $h_{\text{ex}}\mu_\varepsilon = \sum_i \mu_\varepsilon^i$, with μ_ε^i a uniform measure on $\partial B(a_i^\varepsilon, \varepsilon)$ of mass 2π, and for all $i \ne j$, $|a_i^\varepsilon - a_j^\varepsilon| > 4\varepsilon$. We let

$$\text{(II.13)} \qquad \rho_\varepsilon(x) = \begin{cases} 0 & \text{if } |x - a_i^\varepsilon| \le \varepsilon \text{ for some } i, \\ \left(\dfrac{|x - a_i^\varepsilon|}{\varepsilon} - 1\right) & \text{if } \varepsilon < |x - a_i^\varepsilon| < 2\varepsilon \text{ for some } i, \\ 1 & \text{otherwise.} \end{cases}$$

The function φ_ε needs only to be defined modulo 2π, and where $\rho_\varepsilon \ne 0$, i.e. on the set $\Omega_\varepsilon = \Omega \setminus \cup_i B(a_i^\varepsilon, \varepsilon)$. Choosing a point $x_0 \in \Omega_\varepsilon$, we let $\forall x \in \Omega_\varepsilon$

$$\text{(II.14)} \qquad \varphi_\varepsilon(x) = \oint_{(x_0,x)} A_\varepsilon.\tau - \nabla(h_\varepsilon - p h_{\text{ex}}).\nu \, d\ell,$$

where (x_0, x) is any curve joining x_0 to x in Ω_ε and (τ, ν) is the Frénet frame on the curve. By construction,

$$\text{(II.15)} \qquad \nabla\varphi_\varepsilon - A_\varepsilon = -\nabla^\perp(h_\varepsilon - p h_{\text{ex}}),$$

in Ω_ε. Note that the function φ_ε is well defined modulo 2π. Indeed if $\omega \subset \Omega$ is such that $\partial\omega \subset \Omega_\varepsilon$,

$$\int_{\partial\omega} A_\varepsilon.\tau - \nabla(h_\varepsilon - p h_{\text{ex}}).\nu = \int_\omega -\Delta(h_\varepsilon - p h_{\text{ex}}) + h_\varepsilon,$$

the right hand side being equal to $h_{\text{ex}}\mu_\varepsilon(\omega)$ by (II.10). This quantity is in turn equal to $2\pi k$, where k is the number of points a_i^ε in ω. Thus $e^{i\varphi_\varepsilon(x)}$ defined by (II.14) does not depend on the particular curve (x_0, x) chosen.

Step 4

We estimate $J(u_\varepsilon, A_\varepsilon)$. It follows easily from (II.13) — and the fact that from Proposition II.2 there are $n(\varepsilon) \sim \frac{h_{\text{ex}}\mu(\Omega)}{2\pi}$ points a_i^ε — that

$$\frac{1}{2}\int_\Omega |\nabla\rho_\varepsilon|^2 + \frac{1}{2\varepsilon^2}\left(1 - {\rho_\varepsilon}^2\right)^2 \le C h_{\text{ex}}$$

and therefore that

$$\limsup_{\varepsilon\to 0}\frac{1}{h_{\rm ex}^2}\left(\frac{1}{2}\int_\Omega |\nabla\rho_\varepsilon|^2+\frac{1}{2\varepsilon^2}\left(1-{\rho_\varepsilon}^2\right)^2\right)=0. \tag{II.16}$$

Moreover, from (II.15), ${\rho_\varepsilon}^2|\nabla\varphi_\varepsilon - A_\varepsilon|^2\le|\nabla(h_\varepsilon - ph_{\rm ex})|^2$. Therefore, adding (II.11) and (II.16),

$$\limsup_{\varepsilon\to 0}\frac{1}{h_{\rm ex}^2}J_\varepsilon(u_\varepsilon,A_\varepsilon)\le I(\mu),$$

proving Proposition II.1.

II.2 Proof of Proposition II.2

Step 1

We claim that it suffices to prove the proposition for a measure μ with density u verifying $1/C<u(x)<C$ a.e. for some $C>0$.

Indeed if the density u of μ is an arbitrary positive function we may define truncated measures μ_n with densities $u_n=(u\wedge n)\vee 1/n$. Applying the proposition to each μ_n we get approximating measures μ_n^ε tending to μ_n as $\varepsilon\to 0$. Taking a diagonal sequence we may then construct a sequence $\mu_\varepsilon\to\mu$ satisfying properties (1), (2) of Proposition II.2 and such that

$$\limsup_{\varepsilon\to 0}\frac{1}{2}\int_{\Omega\times\Omega} G(x,y)\,d\mu_\varepsilon(x)\,d\mu_\varepsilon(y)\le\liminf_{n\to+\infty} I(\mu_n).$$

That the right-hand side is less than or equal to $I(\mu)$ follows — as in Step 1 of the proof of Proposition II.1 — from the weak convergence $\mu_n\to\mu$ and the fact that

$$\liminf_{n\to+\infty}\frac{1}{2}\int_{\Omega\times\Omega} G(x,y)\,d\mu_n(x)\,d\mu_n(y)\le\frac{1}{2}\int_{\Omega\times\Omega} G(x,y)\,d\mu(x)\,d\mu(y).$$

This last inequality is proved by noting that since G is positive,

$$G(x,y)\,d\mu_n\,d\mu_n\le G(x,y)\,d\left(\mu+\frac{1}{n}\right)\,d\left(\mu+\frac{1}{n}\right),$$

integrating, and letting $n\to+\infty$.

Step 2

From now on we assume the bound $1/C<u<C$ holds for the density of μ, and define the approximating measures μ_ε.

We choose a function $\delta(\varepsilon)$ such that

$$h_{\rm ex}^{-1/2}\ll\delta\ll 1. \tag{II.17}$$

Then for each ε we define the family of squares

$$\{K_i\}_{i\in I} = \{[k\delta,(k+1)\delta] \times [l\delta,(l+1)\delta] \subset \Omega \mid k,l \in \mathbb{Z}\}.$$

For each $i \in I$ we let

$$n_i = \left[\frac{h_{\text{ex}}}{2\pi}\mu(K_i)\right], \tag{II.18}$$

where $[x]$ denotes the greatest integer less than or equal to x. Note that from our assumption on μ,

$$\delta^2/C \le \mu(K_i) \le C\delta^2. \tag{II.19}$$

It is possible to place n_i points in each K_i evenly enough so that their mutual distance, as well as their distance to the boundary of K_i is of the order of $\delta/\sqrt{n_i}$. We call the resulting family of points $(a_i^\varepsilon)_i$ (we will specify more precisely how we place the points below). Note that in view of (II.17), (II.18), (II.19), if $i \neq j$ then

$$|a_i^\varepsilon - a_j^\varepsilon| > \frac{1}{C\sqrt{h_{\text{ex}}}} \gg \varepsilon. \tag{II.20}$$

Also, from (II.18), the total number of points $n(\varepsilon) = \sum_i n_i$ is such that

$$n(\varepsilon) \le \frac{h_{\text{ex}}}{2\pi}\sum_i \mu(K_i) \le n(\varepsilon) + \text{Card}(I).$$

But $\sum_i \mu(K_i) \sim \mu(\Omega)$ and $\text{Card}(I) \sim \frac{|\Omega|}{\delta^2} \ll h_{\text{ex}}$, from (II.17). Therefore

$$n(\varepsilon) \sim \frac{h_{\text{ex}}}{2\pi}\mu(\Omega),$$

which, together with (II.20), shows that property (1) of Proposition II.2 is verified. We define μ_ε^i to be the uniform measure on $C_\varepsilon^i = \partial B(a_i^\varepsilon, \varepsilon)$ of total mass 2π, and

$$\mu_\varepsilon = \frac{1}{h_{\text{ex}}}\sum_{i=1}^{n(\varepsilon)} \mu_\varepsilon^i.$$

Step 3

It remains to prove properties (2) and (3) for μ_ε. The weak convergence of (2) is clear from the construction of μ_ε, since

$$\mu_\varepsilon(K_i) = \frac{2\pi}{h_{\text{ex}}} n_i \approx \mu(K_i)$$

and since the size of the squares K_i tends to 0. The rest of the proof is devoted to proving property (3).

Fix some $\eta > 0$ and let Δ_η denote an η-neighbourhood of the diagonal in $\Omega \times \Omega$. Since $G(.,.)$ is continuous outside Δ_η, it follows from the weak convergence $\mu_\varepsilon \to \mu$ — which implies $\mu_\varepsilon \otimes \mu_\varepsilon \to \mu \otimes \mu$ — that

$$\text{(II.21)} \qquad \lim_{\varepsilon \to 0} \frac{1}{2} \int_{\Omega \times \Omega \setminus \Delta_\eta} G \, d\mu_\varepsilon \, d\mu_\varepsilon = \frac{1}{2} \int_{\Omega \times \Omega \setminus \Delta_\eta} G \, d\mu \, d\mu.$$

Therefore it suffices to prove that

$$\text{(II.22)} \qquad \limsup_{\varepsilon \to 0} \frac{1}{2} \int_{\Delta_\eta} G \, d\mu_\varepsilon \, d\mu_\varepsilon \le \frac{\lambda}{2} \mu(\Omega) + C(\eta),$$

where $C(\eta)$ is a function tending to 0 with η. Indeed if (II.22) is true, then adding it to (II.21) and letting $\eta \to 0$ proves property (3) of Proposition II.2.

Step 4

We prove (II.22). Let I_η be the set of pairs of indices (i,j) such that $C_\varepsilon^i \times C_\varepsilon^j \cap \Delta_\eta \neq \varnothing$ (I_η depends in fact on ε, but we drop the ε in our notation). We have

$$\text{(II.23)} \qquad \frac{1}{2} \iint_{\Delta_\eta} G \, d\mu_\varepsilon \, d\mu_\varepsilon \le \frac{1}{h_{\text{ex}}^2} \left(\sum_{\substack{(i,j) \in I_\eta \\ i \neq j}} \frac{1}{2} \iint G \, d\mu_\varepsilon^i \, d\mu_\varepsilon^j + \sum_{i=1}^{n(\varepsilon)} \frac{1}{2} \iint G \, d\mu_\varepsilon^i \, d\mu_\varepsilon^i \right).$$

We first treat the second sum. From the definition of μ_ε^i

$$\text{(II.24)} \qquad \frac{1}{2} \iint G \, d\mu_\varepsilon^i \, d\mu_\varepsilon^i = \frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} G \left(a_i^\varepsilon + \varepsilon e^{i\theta_1}, a_i^\varepsilon + \varepsilon e^{i\theta_2} \right) \, d\theta_1 \, d\theta_2.$$

We split the second sum in (II.23) in two by letting $J_\eta = \{ i \mid d(a_i^\varepsilon, \partial\Omega) < \eta \}$. Then because of (II.19), $\# J_\eta < n(\varepsilon) |\{ x \mid d(x, \partial\Omega) < \eta \}| < C \eta n(\varepsilon)$ and then, using the upper bound (3) of Lemma II.1 together with (II.24),

$$\text{(II.25)} \qquad \sum_{i \in J_\eta} \frac{1}{2} \iint G \, d\mu_\varepsilon^i \, d\mu_\varepsilon^i \le C \eta \pi n(\varepsilon) \left(|\log \varepsilon| + 1 \right).$$

On the other hand, from Lemma II.1, (2), there exists a constant $C(\eta)$ such that if x, y are at a distance at least $\eta/2$ from the boundary of Ω, then

$$G(x,y) < \frac{1}{2\pi} \log \frac{1}{|x-y|} + C(\eta),$$

thus if $\varepsilon < \eta/2$ and $i \notin J_\eta$, the same inequality holds true for any $(x,y) \in C_\varepsilon^i \times C_\varepsilon^i$. Integrating over $C_\varepsilon^i \times C_\varepsilon^i$ yields, in view of (II.24),

$$\frac{1}{2}\iint G\, d\mu_\varepsilon^i\, d\mu_\varepsilon^i < \pi|\log\varepsilon| + C(\eta).$$

Then, summing over indices $i \notin J_\eta$ yields

$$\sum_{i\notin J_\eta}\frac{1}{2}\iint G\, d\mu_\varepsilon^i\, d\mu_\varepsilon^i \leq n(\varepsilon)\left(\pi|\log\varepsilon| + C(\eta)\right). \tag{II.26}$$

Summing (II.26) and (II.25), and taking the limit as $\varepsilon \to 0$, we find

$$\limsup_{\varepsilon\to 0}\frac{1}{h_{\text{ex}}^2}\left(\sum_{i=1}^{n(\varepsilon)}\frac{1}{2}\iint G\, d\mu_\varepsilon^i\, d\mu_\varepsilon^i\right) \leq \frac{\lambda}{2}\mu(\Omega) + C\eta, \tag{II.27}$$

using the fact that $\lim_{\varepsilon\to 0}\frac{n(\varepsilon)}{h_{\text{ex}}(\varepsilon)} = \frac{\mu(\Omega)}{2\pi}$ and $\lambda = \lim_{\varepsilon\to 0}\frac{|\log\,\varepsilon|}{h_{\text{ex}}}$.

It remains to treat the first sum in (II.23). First we get more specific as to how the points a_i^ε are placed. Each square K_i is partitioned into n_i rectangles by induction as follows:

1) The first subdivision is $\mathcal{S}_1 = \{K_i\}$, it consists of one rectangle.

2) Given a subdivision $\mathcal{S}_n$ of K_i in n rectangles, $\mathcal{S}_{n+1}$ is obtained by dividing the largest rectangle in $\mathcal{S}_n$ into two rectangles of same size, either vertically or horizontally according to which yields the rectangles with aspect ratio closest to 1.

This results in a partition of each K_i, and then a partition of $\cup_i K_i$, into a family of $n(\varepsilon)$ rectangles that we call $\mathcal{S}$. A point is placed at the center of each rectangle in $\mathcal{S}$. There are two fundamental facts that the reader may check easily for a rectangle $R \in \mathcal{S}$ with width w and height h :

$$\frac{w}{h} \in \{1/2, 1, 2\}, \qquad \frac{1}{Cn(\varepsilon)} < |R| < \frac{C}{n(\varepsilon)}, \tag{II.28}$$

where C does not depend on ε. From this we deduce

(a) There exists a $C > 0$ independent of ε such that for any pair of distinct rectangles $R_i, R_j \in \mathcal{S}$ with centers a_i^ε, a_j^ε ; any pair $(x,y) \in R_i \times R_j$ and any $(\theta_1, \theta_2) \in [0, 2\pi] \times [0, 2\pi]$,

$$|x - y| < C\left|a_i^\varepsilon + \varepsilon e^{i\theta_1} - a_j^\varepsilon + \varepsilon e^{i\theta_2}\right|. \tag{II.29}$$

This follows from the boundedness of the aspect ratio of the rectangles and the fact that the distance between the centers is at least 4ε.

(b) For any rectangle $R \in \mathcal{S}_n$,

$$\text{(II.30)} \qquad \frac{2\pi}{h_{\rm ex}} < C\mu(R).$$

This is because $C|R| \geq 1/n(\varepsilon)$ and $n(\varepsilon) \approx \mu(\Omega)h_{\rm ex}/2\pi$. Using the lower bound $u > 1/C$ on the density of μ yields (II.30).

Now take $(i,j) \in J_\eta$ a pair of distinct indices. Then, from the mean value theorem,

$$\text{(II.31)} \qquad \begin{aligned} \frac{1}{4\pi^2} \iint G\, d\mu_\varepsilon^i\, d\mu_\varepsilon^j &= G\left(a_i^\varepsilon + \varepsilon e^{i\theta_1}, a_j^\varepsilon + \varepsilon e^{i\theta_2}\right), \\ \frac{1}{\mu(R_i)\mu(R_j)} \iint_{R_i \times R_j} G\, d\mu\, d\mu &= G(x,y), \end{aligned}$$

for some $(x,y) \in R_i \times R_j$ and $(\theta_1, \theta_2) \in [0, 2\pi] \times [0, 2\pi]$. But from Lemma II.1, (3) and (II.29),

$$\text{(II.32)} \qquad G\left(a_i^\varepsilon + \varepsilon e^{i\theta_1}, a_j^\varepsilon + \varepsilon e^{i\theta_2}\right) < CG(x,y).$$

Combining (II.30), (II.31) and (II.32) yields

$$\frac{1}{h_{\rm ex}^2} \iint G\, d\mu_\varepsilon^i\, d\mu_\varepsilon^j < C \iint_{R_i \times R_j} G\, d\mu\, d\mu,$$

and then

$$\text{(II.33)} \qquad \frac{1}{h_{\rm ex}^2} \sum_{\substack{(i,j)\in I_\eta \\ i \neq j}} \frac{1}{2} \iint G\, d\mu_\varepsilon^i\, d\mu_\varepsilon^j \leq C\mu(\Delta_{2\eta}),$$

since $\cup_{i,j} R_i \times R_j \subset \Delta_{2\eta}$ if ε is small enough.But $\lim_{\eta\to 0} \mu(\Delta_{2\eta}) = 0$, thus summing (II.33), (II.27) in view of (II.23) yields (II.22), completing this step and the proof of Proposition II.2. □

III Lower bound

We consider as usual $(u_\varepsilon, A_\varepsilon)$ minimizers of J, and h_ε the associated magnetic fields. (We will often drop the subscripts ε). We begin with the

Proof of Proposition I.1 : It is clear, as already said, that, by testing J with the configuration $(u \equiv 1, A \equiv 0)$, the minimum of J is less than $Ch_{\rm ex}^2$, for some constant C independent of ε. Then, writing $\rho = |u_\varepsilon|$,

$$\int_\Omega |\nabla \rho|^2 + \frac{1}{2\varepsilon^2}\left(1 - \rho^2\right)^2 \leq C|\log \varepsilon|^2.$$

For any t let $\Omega_t = \{x/\rho(x) < t\}$ and let $\gamma_t = \partial\Omega_t$. Adapting Lemma IV.2 of [SS2] we find

$$\int_{C\varepsilon}^{1} r(\gamma_t)\frac{1-t^2}{\varepsilon}\,dt \le \int_{\Omega} |\nabla\rho|^2 + \frac{1}{2\varepsilon^2}\left(1-\rho^2\right)^2,$$

where C above depends only on Ω and $r(\gamma_t)$ is the radius of γ_t, i.e. the infimum over all finite coverings of γ_t by balls $B_1, \dots, B_k$ of the sum $r_1 + \cdots + r_k$, where r_i is the radius of B_i. Therefore,

$$\int_{C\varepsilon}^{1} r(\gamma_t)\frac{1-t^2}{\varepsilon}\,dt \le C|\log\varepsilon|^2.$$

From this we deduce, using the mean value theorem, that

$$\text{(III.1)} \qquad \exists t \in \left[1 - \frac{1}{|\log\varepsilon|}, 1\right] \quad \text{such that} \quad r(\gamma_t) < C\varepsilon|\log\varepsilon|^3.$$

Now we apply Proposition IV.1 of [SS2] on $\Omega \setminus \Omega_t$, taking $v = e^{i\varphi} = \frac{u_\varepsilon}{|u_\varepsilon|}$, $\sigma = \min(|\log\varepsilon|^{-6}, \frac{1}{2}h_{\text{ex}}^{-1/2})$ and $A = A_\varepsilon$. The conclusion is that there exist balls $\{B_i\}_{i\in I_\varepsilon}$ such that (I.10), (I.11) of the proposition hold and

$$\text{(III.2)} \qquad \frac{1}{2}\int_{B_i\setminus\Omega_t} |\nabla\varphi - A|^2 + \frac{1}{2}\int_{B_i} |h_\varepsilon - h_{\text{ex}}|^2 \ge \pi|d_i|\left(\log\frac{\sigma}{C\varepsilon|\log\varepsilon|^3} - C\right)_+,$$

where d_i is the winding number of $\frac{u_\varepsilon}{|u_\varepsilon|}$ restricted to ∂B_i if $B_i \Subset \Omega$, and zero otherwise. We prove that this implies (I.12).

First $\sigma \ge |\log\varepsilon|^{-p}$ for some $p > 0$, thus the right-hand side above may be rewritten as

$$\text{(III.3)} \qquad \pi|d_i||\log\varepsilon|(1 - o(1)).$$

Also, recall that the second Ginzburg-Landau equation is

$$\text{(III.4)} \qquad -\nabla^\perp(h_\varepsilon - ph_{\text{ex}}) = \rho^2(\nabla\varphi - A).$$

Replacing the right-hand side of (III.2) by (III.3), and on the left using (III.4) and the fact that, from (III.1), $\rho \ge 1 - |\log\varepsilon|^{-1}$ on $\Omega\setminus\Omega_t$, we find

$$\begin{aligned}
\frac{1}{2}\int_{B_i\setminus\Omega_t} |\nabla(h_\varepsilon - ph_{\text{ex}})|^2 + \frac{1}{2}\int_{B_i} |h_\varepsilon - h_{\text{ex}}|^2 &\ge \frac{1}{2}\int_{B_i\setminus\Omega_t} \rho^4|\nabla\varphi - A|^2 + |h_\varepsilon - h_{\text{ex}}|^2 \\
&\ge \frac{1}{2}\left(1 - \frac{C}{|\log\varepsilon|}\right)\left(\int_{B_i\setminus\Omega_t} |\nabla\varphi - A|^2 + \int_{B_i} |h_\varepsilon - h_{\text{ex}}|^2\right) \\
\text{(III.5)} \qquad &\ge \pi|d_i||\log\varepsilon|(1 - o(1)).
\end{aligned}$$

On the other hand, we have the a priori estimate

$$\frac{1}{2}\int_{\Omega} |\nabla(h_\varepsilon - p h_{\mathrm{ex}})|^2 \le \frac{1}{2}\int_{\Omega} |\nabla_A u|^2 \le C h_{\mathrm{ex}}^2$$

hence

$$\|h_\varepsilon\|_{H^1(\Omega)} \le C h_{\mathrm{ex}} \le C|\log \varepsilon|,$$

therefore,

$$\begin{aligned} \int_{B_i} |h_\varepsilon - h_{\mathrm{ex}}|^2 &\le C r_i \|h_\varepsilon - h_{\mathrm{ex}}\|_{L^4}^2 \\ &\le C\frac{|\log \varepsilon|^2}{|\log \varepsilon|^6} = o(1), \end{aligned}$$

where we have used (I.11).
We conclude that (III.5) can be rewritten as (I.12) :

$$\frac{1}{2}\int_{B_i} |\nabla(h_\varepsilon - p h_{\mathrm{ex}})|^2 \ge \pi |d_i||\log \varepsilon|(1 - o(1)).$$

□

From now on, we treat the case $\lambda > 0$ and we consider the vortices (a_i, d_i) of $(u_\varepsilon, A_\varepsilon)$ given by Proposition I.1. We define

$$\mu_\varepsilon = \frac{2\pi \sum_{i \in I_\varepsilon} d_i \delta_{a_i}}{h_{\mathrm{ex}}}. \tag{III.6}$$

In the sequel, $\mathcal{M}$ will denote the set of Radon measures on Ω.

If we choose the Coulomb gauge

$$\begin{cases} \operatorname{div} A = 0 & \text{in } \Omega \\ A.n = 0 & \text{on } \partial\Omega, \end{cases}$$

we have the a priori bounds :

$$\|A\|_{L^\infty} \le C h_{\mathrm{ex}}, \quad \|\nabla u\|_{L^2} \le C h_{\mathrm{ex}}, \qquad \|u\|_{L^\infty} \le 1. \tag{III.7}$$

(See [S1] or [SS1] for the proofs). We also recall that, as in (I.23),

$$J(u, A) \ge \frac{1}{2}\int_{\Omega} |\nabla(h - p h_{\mathrm{ex}})|^2 + |h - p h_{\mathrm{ex}}|^2 + \frac{1}{2\varepsilon^2}(1 - \rho^2)^2. \tag{III.8}$$

We begin by extracting weakly convergent subsequences thanks to the a priori bounds.

Lemma III.1 *From any sequence $\varepsilon_n \to 0$, we can extract a subsequence such that there exist $h_0 \in H^1_p(\Omega)$ and $\mu_0 \in \mathcal{M}$ such that*

(III.9)

$$\frac{h_{\varepsilon_n}}{h_{\rm ex}} - p \rightharpoonup h_0 - p \text{ weakly in } H^1_0(\Omega), \text{ and } \frac{h_{\varepsilon_n}}{h_{\rm ex}} - p \longrightarrow h_0 - p \text{ strongly in } W^{1,q}_0(\Omega), \forall q < 2,$$

$$\text{(III.10)} \qquad \mu_{\varepsilon_n} \to \mu_0 \quad \text{in the sense of measures,}$$

with

$$\text{(III.11)} \qquad -\Delta(h_0 - p) + h_0 = \mu_0.$$

Hence $\mu_0 \in H^{-1} \cap \mathcal{M}$.

Proof : We have the upper bound :

$$J(u, A) \le C h^2_{\rm ex}.$$

Hence,

$$\text{(III.12)} \qquad J(u, A) \le \frac{C}{\lambda} h_{\rm ex} |\log \varepsilon|.$$

Comparing (III.8) and (III.12), we deduce that $\frac{h}{h_{\rm ex}} - p$ is bounded in $H^1_0(\Omega)$, hence, modulo a subsequence,

$$\text{(III.13)} \qquad \frac{h}{h_{\rm ex}} - p \rightharpoonup h_0 - p \quad \text{in } H^1_0.$$

On the other hand, $\{B_i\}_i$ being the family of balls constructed in Proposition I.1, still from (III.8),

$$\text{(III.14)} \quad J(u, A) \ge \sum_i \int_{B_i} |\nabla(h - p h_{\rm ex})|^2 + \frac{1}{2} \int_{\Omega \setminus \cup_i B_i} |\nabla(h - p h_{\rm ex})|^2 + |h - p h_{\rm ex}|^2 + o(1).$$

Hence, combining (III.12), (III.14) and (I.12), we get

$$\frac{C}{\lambda} \ge \frac{J(u, A)}{h_{\rm ex} |\log \varepsilon|} \ge \frac{1}{2 h_{\rm ex} |\log \varepsilon|} \sum_i \int_{B_i} |\nabla(h - p h_{\rm ex})|^2 \ge \frac{\pi \sum_i |d_i|}{h_{\rm ex}} + o(1).$$

Thus, μ_ε is a bounded family of measures, we can then extract a subsequence converging to μ_0 in the sense of measures.

The next step is to compare μ_0 and h_0. For that purpose we prove the following

$$-\Delta\left(\frac{h}{h_{\text{ex}}}-p\right)+\frac{h}{h_{\text{ex}}}-\frac{2\pi}{h_{\text{ex}}}\sum_i d_i\delta_{a_i}\underset{\varepsilon\to 0}{\longrightarrow}0 \ \text{ in } W^{-1,q}(\Omega),\ \forall q<2. \tag{III.15}$$

First, the second Ginzburg-Landau equation holds :

$$-\nabla^\perp(h-ph_{\text{ex}})=(iu,\nabla u)-|u|^2A,$$

and taking the curl,

$$-\Delta(h-ph_{\text{ex}})=\operatorname{curl}(iu,\nabla u)-\operatorname{curl}(|u|^2A)$$

$$-\Delta\left(\frac{h}{h_{\text{ex}}}-p\right)+\frac{h}{h_{\text{ex}}}=\frac{\operatorname{curl}(iu,\nabla u)}{h_{\text{ex}}}+\frac{\operatorname{curl}((1-|u|^2)A)}{h_{\text{ex}}}. \tag{III.16}$$

On the one hand, for any $\xi\in H_0^1(\Omega)$,

$$\int_\Omega \xi\operatorname{curl}((1-|u|^2)A)=\int_\Omega A(1-|u|^2).\nabla^\perp\xi.$$

Then, using Cauchy-Schwartz and the a-priori bounds (III.7),

$$\begin{aligned}\left|\int_\Omega \xi\operatorname{curl}((1-|u|^2)A)\right| &\le C\|\nabla\xi\|_{L^2}\|A\|_{L^\infty}\|1-|u|^2\|_{L^2}\\ &\le Ch_{\text{ex}}^2\varepsilon\|\nabla\xi\|_{L^2},\end{aligned}$$

the right-hand side tending to zero when $\varepsilon\to 0$. This means that

$$\operatorname{curl}((1-|u|^2)A)\longrightarrow 0 \quad \text{in } H^{-1}(\Omega). \tag{III.17}$$

On the other hand,

$$\frac{1}{h_{\text{ex}}}\left(\operatorname{curl}(iu,\nabla u)-2\pi\sum_i d_i\delta_{a_i}\right)\longrightarrow 0 \quad \text{in } W^{-1,q}(\Omega),\ \forall q<2. \tag{III.18}$$

This is a consequence of the proof of Lemma II.3 of [SS1] (see [ASS] for a more detailed proof).
Since $\frac{h}{h_{\text{ex}}}-p$ is bounded in $H_0^1(\Omega)$, the left-hand side of (III.16) is bounded in $H^{-1}(\Omega)$, hence in view of (III.17), $\frac{1}{h_{\text{ex}}}\operatorname{curl}(iu,\nabla u)$ is bounded in $H^{-1}(\Omega)$. Therefore, from (III.18), $\mu_\varepsilon=\frac{2\pi}{h_{\text{ex}}}\sum_i d_i\delta_{a_i}$ is bounded in $W^{-1,q}(\Omega)$, $\forall q<2$. It is also bounded in the sense of measures, hence, by a theorem of Murat (see [Mu]), it is compact in $W^{-1,r}(\Omega),\forall r<q<2$. Combining this with (III.17) and (III.18), we obtain that the right-hand side of (III.16) is compact in $W^{-1,q}(\Omega)$, $\forall q<2$, and converges to μ_0. Thus,

$$-\Delta\left(\frac{h}{h_{\text{ex}}}-p\right)+\frac{h}{h_{\text{ex}}}-\mu_0\underset{\varepsilon\to 0}{\longrightarrow}0 \quad \text{in } W^{-1,q}(\Omega),\ \forall q<2.$$

We deduce that $\frac{h}{h_{\rm ex}} - p \to h_0 - p$ strongly in $W_0^{1,q}(\Omega)$ $\forall q < 2$, and passing to the limit, h_0 satisfies

$$-\Delta(h_0 - p) + h_0 = \mu_0.$$

□

Lemma III.2 *μ_0 and h_0 being defined in the previous lemma,*

$$\liminf_{n\to\infty} \frac{J(u_{\varepsilon_n}, A_{\varepsilon_n})}{h_{\rm ex}^2} \geq \frac{\lambda}{2}\int_\Omega |\mu_0| + \frac{1}{2}\int_\Omega |\nabla(h_0 - p)|^2 + |h_0 - p|^2 = E(h_0),$$

where $E(.)$ was defined in (I.2).

Proof : We start again from the lower bound (III.14) :

$$J(u,A) \geq \sum_i \frac{1}{2}\int_{B_i} |\nabla(h - ph_{\rm ex})|^2 + \frac{1}{2}\int_{\Omega\setminus\cup_i B_i} |\nabla(h - ph_{\rm ex})|^2 + |h - ph_{\rm ex}|^2 + o(1).$$

Using (I.5), this can be rewritten

(III.19) $$\frac{J(u,A)}{h_{\rm ex}^2} \geq \frac{\pi\sum_i |d_i||\log\varepsilon|}{h_{\rm ex}^2} + \frac{1}{2}\int_{\Omega\setminus\cup_i B_i} \left|\nabla\frac{h - ph_{\rm ex}}{h_{\rm ex}}\right|^2 + \left|\frac{h - ph_{\rm ex}}{h_{\rm ex}}\right|^2 + o(1).$$

On the other hand, we know from (I.11) that

$$\sum_{i\in I_\varepsilon} r_i \leq \frac{1}{|\log\varepsilon|^6}.$$

Therefore, one can extract from ε_n a subsequence such that

(III.20) $$\sum_{n=0}^{\infty}\left(\sum_{i\in I_{\varepsilon_n}} r_i\right) < \infty.$$

Let us denote

$$\mathcal{A}_N = \bigcup_{n\geq N}\bigcup_{i\in I_{\varepsilon_n}} B_i.$$

From (III.20), we have

(III.21) $$\text{meas}(\mathcal{A}_N) \to 0 \quad \text{as } N\to\infty.$$

Now, we can write

$$\begin{aligned}
\liminf_{n\to\infty} \frac{1}{h_{\text{ex}}^2} \int_{\Omega\setminus\cup_i B_i} |\nabla(h - p h_{\text{ex}})|^2 + |h - p h_{\text{ex}}|^2 &\geq \liminf_{n\to\infty} \frac{1}{h_{\text{ex}}^2} \int_{\Omega\setminus\mathcal{A}_N} |\nabla(h - p h_{\text{ex}})|^2 + |h - p h_{\text{ex}}|^2 \\
&\geq \int_{\Omega\setminus\mathcal{A}_N} |\nabla(h_0 - p)|^2 + |h_0 - p|^2,
\end{aligned}$$

by definition of $\mathcal{A}_N$ and by lower-semicontinuity. Then, letting $N \to \infty$, as we have (III.21), we deduce

$$\text{(III.22)} \qquad \liminf_{n\to\infty} \frac{1}{h_{\text{ex}}^2} \int_{\Omega\setminus\cup_i B_i} |\nabla(h - p h_{\text{ex}})|^2 + |h - p h_{\text{ex}}|^2 \geq \int_\Omega |\nabla(h_0 - p)|^2 + |h_0 - p|^2.$$

Returning to (III.19), we now have

$$\begin{aligned}
\liminf_{n\to\infty} \frac{J(u,A)}{h_{\text{ex}}^2} &\geq \liminf_{n\to\infty} \frac{\pi \sum_i |d_i| |\log \varepsilon_n|}{h_{\text{ex}}^2} + \frac{1}{2}\int_\Omega |\nabla(h_0 - p)|^2 + |h_0 - p|^2 \\
&\geq \liminf_{n\to\infty} \frac{1}{2}\int_\Omega |\mu_{\varepsilon_n}| \frac{|\log \varepsilon_n|}{h_{\text{ex}}} + \frac{1}{2}\int_\Omega |\nabla(h_0 - p)|^2 + |h_0 - p|^2,
\end{aligned}$$

and by lower semi-continuity again,

$$\liminf_{n\to\infty} \frac{J(u,A)}{h_{\text{ex}}^2} \geq \frac{\lambda}{2}\int_\Omega |\mu_0| + \frac{1}{2}\int_\Omega |\nabla(h_0 - p)|^2 + |h_0 - p|^2 = E(h_0).$$

□

We will prove in the next section that h_0 is in fact the *unique* minimizer h_* of E, by using the upper bound of Proposition II.1, but we first need some properties of h_*.

Proposition III.1 *The minimum of E over V is uniquely achieved by $h_* \in C^{1,\alpha}(\Omega)$ ($\forall \alpha < 1$) satisfying the following properties:*

$$\text{(III.23)} \qquad \begin{cases} h_* \geq p - \dfrac{\lambda}{2} \\ h_* = p \text{ on } \partial\Omega \\ \left(h_* - \left(p - \dfrac{\lambda}{2}\right)\right)(-\Delta(h_* - p) + h_*) = 0 \text{ in } \Omega \\ \mu = -\Delta(h_* - p) + h_* \geq 0, \end{cases}$$

problem (III.23) being equivalent to (P).

We split the proof of this proposition into several lemmas. We denote by μ_+ the positive part of μ, and μ_- its negative part, so that $\mu = \mu_+ - \mu_-$.

Lemma III.3

$$h_* = p - \frac{\lambda}{2} \quad \mu_+ - \text{a.e.}$$

$$h_* = p + \frac{\lambda}{2} \quad \mu_- - \text{a.e.}$$

$$p - \frac{\lambda}{2} \leq h_* \leq p + \frac{\lambda}{2} \text{ on } \Omega.$$

Proof : First, by lower semi-continuity, it is easy to see that the minimum of E is achieved. In addition, there is a unique minimizer by convexity of E.
Also, recall that from Section II, we may write

$$E(h_*) = I(\mu) = \frac{\lambda}{2}\int_\Omega |\mu| + \frac{1}{2}\int_{\Omega\times\Omega} G(x,y)d(\mu-p)(x)d(\mu-p)(y), \tag{III.24}$$

where $G(x,y)$ is defined in (II.3). The functional $I(.)$ is defined over $\mathcal{M}\cap H^{-1}$. Now, as μ minimizes I, we can make variations $(1+tf)\mu$, where $f \in C^0(\Omega)$; the first order in $t \to 0$ has to vanish :

$$\forall t \in \mathbb{R} \qquad \frac{\lambda}{2}\int_\Omega |(1+tf)\mu| + \frac{1}{2}\int_{\Omega\times\Omega} G(x,y)d((1+tf)\mu-p)(x)d((1+tf)\mu-p)(y)$$
$$\geq \frac{\lambda}{2}\int_\Omega |\mu| + \frac{1}{2}\int_{\Omega\times\Omega} G(x,y)d(\mu-p)(x)d(\mu-p)(y),$$

hence, by symmetry of G, looking at the first order,

$$\frac{\lambda}{2}\int_\Omega f|\mu| + \int_{\Omega\times\Omega} G(x,y)d(f\mu)(x)d(\mu-p)(y) = 0.$$

Using Fubini's theorem, this means that

$$\frac{\lambda}{2}\int_\Omega f|\mu| + \int_\Omega (h_*-p)f\mu = 0.$$

This is true for any $f \in C^0(\Omega)$, hence

$$\frac{\lambda}{2}|\mu| + (h_*-p)\mu = 0 \text{ in } \mathcal{M}.$$

We deduce the two assertions

$$\begin{cases} h_* = p - \frac{\lambda}{2} & \mu_+ - \text{a.e.} \\ h_* = p + \frac{\lambda}{2} & \mu_- - \text{a.e.} \end{cases} \tag{III.25}$$

On the other hand, one can also consider variations $\mu+\nu$ for $\nu \in \mathcal{M}\cap H^{-1}$ such that $\nu \perp \mu$ (which means that μ and ν are mutually singular). Then,

$$\int_\Omega |\mu+\nu| = \int_\Omega |\mu| + \int_\Omega |\nu|,$$

and

$$\begin{aligned}\frac{\lambda}{2}\int_\Omega |\mu+\nu| &+ \frac{1}{2}\int_{\Omega\times\Omega} G(x,y)d(\mu+\nu-p)(x)d(\mu+\nu-p)(y)\\ &\geq \frac{\lambda}{2}\int_\Omega |\mu| + \frac{1}{2}\int_{\Omega\times\Omega} G(x,y)d(\mu-p)(x)d(\mu-p)(y).\end{aligned}$$

Consequently, as before,

$$\frac{\lambda}{2}\int_\Omega |\nu| + \int_\Omega (h_*-p)\nu \geq 0,\ \forall \nu \perp \mu. \tag{III.26}$$

(III.25) and (III.26) imply that

$$p-\frac{\lambda}{2} \leq h_* \leq p+\frac{\lambda}{2} \quad \text{a.e.}$$

□

Lemma III.4 *A minimal μ is a positive measure, absolutely continuous with respect to the Lebesgue measure.*

As $(h_*-p)\in H^1_0(\Omega)$, $(h_*-p)_+ \in H^1_0(\Omega)$, and, since $\mu \in H^{-1}$, we can consider the integral

$$\int_\Omega \mu(h_*-p)_+ = \int_\Omega \mu_+(h_*-p)_+ - \int_\Omega \mu_-(h_*-p)_+.$$

But, from Lemma III.3, $(h_*-p)_+ = 0$, μ_+-a.e., hence

$$\int_\Omega \mu(h_*-p)_+ = -\int_\Omega \mu_-(h_*-p)_+ \leq 0.$$

On the other hand, integrating by parts,

$$\begin{aligned}\int_\Omega \mu(h_*-p)_+ &= \int_\Omega (-\Delta(h_*-p)+h_*)(h_*-p)_+\\ &= \int_{h_*>p} |\nabla(h_*-p)|^2 + \int_{h_*>p} h_*(h_*-p) \geq 0.\end{aligned}$$

Hence, we deduce that

$$\int_\Omega \mu_-(h_* - p)_+ = 0.$$

But, as

$$(h_* - p)_+ = \frac{\lambda}{2} \quad \mu_- \text{—a.e.},$$

we must have

$$0 = \int_\Omega \mu_-(h_* - p)_+ = \frac{\lambda}{2}\int_\Omega |\mu_-|.$$

Therefore, $\int_\Omega |\mu_-| = 0$ and $\mu_- = 0$. We conclude that $\mu = \mu_+$ and μ is a positive measure. □

We can then say that h_* is solution of

$$\begin{cases} h_* \geq p - \frac{\lambda}{2} \text{ in } \Omega \\ h_* = p \text{ on } \partial\Omega \\ \left(h_* - \left(p - \frac{\lambda}{2}\right)\right)(-\Delta(h_* - p) + h_*) = 0 \text{ in } \Omega \\ \mu = -\Delta(h_* - p) + h_* \geq 0, \end{cases}$$

This is equivalent to problem (P) (see [R] for example, otherwise it can be checked directly).
This equation characterizes a minimizer of $\int_\Omega |\nabla l|^2 + l^2 - 2l\Delta p$ over the set of $l \in H^1_p$ such that $l \geq p - \frac{\lambda}{2}$ (obstacle problem). By convexity of the problem, there is a unique minimizer. In addition, the classical regularity theorem for such problems ([R] p. 141) applies and says that $h_* \in C^{1,\alpha}(\Omega), \forall \alpha < 1$. Proposition III.1 is then proved.
For the convenience of the reader, we include an elementary proof of the fact that $h_* \in C^{1,\alpha}, \forall \alpha < 1$ and that $\mu \ll dx$ (i.e. is absolutely continuous with respect to the Lebesgue measure).

Proof : We define $\psi_\alpha(t)$ over $\mathbb{R}$ by

$$\psi_\alpha(t) = 1 \text{ if } t \leq -\frac{\lambda}{2}$$
$$\psi_\alpha(t) = 0 \text{ if } t \geq -\frac{\lambda}{2} + \alpha$$
$$\psi_\alpha(t) = 1 - \frac{1}{\alpha}\left(t - \frac{\lambda}{2}\right) \text{ if } -\frac{\lambda}{2} \leq t \leq -\frac{\lambda}{2} + \alpha.$$

Let $\xi \in C_0^\infty(\Omega)$ such that $\xi \geq 0$. As $\Delta(h_* - p)$ is in H^{-1} and $\psi_\alpha(h_* - p) \in H^1_0$, we can consider the following integral :

$$-\int_\Omega \Delta(h_* - p)\xi\psi_\alpha(h_* - p) = \int_\Omega \xi\nabla(h_* - p).\nabla(\psi_\alpha(h_* - p)) + \int_\Omega \psi_\alpha(h_* - p)\nabla(h_* - p).\nabla\xi.$$

But, $\nabla(\psi_\alpha(h_* - p)) = -\frac{1}{\alpha}\mathbf{1}_{\{-\frac{\lambda}{2}\leq h_* - p\leq \alpha-\frac{\lambda}{2}\}}\nabla(h_* - p)$, hence

(III.27)

$$-\int_\Omega \Delta(h_* - p)\xi\psi_\alpha(h_* - p) = -\frac{1}{\alpha}\int_{\{-\frac{\lambda}{2}\leq h_* - p\leq \alpha-\frac{\lambda}{2}\}} \xi|\nabla(h_* - p)|^2 + \int_\Omega \psi_\alpha(h_* - p)\nabla(h_* - p).\nabla\xi.$$

We can write a Radon-Nikodym decomposition with respect to the Lebesgue measure dx of $\Delta(h_* - p)$, which is a measure, as $(\Delta(h_* - p))^R + (\Delta(h_* - p))^S$ where

$$(\Delta(h_* - p))^R \ll dx, \quad (\Delta(h_* - p))^S \perp dx.$$

On the one hand, we know that $-\Delta(h_* - p) + h_* = 0$ outside of $\omega_\lambda = \{h_* - p = -\lambda/2\}$, hence the support of $(\Delta(h_* - p))^S$ is included in $\overline{\omega_\lambda}$, and $\psi_\alpha(h_* - p) = 1$ on the support of $(\Delta(h_* - p))^S$. On the other hand, by Lebesgue's dominated convergence theorem, as

$$\psi_\alpha(h_* - p) \underset{\alpha\to 0}{\longrightarrow} \mathbf{1}_{\omega_\lambda} \text{ a.e.},$$

we have

$$\int_\Omega (-\Delta(h_* - p))^R\xi\psi_\alpha(h_* - p) \underset{\alpha\to 0}{\longrightarrow} \int_{\omega_\lambda} (-\Delta(h_* - p))^R\xi,$$

and

$$\int_\Omega \nabla(h_* - p).(\nabla\xi)\psi_\alpha(h_* - p) \underset{\alpha\to 0}{\longrightarrow} \int_{\omega_\lambda} \nabla(h_* - p).\nabla\xi = 0.$$

(Indeed $\nabla(h_* - p) = 0$ a.e. on ω_λ.) Passing to the limit $\alpha \to 0$ in (III.27), we thus have

$$\int_\Omega (-\Delta(h_* - p))^S\xi + \int_{\omega_\lambda} (-\Delta(h_* - p))^R\xi \leq 0.$$

This is true for any $\xi \geq 0$ in C_0^∞, therefore $-\Delta(h_* - p) \leq 0$ on $\overline{\omega_\lambda}$. But $-\Delta(h_* - p) + h_* \geq 0$, hence $(-\Delta(h_* - p))^S = (-\Delta(h_* - p) + h_*)^S \geq 0$, therefore $(-\Delta(h_* - p))^S$ must be 0. Thus, we obtain that $\Delta(h_* - p)$ is absolutely continuous (or L^1), and that

$$\frac{\lambda}{2} - p \leq -h_* \leq -\Delta(h_* - p) \leq 0 \quad \text{on } \omega_\lambda.$$

Hence, Δh_* is bounded on ω_λ. It is also bounded on the complement since, there, $\Delta h_* = h_* + \Delta p$ and h_* is bounded. We conclude that $\Delta h_* \in L^\infty(\Omega)$, in particular μ is absolutely continuous, and $h_* \in C^{1,\alpha}, \forall\alpha < 1$. □

IV Completing the proofs of Theorems 1 and 2 : the final convergence results

We recall that the case $\lambda = 0$ has been totally treated in Corollary II.2. For the case $\lambda > 0$, we start from the result of Lemma III.2, stating that from any sequence $\varepsilon_n \to 0$, we can extract a subsequence, also denoted ε_n such that $\frac{h_{\varepsilon_n}}{h_{\text{ex}}} \rightharpoonup h_0$ and $\mu_{\varepsilon_n} \to \mu_0$, for some $h_0 \in V$, and $\mu_0 = -\Delta(h_0 - p) + h_0$, with

$$\liminf_{n\to\infty} \frac{J(u,A)}{h_{\text{ex}}^2} \geq E(h_0) = \frac{\lambda}{2} \int_\Omega |\mu_0| + \frac{1}{2} \int_\Omega |\nabla h_0 - p|^2 + |h_0 - p|^2.$$

But from Proposition III.1 and Lemma III.4, the minimum of E over V is achieved by a unique h_*, and the measure $\mu = -\Delta(h_* - p) + h_*$ is positive and absolutely continuous. It then follows from the upper bound of Proposition II.1 that

$$\limsup_{n\to\infty} \frac{J(u_{\varepsilon_n}, A_{\varepsilon_n})}{h_{\text{ex}}^2} \leq E(h_*),$$

and then that $h_0 = h_*$, $\mu_0 = \mu$. Also

$$\lim_{n\to\infty} \frac{J(u_{\varepsilon_n}, A_{\varepsilon_n})}{h_{\text{ex}}^2} = E(h_*).$$

Examining the proof of Lemma III.2 and (III.18) especially, we have also proved that

$$\begin{aligned} &\lim_{n\to\infty} \frac{\lambda}{2} \int_\Omega |\mu_{\varepsilon_n}| + \frac{1}{2} \int_{\Omega\setminus\cup_i B_i} \left|\nabla \frac{h_{\varepsilon_n} - p h_{\text{ex}}}{h_{\text{ex}}}\right|^2 + \left|\frac{h_{\varepsilon_n} - p h_{\text{ex}}}{h_{\text{ex}}}\right|^2 \\ &= E(h_*) = \frac{\lambda}{2} \int_\Omega |\mu| + \frac{1}{2} \int_\Omega |\nabla (h_* - p)|^2 + |h_* - p|^2 \end{aligned} \tag{IV.1}$$

In addition, from Lemma III.1, we have $\mu_{\varepsilon_n} \to \mu$ in $\mathcal{M}$, hence, by lower semi-continuity,

$$\liminf_{n\to\infty} \frac{\lambda}{2} \int_\Omega |\mu_{\varepsilon_n}| \geq \frac{\lambda}{2} \int_\Omega |\mu|. \tag{IV.2}$$

From Lemma III.1 too, $\frac{h}{h_{\text{ex}}} \rightharpoonup h_*$ in H^1_p. By compact Sobolev embedding, and arguing as in (III.21), we have

$$\lim_{n\to\infty} \int_{\Omega\setminus\cup_i B_i} \left|\frac{h_{\varepsilon_n}}{h_{\text{ex}}} - p\right|^2 = \int_\Omega |h_* - p|^2. \tag{IV.3}$$

As in (III.21) again,

$$\liminf_{n\to\infty}\int_{\Omega\setminus\cup_i B_i}\left|\nabla\frac{h_{\varepsilon_n}-ph_{\mathrm{ex}}}{h_{\mathrm{ex}}}\right|^2\geq\int_\Omega|\nabla(h_*-p)|^2. \tag{IV.4}$$

Comparing (IV.2), (IV.3) and (IV.4) to (IV.1), we must have

$$\lim_{n\to\infty}\int_\Omega|\mu_{\varepsilon_n}|=\int_\Omega|\mu|=\int_\Omega\mu, \tag{IV.5}$$

and

$$\lim_{n\to\infty}\int_{\Omega\setminus\cup_i B_i}\left|\nabla\frac{h_{\varepsilon_n}-ph_{\mathrm{ex}}}{h_{\mathrm{ex}}}\right|^2=\int_\Omega|\nabla(h_*-p)|^2. \tag{IV.6}$$

In addition, (III.14) also implies that

$$\limsup_{n\to\infty}\int_\Omega\left|\nabla\left(\frac{h_{\varepsilon_n}-ph_{\mathrm{ex}}}{h_{\mathrm{ex}}}\right)\right|^2\leq\lambda\int_\Omega\mu+\int_\Omega|\nabla(h_*-p)|^2. \tag{IV.7}$$

We deduce from (IV.5) that, in addition to having $\mu_{\varepsilon_n}\to\mu$ in $\mathcal{M}$, we have $|\mu_{\varepsilon_n}|\to\mu$ in $\mathcal{M}$. Indeed, for any domain $U\subset\Omega$

$$\liminf\int_U|\mu_{\varepsilon_n}|\geq\liminf\int_U\mu_{\varepsilon_n}=\int_U\mu,$$

and similarly

$$\liminf\int_{\Omega\setminus U}|\mu_{\varepsilon_n}|\geq\int_{\Omega\setminus U}\mu,$$

thus there must be equality. We can hence conclude that

$$\frac{2\pi\sum_i d_i\delta_{a_i}}{h_{\mathrm{ex}}}\to\mu=\left(p-\frac{\lambda}{2}\right)\mathbf{1}_{\omega_\lambda},$$

$$\frac{2\pi\sum_i |d_i|\delta_{a_i}}{h_{\mathrm{ex}}}\to\mu=\left(p-\frac{\lambda}{2}\right)\mathbf{1}_{\omega_\lambda}.$$

In order to complete the proof of Theorem 1, there remains to prove

Proposition IV.1

$$\left|\nabla\left(\frac{h_{\varepsilon_n}}{h_{\mathrm{ex}}}-h_*\right)\right|^2\underset{n\to\infty}{\longrightarrow}\lambda\mu\qquad\text{in }\mathcal{M}. \tag{IV.8}$$

Proof : From the Brezis-Lieb lemma, it suffices to prove that

$$\left|\nabla\left(\frac{h_{\varepsilon_n}}{h_{\rm ex}}-p\right)\right|^2 \underset{n\to\infty}{\longrightarrow} |\nabla(h_*-p)|^2+\lambda\mu \qquad \text{in } \mathcal{M}. \tag{IV.9}$$

Considering any domain $U\subset\Omega$, and arguing as in Lemma III.2, we have

$$\begin{aligned}\liminf_{n\to\infty}\int_U\left|\nabla\left(\frac{h_{\varepsilon_n}}{h_{\rm ex}}-p\right)\right|^2 &= \liminf_{n\to\infty}\int_{U\backslash\cup_i B_i}\left|\nabla\left(\frac{h_{\varepsilon_n}}{h_{\rm ex}}-p\right)\right|^2+\int_{(\cup_i B_i)\cap U}\left|\nabla\left(\frac{h_{\varepsilon_n}}{h_{\rm ex}}-p\right)\right|^2\\ &\geq \int_U|\nabla(h_*-p)|^2+\liminf_{n\to\infty}\int_{(\cup_i B_i)\cap U}\left|\nabla\left(\frac{h_{\varepsilon_n}}{h_{\rm ex}}-p\right)\right|^2. \qquad \text{(IV.10)}\end{aligned}$$

Now, as μ is a regular measure, there exists a family of compact sets $U_\eta\Subset U$ such that $\mu(U)=\lim_{\eta\to 0}\mu(U_\eta)$. For any η, when ε is sufficiently small, all the $B(a_i,r_i)$ such that $a_i\in U_\eta$ are included in U, hence

$$\begin{aligned}\int_{(\cup_i B_i)\cap U}\left|\nabla\left(\frac{h_{\varepsilon_n}}{h_{\rm ex}}-p\right)\right|^2 &\geq \int_{\cup_i B(a_i,r_i)/a_i\in U_\eta}\left|\nabla\left(\frac{h_{\varepsilon_n}}{h_{\rm ex}}-p\right)\right|^2\\ &\geq \frac{2\pi}{h_{\rm ex}^2}\sum_{i/a_i\in U_\eta}|d_i||\log\varepsilon|(1-o(1)),\end{aligned}$$

using (I.5). By definition of μ, this amounts to

$$\liminf_{n\to\infty}\int_{(\cup_i B_i)\cap U}\left|\nabla\left(\frac{h_{\varepsilon_n}}{h_{\rm ex}}-p\right)\right|^2\geq\frac{1}{h_{\rm ex}^2}\left(\int_{U_\eta}|\mu_{\varepsilon_n}|\right)h_{\rm ex}|\log\varepsilon|,$$

hence by convergence of $|\mu_{\varepsilon_n}|$, we are led to

$$\liminf_{n\to\infty}\int_{(\cup_i B_i)\cap U}\left|\nabla\left(\frac{h_{\varepsilon_n}}{h_{\rm ex}}-p\right)\right|^2\geq\lambda\mu(U_\eta). \tag{IV.11}$$

Combining this to (IV.10), we deduce

$$\liminf_{n\to\infty}\int_U\left|\nabla\left(\frac{h_{\varepsilon_n}}{h_{\rm ex}}-p\right)\right|^2\geq\int_U|\nabla(h_*-p)|^2+\lambda\mu(U_\eta),$$

and passing to the limit $\eta\to 0$, we obtain

$$\liminf_{n\to\infty}\int_U\left|\nabla\left(\frac{h_{\varepsilon_n}}{h_{\rm ex}}-p\right)\right|^2\geq\int_U|\nabla(h_*-p)|^2+\lambda\mu(U).$$

Comparing this to (IV.7), we obtain (IV.9). □

Now, all these limits are independent of the chosen subsequence; therefore, the whole sequence converges. Since this is true for any sequence $\varepsilon_n \to 0$, this is true for $\varepsilon \to 0$. This completes all the proofs.

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