Vortex collisions and energy-dissipation rates in the Ginzburg-Landau heat flow

Part II: The dynamics

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Abstract

We deduce from the first part of this paper [S1] estimates on the energy-dissipation rates for solutions of the Ginzburg-Landau heat-flow, which allow to study various phenomena occurring in this flow, among which vortex-collisions; allowing in particular to extend the dynamical law of vortices passed collisions.

1 Introduction and statement of the main results

1.1 Presentation of the problem

We recall from [S1] that we are interested in the following parabolic Ginzburg-Landau equation in 2 dimensions, in the asymptotic limit $\varepsilon \to 0$:

$$(1.1)\qquad \begin{cases} \dfrac{\partial_t u}{|\log \varepsilon|} = \Delta u + \dfrac{1}{\varepsilon^2} u(1-|u|^2) & \text{in } \Omega \times \mathbb{R}_+ \\ u(.,0) = u_\varepsilon^0 & \text{in } \Omega, \end{cases}$$

where Ω is a two-dimensional domain, assumed to be smooth, bounded and simply connected, and where u is a *complex-valued* function, assumed to satisfy either one of the boundary conditions

$$(1.2)\qquad u = g \quad \text{on } \partial\Omega$$

with g a fixed regular map from Ω to $\mathbb{S}^1$, in which case we also assume that Ω is strictly starshaped with respect to a point; or

$$(1.3)\qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega$$

in which case no further assumption is made.

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The Ginzburg-Landau heat flow is an L^2 gradient-flow (or steepest descent) for the Ginzburg-Landau functional

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2}. \tag{1.4}$$

For the motivations to study this equation, and the notion of *vortices*, we refer to the first part of this paper [S1]. It was shown by Bethuel-Brezis-Hélein in [BBH] that under the assumption

$$E_\varepsilon(u) \le C|\log \varepsilon|, \tag{1.5}$$

minimizers (respectively critical points) of E_ε have a bounded number of vortices which converge, as $\varepsilon \to 0$, to a finite set of points which are minimizers (respectively critical points) of an explicit finite-dimensional function called *renormalized energy*, and denoted W. They proved the crucial relation that if u_ε has n limiting vortices at $p_1, \cdots, p_n$, of degrees D_i,

$$E_\varepsilon(u_\varepsilon) \ge \pi n|\log \varepsilon| + W_{\mathbf{D}}(p_1, \cdots, p_n) + n\gamma + o(1), \tag{1.6}$$

where γ is a universal constant introduced in [BBH] (it is the energy of the profile of the 1-vortex solution in the plane). The main term in $W_{\mathbf{D}}$

$$-\pi \sum_{i\neq j} D_i D_j \log|p_i - p_j| \tag{1.7}$$

contains the interaction between the vortices and indicates that vortices of opposite sign attract each other, while vortices of same sign repel each other.

The dynamics of the vortices under the heat-flow (1.1) has also been studied, and it was established by Lin [Li] and Jerrard-Soner [JS] (see also Spirn [Sp] for the equation with magnetic field), that, as could be expected, the limiting vortices p_i of the solutions of (1.1) evolve (in that time-scale) according to the gradient of the renormalized energy W, i.e. according to the set of ODE's

$$\begin{cases} \dfrac{dp_i}{dt} = -\dfrac{1}{\pi}\nabla_i W_{\mathbf{D}}(p_1, \cdots, p_n)(t) \\ p_i(0) = p_i^0 \end{cases} \tag{1.8}$$

This was established under the following set of restrictions:

1. The initial vortices all have degree ± 1 and are well-separated.
2. The initial data is assumed to be "well-prepared" i.e.

   $$E_\varepsilon(u_\varepsilon^0) \le \pi n|\log \varepsilon| + C, \tag{1.9}$$

   where n is the number of initial vortices.
3. There are no collisions (or we work until the first collision-time under the law (1.8)).
4. The vortices cannot exit Ω.

A different proof via a Γ-convergence or energy-based method was later given by Sandier-Serfaty in [SS2], under the same conditions and the slightly stronger "very well-prepared" assumption

$$E_\varepsilon(u_\varepsilon^0) \le \pi n |\log \varepsilon| + W_{\mathbf{D}}(p_1^0, \cdots, p_n^0) + n\gamma + o(1). \tag{1.10}$$

All these results were valid only up to collision-time under the law (1.8); but, if there are initially vortices of opposite degrees, then this law does generically generate collisions (see the form (1.7)). Collisions create a problem in the analysis because the "well-preparedness" breaks down during a collision. They are probably one of the most interesting phenomena in these Ginzburg-Landau dynamics, and Ginzburg-Landau itself is one of the simplest models in which collisions of vortices can be studied.

In this paper, we are interested in giving results relaxing the assumptions above. The main objectives of this work are to study collisions, which were not well-understood, to determine how and how fast the energy dissipates during such collisions, to give a dynamical law after blow-up, and to see how the dynamical law of the vortices can be continued /extended after collisions. We also show how the well-prepared assumption can be weakened, and relax the separation hypothesis, for example dealing with the possible separation of two +1 vortices which are initially very close. We use our study of Part I [S1] of "pathological situations" i.e those for which we have a group of vortices which are far from the others, and degrees d_i and $(\sum_i d_i)^2 \neq \sum_i d_i^2$ in the group, which we called an *"unbalanced cluster of vortices"*.

While this work was in completion, very similar issues were addressed by Bethuel, Orlandi and Smets in [BOS1] and later on, their study was completed in [BOS2]. Prior to all these works, the only partial result on collisions was the paper of Bauman-Chen-Phillips-Sternberg [BCPS], where they studied the situation in the whole plane with quite rigid conditions at infinity.

The first paper of Bethuel-Orlandi-Smets [BOS1] gives a geometric measure theoretical description of the limiting vortex-trajectories under very general assumptions (a simple bound $E_\varepsilon(u_\varepsilon) \le C|\log \varepsilon|$), including possible splittings, collisions and recombinations, and results of annihilation in the case of collisions with total degree 0; it also exhibit a phenomenon of "phase-vortex interaction" occurring (only) in infinite samples (their setup is the whole plane), which can create a drift of the vortices. Their later paper contains some results more similar to the present paper, it shows that the limiting trajectories of the vortices are rectifiable, and derives their limiting motion law outside of collision times, via the "balanced" property and a quantization of the energy like the one we mentioned in Part I [S1], relation (1.31) here.

Some of the main differences between our work and theirs is that we handle boundary conditions in bounded domains, and that our method, inspired by [SS2], is rather energy-based than PDE-based: it relies on examining the energy-dissipation rates through the study made in Part I [S1] of the perturbed Ginzburg-Landau equation

$$\begin{cases} \Delta u + \frac{u}{\varepsilon^2}(1-|u|^2) = f_\varepsilon & \text{in } \Omega \\ u = g \ (resp. \ \frac{\partial u}{\partial \nu} = 0) & \text{on } \partial\Omega. \end{cases} \tag{1.11}$$

under the hypotheses $E_\varepsilon(u_\varepsilon) \le M|\log \varepsilon|$, $|u_\varepsilon| \le 1$, and $|\nabla u_\varepsilon| \le \frac{M}{\varepsilon}$; and on characterizing precisely the value of the energy and location of the zeroes of u_ε during the dynamics.

1.2 Methodology

Let us first recall the method of the paper [SS2]. It is written as an abstract scheme, which we will not fully quote here (refer to [SS2]), but rather we describe here its implementation for Ginzburg-Landau.

The idea is to use the fact that, given a priori the number of limiting vortices n, and their degrees $D_i = \pm 1$, $E_\varepsilon - \pi n|\log \varepsilon|$ Γ-converges to W, combined with some additional estimates on the C^1 structure of the energy landscape. For the meaning of Γ-convergence, we need to specify a sense of convergence: we say $u_\varepsilon \overset{S}{\rightharpoonup} (p_1, \cdots, p_n)$ if

$$\mu_\varepsilon := \mathrm{curl}\,(iu_\varepsilon, \nabla u_\varepsilon) \rightharpoonup 2\pi \sum_{i=1}^{n} D_i \delta_{p_i}, \tag{1.12}$$

where $(.,.)$ denotes, here as well as in the sequel, the scalar product in $\mathbb{C}$ identified with $\mathbb{R}^2$. This is the convergence of the Jacobian determinant, or vorticity of u_ε (exactly as in fluid mechanics), its role in Ginzburg-Landau has been first emphasized by Jerrard-Soner [JS2] and Alberti [Al] (see also [SS3]) and has been commonly used since then. It allows to trace down the vortices and find the limiting vortices p_i. The best compactness for μ_ε one obtains is in a weak norm: in the dual of $C_c^{0,\gamma}(\Omega)$, but this is not important here. Observe that the p_i's are the limits as $\varepsilon \to 0$ of the zeroes of u_ε, not the zeroes themselves, and the degrees are the limits of the total degrees of the zeroes converging to each p_i.

The equation (1.1) can be seen as the gradient flow

$$\partial_t u_\varepsilon = -\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon) \tag{1.13}$$

where $\nabla_{X_\varepsilon} E_\varepsilon$ denotes the gradient of E_ε with respect to the Hilbert structure

$$\|.\|_{X_\varepsilon}^2 = \frac{1}{|\log \varepsilon|} \|.\|_{L^2(\Omega)}^2.$$

With these notations, if u_ε is a solution of the flow (1.1), the energy-dissipation rate is

$$\begin{aligned} -\frac{d}{dt} E_\varepsilon(u_\varepsilon(x,t)) &= -\langle \partial_t u_\varepsilon, \nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon)\rangle_{X_\varepsilon} \\ &= \|\partial_t u_\varepsilon\|_{X_\varepsilon}^2 = \|\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon)\|_{X_\varepsilon}^2 \qquad (1.14) \\ &= \frac{1}{2}\|\partial_t u_\varepsilon\|_{X_\varepsilon}^2 + \frac{1}{2}\|\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon)\|_{X_\varepsilon}^2. \qquad (1.15) \end{aligned}$$

The main idea of [SS2] is in writing this energy-dissipation as (1.15) and in proving that two additional relations hold.

The first relation (which we recalled in Part I) was: if $\mathrm{curl}\,(iu_\varepsilon, \nabla u_\varepsilon) \rightharpoonup 2\pi \sum_i D_i \delta_{p_i}$ as $\varepsilon \to 0$, then

(1.16)

$$\varliminf_{\varepsilon \to 0} \int_\Omega |\log \varepsilon| \left| \Delta u_\varepsilon + \frac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2) \right|^2 = \varliminf_{\varepsilon \to 0} \|\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon)\|_{X_\varepsilon}^2 \geq \frac{1}{\pi} \sum_i |\nabla_i W_{\mathbf{D}}(p_1, \cdots, p_n)|^2,$$

which relates the slope of the energy at a configuration to the slope of the renormalized energy at the underlying vortices.

The other relation is that, under the assumption $E_\varepsilon(u_\varepsilon) \le \pi n |\log \varepsilon| + O(1)$, if for every $t \in [0,T]$, $\mu_\varepsilon(t) = \mathrm{curl}\,(iu_\varepsilon, \nabla u_\varepsilon)(t) \rightharpoonup 2\pi \sum_{i=1}^n D_i \delta_{p_i(t)}$ then

$$\lim_{\varepsilon\to 0} \frac{1}{|\log \varepsilon|} \int_{\Omega\times[0,T]} |\partial_t u_\varepsilon|^2 = \lim_{\varepsilon \to 0} \int_0^T \|\partial_t u_\varepsilon\|_{X_\varepsilon}^2 \ge \pi \sum_{i=1}^n \int_0^T |d_t p_i|^2 \, dt \tag{1.17}$$

This lower bound is sharp and relates the kinetic energy $\int_\Omega |\partial_t u|^2$ to the velocity of the underlying vortices. It comes as a corollary of a more general result called "product-estimate", valid for any configuration (not necessarily solving (1.1)), proved in [SS1], which we use again several times in this paper, and whose time-dependent version is ($\mathcal{M}(\Omega)$ denotes the space of bounded Radon measures on Ω):

Theorem 1 ("Product estimate", time-dependent version, see [SS1]). *Let $u_\varepsilon(x,t)$ be defined over $\Omega \times [0,T]$ and be such that*

$$\begin{cases} \forall t \in [0,T], \quad E_\varepsilon(u_\varepsilon(t)) \le C|\log \varepsilon| \\ \displaystyle\int_{\Omega\times[0,T]} |\partial_t u_\varepsilon|^2 \le C|\log \varepsilon|. \end{cases} \tag{1.18}$$

Then, V_ε being defined by

$$V_\varepsilon = (\partial_2(iu_\varepsilon, \partial_t u_\varepsilon) - \partial_t(iu_\varepsilon, \partial_2 u_\varepsilon), -\partial_1(iu_\varepsilon, \partial_t u_\varepsilon) + \partial_t(iu_\varepsilon, \partial_1 u_\varepsilon)) \tag{1.19}$$

there exist $\mu \in L^\infty([0,T], \mathcal{M}(\Omega))$ of the form

$$\mu(t) = 2\pi \sum_i D_i(t) \delta_{p_i(t)} \qquad d_i(t) \in \mathbb{Z},$$

and $V \in L^2([0,T], \mathcal{M}(\Omega))$ such that, after extraction,

$$\begin{aligned} \mu_\varepsilon &\rightharpoonup \mu \quad \text{in } (C_c^{0,\gamma}([0,T]\times\Omega))', \ \forall \gamma > 0, \\ V_\varepsilon &\rightharpoonup V \quad \text{in } (C_c^{0,\gamma}([0,T]\times\Omega))', \ \forall \gamma > 0, \end{aligned}$$

with

$$\partial_t \mu + \mathrm{div}\ V = 0. \tag{1.20}$$

Moreover, for any $X \in C_c^0([0,T]\times\Omega, \mathbb{R}^2)$ and $f \in C_c^0([0,T]\times\Omega)$, we have

$$\lim_{\varepsilon\to 0} \frac{1}{|\log \varepsilon|^2} \int_{\Omega\times[0,T]} |X\cdot\nabla u_\varepsilon|^2 \int_{\Omega\times[0,T]} f^2 |\partial_t u_\varepsilon|^2 \ge \frac14 \left| \int_{\Omega\times[0,T]} V \cdot fX \right|^2. \tag{1.21}$$

Observe that taking $|X| \le 1$ and $|f| \le 1$, for solutions of (1.1) assuming $\int_\Omega |\nabla u_\varepsilon|^2 \le C|\log \varepsilon|$, we have $\frac{1}{|\log \varepsilon|} \int_{\Omega\times[0,T]} |X\cdot\nabla u_\varepsilon|^2 \le CT$, while $\frac{1}{|\log \varepsilon|}\int_{\Omega\times[0,T]} |\partial_t u_\varepsilon|^2 = E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(T))$, thus the relation (1.21) states essentially that for such solutions,

$$|p_i(T) - p_i(0)|^2 \le CT\left(E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(T))\right) \tag{1.22}$$

or in condensed notation

$$|\Delta p|^2 \le C \Delta T \Delta E \tag{1.23}$$

relating Δp the difference in position p, ΔT the difference in time, and ΔE the difference in energy. This crucial relation reflects of course the parabolic scaling and will be used to bound from below collision times.
The product estimate also allowed us to define limiting continuous (in fact $C^{0,\frac{1}{2}}$) vortex-trajectories as follows.

Proposition 1.1 (Vortex trajectories - see [SS2]). *Let $u_\varepsilon(x,t)$ be defined over $\Omega \times \mathbb{R}_+$ and such that (1.18) holds with $T = +\infty$ (in particular these hold for u_ε solving (1.1) with (1.27) holding). Then, after extraction of a subsequence, there exist points $p_i(t)$ and integers $D_i(t) \in \mathbb{Z}$ and $n(t) \in \mathbb{N}$ such that*

$$\operatorname{curl}(iu_\varepsilon, \nabla u_\varepsilon)(t) \rightharpoonup \mu(t) = 2\pi \sum_{i=1}^{n(t)} D_i(t)\delta_{p_i(t)} \quad \text{as } \varepsilon \to 0,$$

moreover $t \mapsto \langle \zeta, \mu(t)\rangle \in H^1((0,\infty))$ for every $\zeta \in C^1_c(\Omega)$. If in addition, for a given τ, $\sum_{i=1}^{n(t)} |D_i(t)| \le \sum_{i=1}^{n(\tau)} |D_i(\tau)|$ for every $t \ge \tau$, and $D_i(\tau) = \pm 1$ with the $p_i(\tau)$ distinct, then there exists $T_* > \tau$ such that for every $t \in [\tau, T_*)$

$$\mu(t) = 2\pi \sum_{i=1}^{n(\tau)} D_i(\tau)\delta_{p_i(t)}$$

where the $p_i(t)$ are distinct points and $p_i \in H^1((\tau, T_*), \Omega)$. Moreover, if $T_* < \infty$ then

$$\lim_{t\to T_*^-} \min\left(\min_{i\neq j} |p_i(t) - p_j(t)|, \min_i dist(p_i(t), \partial\Omega)\right) = 0.$$

Returning to our two relations (1.16) and (1.17), once they are proved, we combine them with (1.15), and integrate in time, which leads to

$$\begin{aligned}
E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(T)) &\ge \frac{1}{2}\|\partial_t u_\varepsilon\|^2_{X_\varepsilon} + \|\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon)\|^2_{X_\varepsilon} \\
&\ge \frac{1}{2}\sum_i \int_0^T \pi |d_t p_i|^2 + \frac{1}{\pi}|\nabla_i W_{\mathbf{D}}(p)|^2 + o(1) \\
&\ge \sum_i \int_0^T \langle -d_t p_i, \nabla_i W_{\mathbf{D}}(p)\rangle + o(1) \qquad (1.24)\\
&\ge W_{\mathbf{D}}(p_1(0), \cdots, p_n(0)) - W_{\mathbf{D}}(p_1(T), \cdots, p_n(T)) + o(1).
\end{aligned}$$

When $u_\varepsilon(0)$ is "very well-prepared", this implies that $E_\varepsilon(u_\varepsilon(T)) \le \pi n|\log \varepsilon| + W_{\mathbf{D}}(p_i(T)) + n\gamma + o(1)$. But the Γ-convergence of E_ε yields the opposite inequality, hence there is equality, in particular equality in the Cauchy-Schwarz relation (1.24) which allows to retrieve the dynamical law $\partial_t p_i = -\frac{1}{\pi}\nabla_i W_{\mathbf{D}}(p_1, \cdots, p_n)$, as long as the number of vortices remains fixed. An important fact which follows is that "very well-preparedness" is preserved through the flow, i.e. we always have $E_\varepsilon(u_\varepsilon(t)) = \pi n|\log \varepsilon| + W_{\mathbf{D}}(p_1(t), \cdots, p_n(t)) + n\gamma + o(1)$.

Part of what we do here is to prove that this scheme can be carried out even after blow-up in space-time, allowing to treat the situation when vortices are at a distance $l \ll 1$, as long as l is not too small. This will be the object of Theorems 4 and 6.

Let us now turn to the other part of the approach. Vortices colliding corresponds to the more general fact that several vortices converge to the same limit as $\varepsilon \to 0$, with possible (but not necessarily) limiting degree 0. When vortices are well separated, then time needs to be accelerated as in (1.1) in order to see vortex-motion, as first observed in [RS]. But this is not true when vortices become very close, because formally the phase-excess φ of the solution u_ε then decays according to an accelerated heat equation $\frac{\partial_t \varphi}{|\log \varepsilon|} = \Delta\varphi$, as pointed out in [Li, JS, BOS1], thus in the faster time-scale $\frac{1}{|\log \varepsilon|}$, while the other remote vortices should not move. The task will thus consist in retrieving these phenomena quantitatively.

For solutions of the gradient-flow, we have seen that the energy dissipation-rate is

$$-\frac{d}{dt}E_\varepsilon(u_\varepsilon(t)) = \|\partial_t u_\varepsilon\|^2_{X_\varepsilon} = \|\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon)\|^2_{X_\varepsilon}. \tag{1.25}$$

If we write for simplicity that (1.11) holds, with $f_\varepsilon = \frac{\partial_t u_\varepsilon}{|\log \varepsilon|}$, we have $f_\varepsilon = -\frac{1}{|\log \varepsilon|}\nabla_{X_\varepsilon} E_\varepsilon(u)$ in the previous notations, and

$$\|\nabla_{X_\varepsilon} E_\varepsilon(u)\|^2_{X_\varepsilon} = |\log \varepsilon| \|f_\varepsilon\|^2_{L^2(\Omega)} \tag{1.26}$$

Combining this to (1.25), we see that knowing $\|f_\varepsilon\|_{L^2}$ gives the energy-dissipation rate (in time), or rather $-\frac{1}{|\log \varepsilon|}\frac{d}{dt}E_\varepsilon$. If $\|f_\varepsilon\|_{L^2}$ is large, then the energy dissipates fast, thus decreasing to a point which allows to rule out certain configurations (for example if E_ε decreases so much that $E_\varepsilon \le C$ then there can be no more vortices). On the other hand, if f_ε is small, then the behavior of vortices can be controlled through the results obtained in Part I [S1]. The idea is thus to use this alternative in a *quantitative* way, in order to obtain information on vortex-collisions or other pathological situations.

Let us recall one of the main results of Part I (see Theorem 1 in [S1]): assuming that u_ε solves (1.11) and under the additional hypotheses

$$E_\varepsilon(u_\varepsilon) \le M|\log \varepsilon| \tag{1.27}$$

$$|u_\varepsilon| \le 1 \quad |\nabla u_\varepsilon| \le \frac{M}{\varepsilon} \tag{1.28}$$

$$\|f_\varepsilon\|^2_{L^2(\Omega)} \le \frac{1}{\varepsilon^\beta} \quad \text{for some } \beta < 2, \tag{1.29}$$

then we can find what we called a "good collection" of vortices and degrees (a_i, d_i) of u_ε, and we have

$$\forall \alpha < 1, \qquad \alpha\pi \sum_{i=1}^n d_i^2 \le \frac{E_\varepsilon(u_\varepsilon)}{|\log \varepsilon|} + C|\log \varepsilon|^{7/2}\varepsilon^{1-\alpha}\|f_\varepsilon\|_{L^2(\Omega)} + o(1), \tag{1.30}$$

and

$$o(1) \le E_\varepsilon(u_\varepsilon) - \left(\pi \sum_{i=1}^n d_i^2 \log\frac{1}{\varepsilon} + W_{\mathbf{d}}(a_1, \cdots, a_n) + \sum_{i=1}^n \gamma(V_i)\right) \le C\|f_\varepsilon\|^2_{L^2(\Omega)} + o(1), \tag{1.31}$$

where $\gamma(V_i)$ are constants depending on the d_i's and equal to γ when $d_i = \pm 1$.

This allows to deduce two important ingredients: an upper bound on the number of actual zeroes of u_ε from (1.30), and a *differential inequality on the energy* through (1.31), which is optimal, and allows to retrieve the fast parabolic scaling.

1.3 Main results on the dynamics

Several of our results give information on the vortices of the solutions u_ε at the ε-level, giving asymptotic time-scales of collisions and of energy-dissipation. This is of course a little more precise than just characterizing the trajectories of the limiting vortices, which we do in Theorem 5. We also derive the dynamical law after blow-up (at any not too small scale) during collisions, which is also more precise.

The first application of the theorems proved in [S1] consists in showing that the "very-well prepared assumption" that was used in [SS2] is not restrictive since "well-prepared" data becomes instantaneously (i.e. in $o(1)$ time) "very well-prepared", by fast dissipation of the energy-excess obtained in (1.31). In fact, we can further relax the well-prepared assumption through the following.

Theorem 2 (Instantaneous "very-well preparedness"). *Assume that u_ε is a solution of (1.1) such that (1.28) holds and*

$$\operatorname{curl}(iu_\varepsilon^0, \nabla u_\varepsilon^0) \rightharpoonup 2\pi \sum_{i=1}^{n} D_i \delta_{p_i^0} \quad \textit{as} \ \ \varepsilon \to 0, \tag{1.32}$$

with $D_i = \pm 1$ and the p_i^0 are distinct points, and such that

$$E_\varepsilon(u_\varepsilon^0) \le \pi n |\log \varepsilon| + \frac{|\log \varepsilon|}{(\log \ |\log \varepsilon|)^\beta} \tag{1.33}$$

for some $\beta > 1$. Then, there exists a time $T_\varepsilon \le C\frac{\log \ |\log \varepsilon|}{|\log \varepsilon|}$ such that for every $t_\varepsilon \in [0, T_\varepsilon]$, we have

$$\operatorname{curl}(iu_\varepsilon, \nabla u_\varepsilon)(t_\varepsilon) \rightharpoonup 2\pi \sum_{i=1}^{n} D_i \delta_{p_i^0} \quad \textit{as} \ \ \varepsilon \to 0, \tag{1.34}$$

and

$$E_\varepsilon(u_\varepsilon(T_\varepsilon)) \le \pi n |\log \varepsilon| + W_{\mathbf{D}}(p_1^0, \cdots, p_n^0) + n\gamma + o(1). \tag{1.35}$$

That is, under these weaker assumptions (an energy-excess $\gg 1$ is allowed in (1.33)), in a time $T_\varepsilon = o(1)$, the initial vortices have not moved, and u_ε has become well-prepared, i.e. all excess-energy has dissipated; one can then apply the previous results [JS, Li, SS2] starting at the time T_ε and retrieve the same dynamical law (1.8).

As an application of Theorem 2 of [S1], we get the next result, which allows to continue the dynamics after vortex-collisions. Let us assume that we are in the following generic case: u_ε has a dipole of vortices of degree ± 1 colliding, i.e. which are at a distance $l \ll 1$ (as $\varepsilon \to 0$) from each other and converging to a point p_{dip} as $\varepsilon \to 0$, and n other vortices of degree ± 1, converging to distinct points $p_1, \cdots, p_n$, distinct from p_{dip}. This situation implies that

$$\operatorname{curl}(iu_\varepsilon^0, \nabla u_\varepsilon^0) \rightharpoonup 2\pi \sum_{i=1}^{n} D_i \delta_{p_i^0} \quad \text{as } \varepsilon \to 0, \tag{1.36}$$

with $D_i = \pm 1$. We may also assume that there exists $p_\varepsilon \to p_{dip}$ such that, considering $\overline{u_\varepsilon}(x) = u_\varepsilon(p_\varepsilon + lx, 0)$ we have

$$\operatorname{curl}(i\overline{u_\varepsilon}, \nabla \overline{u_\varepsilon}) \rightharpoonup 2\pi \left(\delta_{b_+} - \delta_{b_-}\right) \quad \text{as } \varepsilon \to 0 \tag{1.37}$$

where $|b_+ - b_-| = 1$.

We may also assume that this situation is inherited from a well-prepared data at a previous time, so we may assume that u_ε is well-prepared with respect to these vortices, i.e. $E_\varepsilon(u_\varepsilon) \le \pi n |\log \varepsilon| + 2\pi \log \frac{l}{\varepsilon} + O(1)$.

Theorem 3 (Collisions). *Let u_ε be a solution of (1.1) such that at time 0, (1.28) and (1.36)–(1.37) hold, and*

$$E_\varepsilon(u_\varepsilon^0) \le \pi n |\log \varepsilon| + 2\pi \log \frac{l}{\varepsilon} + O(1),$$

with $l = o(1)$. Then there exists a first time $T_1 \le C_1 l^2 + C_2 |\log \varepsilon|^4 e^{-2\sqrt{|\log \varepsilon|}} \le o(1)$ for which $u_\varepsilon(T_1)$ has exactly n zeroes (i.e. the dipole has collided). If $l \ge \varepsilon^\beta$ with $\beta < 1$, then also $T_1 \ge C_3 l^2$. Moreover, there exists a time $T_2 \le T_1 + C_4 \frac{\log |\log \varepsilon|}{|\log \varepsilon|} \le o(1)$ such that for every $t_\varepsilon \le T_2$, we have

$$\operatorname{curl}(iu_\varepsilon, \nabla u_\varepsilon)(t_\varepsilon) \rightharpoonup 2\pi \sum_{i=1}^n D_i \delta_{p_i^0} \quad \text{as } \varepsilon \to 0 \tag{1.38}$$

and

$$E_\varepsilon(u_\varepsilon(T_2)) \le \pi n |\log \varepsilon| + W_{\mathbf{D}}(p_1^0, \cdots, p_n^0) + n\gamma + o(1) \tag{1.39}$$

The relation (1.38) indicates that the vortices not involved in the collision have not moved during the time $T_2 = o(1)$, and (1.39) that u_ε has become well-prepared again relative to those vortices within that time. Thus all excess-energy carried by the colliding vortices has dissipated in $o(1)$ time, and the previously known results apply after that time T_2, i.e. one may continue and retrieve the dynamical law with the remaining vortices. Moreover, our result shows that the actual collision of the zeroes should happen in $O(l^2)$ time (the lower bound on the collision time is simply provided by an appropriate version of the "product-estimate" Theorem 1 or (1.23)), in agreement with the expectation that the distance between colliding vortices decreases like $\sqrt{T_* - t}$, if they interact according to the expected law $\frac{da_i}{dt} = \frac{1}{\pi} \frac{a_j - a_i}{|a_j - a_i|^2}$, while the leftover energy-excess dissipates in $\frac{\log |\log \varepsilon|}{|\log \varepsilon|}$ time, in agreement with the time-scaling of the equation. A further justification is given by the result of Theorem 4 below.

Analogous results can be derived from Theorems 1 and 2 of [S1] for other "bad" situations when vortices accumulate in an unbalanced cluster i.e. with $\sum_i d_i^2 \neq (\sum_i d_i)^2$, for example two repulsing $+1$ starting at a distance l. These results are are given in Sections 3.2, 3.3.

The next result consists in analyzing the vortex-collisions or vortex-separation by blow-up, in order to retrieve some dynamical law. Thanks to Theorem 1 of [S1], which allows to control errors, the analysis of [SS2] which we presented above carries through after blow-up, as long as the blow-up scale l satisfies $\log^4 l \le O(|\log \varepsilon|)$. We assume that, blowing-up around p^ε, we see blown-up limit vortices b_k, and give the dynamical law of the b_k's. This is the result, where for simplicity of statement we assume there is a unique point p of accumulation of the vortices (a more general result is given further down in the paper, see Theorem 6). Observe it is valid for any number of vortices and any interaction (attractive or repulsive).

Theorem 4 (Exact dynamical law after blow-up). *Assume u_ε is a solution to (1.1) with (1.27) and (1.28). Assume $l = o(1)$ with $\log^4 l \le C|\log \varepsilon|$, and the points $p^\varepsilon \to p$ are such that, defining $\overline{u_\varepsilon}(x,t) = u_\varepsilon(p^\varepsilon + lx, l^2 t)$, we have*

$$\mathrm{curl}\,(i\overline{u_\varepsilon}, \nabla \overline{u_\varepsilon})(0) \rightharpoonup 2\pi \sum_{k=1}^{n} D_k \delta_{b_k^0} \quad \text{as } \varepsilon \to 0 \tag{1.40}$$

with $D_k = \pm 1$, and assume

$$E_\varepsilon(u_\varepsilon^0) \le \pi n |\log \varepsilon| + W_{\sum_{\mathbf{k}} \mathbf{D_k}}(p) - \pi \sum_{k \ne k'} D_k D_{k'} \log(l|b_k^0 - b_{k'}^0|) + n\gamma + r_\varepsilon \tag{1.41}$$

with either $r_\varepsilon \le o(1)$ or $r_\varepsilon \le \frac{l^2|\log \varepsilon|}{(\log |\log \varepsilon|)^\beta}$ for some $\beta > 1$. Then, there exist $H^1((0,T^*))$ trajectories $b_k(t)$ such that, for every $t \in [0, T^*)$,

$$\mathrm{curl}\,(i\overline{u_\varepsilon}, \nabla \overline{u_\varepsilon})(t) \rightharpoonup 2\pi \sum_{k=1}^{n} D_k \delta_{b_k(t)} \quad \text{as } \varepsilon \to 0$$

where b_k solves the dynamical law

$$\begin{cases} \dfrac{db_k}{dt} = -\dfrac{1}{\pi} \displaystyle\sum_{k' \ne k} D_{k'} D_k \frac{b_{k'} - b_k}{|b_{k'} - b_k|^2} \\ b_k(0) = b_k^0 \end{cases} \tag{1.42}$$

and T^* is the first collision-time under this law. Moreover, for every $t \in (0, T^*)$, we have

$$E_\varepsilon(u_\varepsilon(l^2 t)) \le \pi n |\log \varepsilon| + W_{\sum_{\mathbf{k}} \mathbf{D_k}}(p) - \pi \sum_{k,k'} D_k D_{k'} \log(l|b_k(t) - b_{k'}(t)|) + n\gamma + o(1), \tag{1.43}$$

as $\varepsilon \to 0$.

1.4 The dynamical law of the limiting vortices

Combining easily the results of the previous theorems, we can extend the dynamical law of the limiting vortices (1.8) passed collision times, provided there are only "simple" or "dual" collisions.

Definition 1. *In Proposition 1.1, we say the collision(s) at time T_* are simple if for every i, $Card\{j \ne i / \lim_{t \to T_*^-} |p_j(t) - p_i(t)| = 0\} = 1$ or 0 if $p_i(t) \to \partial\Omega$.*

We now state the dynamical law, assuming for simplicity that we are in the case of the Dirichlet boundary condition (which allows to rule out the case of vortices exiting Ω). The terminology follows that of Proposition 1.1, and the statement is meant to be applied iteratively to $k = 0, 1, 2, \cdots$

Theorem 5 (Global in time dynamical law). *Let u_ε solve (1.1) with Dirichlet boundary condition and be such that (1.28), (1.32) and (1.33) hold. Setting $T_0 = 0$, there exist collision times $0 < T_1 < T_2 < \cdots < T_k \cdots \le \infty$ such that if either $k = 0$ or the collisions at times*

$T_1, \cdots, T_k$ are simple then, denoting by p_i^k the distinct points in Ω and $D_i^k = \pm 1$ the integers such that

$$\mu(t) \rightharpoonup 2\pi \sum_{i=1}^{n_k} D_i^k \delta_{p_i^k} \quad \text{as } t \to T_k^-,$$

we have

$$\forall t \in [T_k, T_{k+1}) \qquad \operatorname{curl}(iu_\varepsilon, \nabla u_\varepsilon)(t) \rightharpoonup \mu(t) = 2\pi \sum_{i=1}^{n_k} D_i^k \delta_{p_i(t)} \quad \text{as } \varepsilon \to 0$$

where the $p_i(t)$ solve the initial value problem

$$\begin{cases} \dfrac{dp_i}{dt} = -\dfrac{1}{\pi} \nabla_i W_{\mathbf{D}}(p_1, \cdots, p_{n_k})(t) \\ p_i(T_k) = p_i^k, \end{cases} \tag{1.44}$$

and $T_{k+1} \le \infty$ is the first collision time under this law. Moreover, for every $t \in (T_k, T_{k+1})$

$$E_\varepsilon(u_\varepsilon(t)) = \pi n_k |\log \varepsilon| + W_{\mathbf{D}}(p_1(t), \cdots, p_{n_k}(t)) + o(1) \quad \text{as } \varepsilon \to 0. \tag{1.45}$$

Finally, $n_k \le n_{k-1} - 2$ hence the number of simple collisions is bounded by $n_0/2 = n/2$.

We may sum this theorem up by the following principle: *If u_ε is a solution of (1.1) such that (1.28), (1.32) and (1.33) hold, then, as long as there are only simple (and not multiple) collisions, the dynamical law of its vortices is given by (1.8), where, when two vortices collide, they should be erased from the list, and the law (1.8) should afterwards be understood as the law with the remaining vortices.*

Let us finally point out that in the course of the paper, we also prove general and sharp lower bounds for the Ginzburg-Landau energy in terms of vortices, see Section 4.2.

1.5 Perspectives

As we mentioned, one cannot rule out, even though they are not generic, the possibility of multiple collisions under the law (1.8), i.e. of more than two vortices meeting at the same time and place, with mutual distances of same order. One would first need to classify all the types of collisions that are possible under (1.8). Of particular difficulty is the case of collisions of a group of "balanced" vortices with $\sum_i d_i^2 = (\sum_i d_i)^2$, because this does not seem to dissipate any energy. This may be related to the conjecture of Ovchinnikov-Sigal [OS] of existence of nonradial solutions of Ginzburg-Landau in the whole plane, that is with several vortices satisfying $\sum_i d_i^2 = (\sum_i d_i)^2$. The other (i.e. unbalanced) collisions can be treated in the same way as here for dual collisions.

We have not written down every possible result that can be obtained through our method but rather we have tried to treat the most striking cases, and explain in the course of the paper how to generalize to other situations. Contrarily to [BOS1, BOS2], our study does not really allow to relax further the prepared assumption (1.33) into (1.27) nor to relax the hypothesis $D_i = \pm 1$, because under the only hypothesis $E_\varepsilon(u_\varepsilon^0) \le C|\log \varepsilon|$, the hypothesis (1.32) allows for substructures of vortices converging to each p_i^0. However, Theorems 4 and 6 give an example of how to deal with such cases (see Remark 5.3). Also, in very short time we must have $\|\frac{\partial_t u_\varepsilon}{|\log\,\varepsilon|}\|^2_{L^2} = \|f_\varepsilon\|^2_{L^2} \le \frac{C}{\varepsilon^\beta}$, $\beta < 2$, and then these substructures of vortices are well-defined (see Proposition 2.2 in Part I) and satisfy (1.31). The first difficulty here is in proving that while the small vortex structures form, the p_i^0 do not move (this should be done as in Theorem 3, 4 and 6), the second more delicate one is in understanding what happens to zeroes of degree $\neq \pm 1$ (we know that configurations with vortices of degree > 1 can be stationary even though not stable, on the other hand once we know that a vortex of degree > 1 has split into several vortices, then we can use our method like in Section 3.2). Then, these clusters of vortices should interact according to, typically, Theorem 4 or 6. The closest vortices, at distance l, should collide (or separate) first, in time $O(l^2)$, while the others do not move in that time-scale, then the closest vortices among those left should interact, etc, until, after a $o(1)$ time there should only be vortices at finite distances left, probably near each p_i^0 if $D_i = \pm 1$ — but not necessarily otherwise — and the configuration should become "very-well prepared" according to (1.31).

A delicate open problem would be to completely release the assumption (1.27) and thus the upper bound on the number of vortices.

Finally, it would be interesting to study the law (1.44) and see in particular if the following results hold: in the Dirichlet case, after a finite time (independent of ε), there are $d = \deg g \ge 0$ vortices of degree 1 left; in the Neumann case, after a finite time, there are no vortices left in Ω.

2 First applications to the energy dissipation

We start by presenting the most direct applications of the "static" results of Part I. They rely mainly on studying the energy-decay through a simple differential inequation. We always assume that u_ε solves (1.1) with Dirichlet or Neumann boundary condition, with $E_\varepsilon(u_\varepsilon^0) \le M|\log\,\varepsilon|$, $|u_\varepsilon^0| \le 1$ and $|\nabla u_\varepsilon^0| \le \frac{M}{\varepsilon}$. We recall that the existence and uniqueness of the solution of (1.1) is known, and that standard estimates prove that the above estimates on u_ε^0 remain true at later times, with constants independent of t. Thus the results of Part I, where the error terms only depend on these constants, can be applied, and yield errors independent of time.

2.1 A clearing-out lemma

We start with a first simple result, because it gives the model for the other proofs; it is a sort of clearing-out result (here we use this terminology borrowed from the literature — e.g. Ilmanen's paper on Allen-Cahn — in a loose sense meaning disappearance of all vortices and excess-energy), saying that if initially there is little energy (less than what is needed to create a vortex), then the solution is completely cleaned up in very small time. This may happen for instance with an initial dipole of vortices of degree ± 1 at distance $l \le \varepsilon^\gamma$, $\gamma > \frac{1}{2}$, initially, which can be constructed to have an energy $\le 2\pi \log \frac{l}{\varepsilon} \le 2\pi(1-\gamma)|\log\,\varepsilon|$. The result corresponds to the energy-decay of the phase-excess through the accelerated heat equation.

We recall the definition of W_0 was

$$W_0 = \int_\Omega |\nabla \Phi|^2 \tag{2.1}$$

where $\Phi = 0$ in the Neumann case, and Φ is a harmonic function with $\frac{\partial \Phi}{\partial \nu} = (ig, \frac{\partial g}{\partial \tau})$ on $\partial\Omega$ in the Dirichlet case.

Proposition 2.1 (Clearing-out lemma). *Let u_ε be a solution of (1.1) with Dirichlet or Neumann boundary condition, such that*

$$E_\varepsilon(u_\varepsilon^0) \le \eta|\log\,\varepsilon|$$

with $\eta < \pi$ (this is possible only if $\deg g = 0$ in the Dirichlet case). Then

1. *For any $\gamma < 2 - \eta/\pi$ in the Dirichlet case, resp. $\gamma < 2 - 2\eta/\pi$ in the Neumann case, there exists a time $T_1 \le \varepsilon^\gamma$, such that $\|1 - |u_\varepsilon(T_1)|\|_{L^\infty(\Omega)} = o(1)$.*

2. *There exists a time $T_2 \le C\frac{\log\;|\log\,\varepsilon|}{|\log\,\varepsilon|}$ such that $\forall t_\varepsilon \ge T_2$, $\|1 - |u_\varepsilon(t_\varepsilon)|\|_{L^\infty(\Omega)} = o(1)$, and*

$$E_\varepsilon(u_\varepsilon(t_\varepsilon)) \le W_0 + o(1).$$

Proof. First, recall that the energy decreases in time so we always have $E_\varepsilon(u_\varepsilon(t)) \le \eta|\log\,\varepsilon|$ for $t \ge 0$. Moreover, writing $f_\varepsilon = \frac{\partial_t u_\varepsilon}{|\log\,\varepsilon|}$, we have

$$|\log\,\varepsilon| \int_0^t \|f_\varepsilon\|^2_{L^2(\Omega)}\,dt = E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(t)) \le \eta|\log\,\varepsilon|.$$

Hence, by a mean-value argument, we deduce that, for $\gamma < 2$, there exists a time $T_1 \le \varepsilon^\gamma$ (T_1 depending on ε) such that

$$\|f_\varepsilon\|^2_{L^2} \le \eta\varepsilon^{-\gamma}. \tag{2.2}$$

At time T_1, Proposition 2.2 in [S1] applies, and yields vortices (a_i, d_i). Moreover, (1.30) holds, thus

$$\alpha\pi\sum_i d_i^2 \le \eta + C|\log\,\varepsilon|^{7/2}\varepsilon^{1-\alpha-\gamma/2} + o(1)$$

Taking $\alpha > \frac{\eta}{2\pi}$ and $\gamma < 2 - 2\alpha$, we find, that as ε gets small enough, $\sum_i d_i^2 < 2$, hence $\sum d_i^2 = 0$ or $= 1$. But the d_i's given by Proposition 2.2 in [S1] are all nonzero, hence we deduce that either the set of a_i's is empty, and then $\|1 - |u_\varepsilon(T_1)|\|_{L^\infty(\Omega)} = o(1)$; or there is only one a_i with degree $+1$ or -1. This implies that the total degree of u_ε in Ω is ± 1, which is impossible in the Dirichlet case. In the Neumann case, if there is such a vortex a_i, using Lemma 3.3 in [S1] and examining closely the form of $W_{\mathbf{d}}$, we can show that the energy is bounded from below by $\pi\log\frac{l}{\varepsilon} + O(1)$ where l is dist $(a_i, \partial\Omega)$. This contradicts $E_\varepsilon(u_\varepsilon) \le \eta|\log\,\varepsilon|$ unless dist $(a_i, \partial\Omega) \le \varepsilon^\mu$ for some $\mu \ge 1 - \eta/\pi$. But then, the second assertion of Theorem 2 in [S1] would give $\|f_\varepsilon\|^2_{L^2} \ge \frac{C}{|\log\,\varepsilon|^2\varepsilon^{2\mu}}$. When $\gamma < 2 - 2\eta/\pi < 2\mu$, this contradicts (2.2).

This proves that the only possible case was that the set of a_i's is empty at time T_1, and thus the desired property holds at time T_1.

Let us prove the second property. At any time $t \ge 0$, either $\|f_\varepsilon\|^2_{L^2(\Omega)} \ge \eta|\log\,\varepsilon|$ in which case $E_\varepsilon(u_\varepsilon(t)) \le \|f_\varepsilon\|^2_{L^2(\Omega)}$, or $\|f_\varepsilon\|^2_{L^2(\Omega)} \le \eta|\log\,\varepsilon|$. In the latter case, Proposition 2.2 in [S1] applies and gives vortices (a_i, d_i), and (1.30) yields, for every $\alpha < 1$,

$$\alpha\pi\sum_i d_i^2 \le \eta + o(1)$$

Since $\eta < \pi$, this implies that $\sum_i d_i^2 < 1$ if ε is small enough, hence (using again the fact that the d_i's are nonzero integers) the set of vortices a_i is empty. Applying (1.31) i.e. Theorem 1 of [S1] then yields

$$E_\varepsilon(u_\varepsilon(t)) \le W_0 + C\|f_\varepsilon\|^2_{L^2(\Omega)} + o(1),$$

where the constant and the $o(1)$ only depend on the apriori estimates on u_ε, hence not on t. Changing C if necessary, this means that in all cases, for every $t \ge 0$,

$$E_\varepsilon(u_\varepsilon(t)) \le W_0 + C\|f_\varepsilon\|^2_{L^2(\Omega)} + o(1). \tag{2.3}$$

On the other hand, we have $\frac{dE_\varepsilon}{dt} = -|\log\varepsilon|\|f_\varepsilon\|^2_{L^2(\Omega)}$, hence we may write

$$E_\varepsilon(u_\varepsilon(t)) \le W_0 + o(1) - \frac{C}{|\log\varepsilon|}\frac{dE_\varepsilon(u_\varepsilon(t))}{dt}. \tag{2.4}$$

Solving this ordinary differential inequality, we find

$$E_\varepsilon(u_\varepsilon(t)) \le W_0 + o(1) + \left(E_\varepsilon(u_\varepsilon^0) - W_0 + o(1)\right) e^{-t|\log\varepsilon|/C}. \tag{2.5}$$

Therefore, if $t \ge \frac{c\log|\log\varepsilon|}{|\log\varepsilon|}$ with c well-chosen, we have $e^{-t|\log\varepsilon|/C} \le |\log\varepsilon|^{-2}$ and thus from (2.5), using (1.27),

$$E_\varepsilon(u_\varepsilon(t)) \le W_0 + o(1).$$

On the other hand, it is not difficult to check that

$$\int_\Omega \frac{(1-|u_\varepsilon|^2)^2}{\varepsilon^2} \le C\left(E_\varepsilon(u_\varepsilon) - W_0\right)$$

hence $\int_\Omega \frac{(1-|u_\varepsilon|^2)^2}{\varepsilon^2} \le o(1)$, and since $|\nabla u_\varepsilon| \le \frac{C}{\varepsilon}$, this implies by standard arguments that $|u_\varepsilon| \ge 1 - o(1)$ at any time $t \ge \frac{c\log|\log\varepsilon|}{|\log\varepsilon|}$, hence the result. □

2.2 Proof of Theorem 2

In this subsection, we prove Theorem 2 which shows that, under some weaker assumptions, solutions become "very well-prepared" in short time.

We start with a lemma which will be used several times, and whose proof is very similar to that of Proposition 2.1. It asserts that, under a weak condition on the initial energy, solutions become very well-prepared in time $O(\frac{\log|\log\varepsilon|}{|\log\varepsilon|})$ if we know that their vortices do not move during that time.

Lemma 2.1 (Instantaneous very-well preparedness provided vortices do not move). *Let u_ε be a solution of (1.1) with Dirichlet or Neumann boundary condition, and (1.28). There exists a time $T_\varepsilon = M\frac{\log|\log\varepsilon|}{|\log\varepsilon|}$ such that, if*

$$\forall t_\varepsilon \in [0, T_\varepsilon], \qquad \mathrm{curl}\,(iu_\varepsilon, \nabla u_\varepsilon)(t_\varepsilon) \rightharpoonup 2\pi\sum_{i=1}^n D_i\delta_{p_i^0} \tag{2.6}$$

where the p_i^0's are distinct points in Ω and $D_i = \pm1$, and

$$E_\varepsilon(u_\varepsilon^0) \le \pi(n+\eta)|\log\varepsilon| \tag{2.7}$$

for some $\eta < 1$, then for every $t_\varepsilon \le T_\varepsilon$,

$$E_\varepsilon(u_\varepsilon(t_\varepsilon)) \le \pi n|\log\,\varepsilon| + W_{\mathbf{D}}(p_1^0,\cdots,p_n^0) + n\gamma + C|\log\,\varepsilon|e^{-t_\varepsilon|\log\,\varepsilon|/C} + o(1), \tag{2.8}$$

in particular, if M is large enough,

$$E_\varepsilon(u_\varepsilon(T_\varepsilon)) \le \pi n|\log\,\varepsilon| + W_{\mathbf{D}}(p_1^0,\cdots,p_n^0) + n\gamma + o(1). \tag{2.9}$$

Proof. The strategy is as in the previous proof. For each time, either $\|f_\varepsilon\|^2_{L^2(\Omega)} = \|\frac{\partial_t u_\varepsilon}{|\log\,\varepsilon|}\|^2_{L^2(\Omega)} \gg |\log\,\varepsilon|$, in which case we automatically have

$$E_\varepsilon(u_\varepsilon(t)) \le \pi n|\log\,\varepsilon| + W_{\mathbf{D}}(p_1^0,\cdots,p_n^0) + n\gamma + C\|f_\varepsilon\|^2_{L^2(\Omega)} + o(1), \tag{2.10}$$

or we have $\|f_\varepsilon\|^2_{L^2(\Omega)} \le O(|\log\,\varepsilon|)$. In that second case, Proposition 2.2 in [S1] applies, gives vortices (a_i, d_i), and we may apply (1.30). Combining it with the bound on the energy (2.7), valid for all times, we find, for every $\alpha < 1$,

$$\alpha\pi\sum_i d_i^2 \le \pi(n+\eta) + o(1).$$

Taking α large enough, and using the fact that the d_i's are integers, we find

$$\sum_i d_i^2 \le n.$$

Therefore, the number of points a_i's is bounded by n, with equality if and only if there are n points with $d_i = \pm 1$ for each i. On the other hand, for every $t \in [0, T_\varepsilon]$, we have (2.6), which implies that there exists at least one a_i converging to each p_k^0. Combining this with the above, there can only be one a_i converging to each p_i^0, with degree $d_i = D_i = \pm 1$. But Theorem 1 of [S1] applies at that time, thus from (1.31), we have

$$E_\varepsilon(u_\varepsilon(t)) \le \pi n|\log\,\varepsilon| + W_{\mathbf{D}}(a_1,\cdots,a_n) + n\gamma + C\|f_\varepsilon\|^2_{L^2(\Omega)} + o(1).$$

Combining this with the above convergence of the a_i's, we find that (2.10) holds in this case as well. So for every $t \in [0, T_\varepsilon]$, we have

$$E_\varepsilon(u_\varepsilon(t)) \le \pi n|\log\,\varepsilon| + W_{\mathbf{D}}(p_1^0,\cdots,p_n^0) + n\gamma - \frac{C}{|\log\,\varepsilon|}\frac{dE_\varepsilon(u_\varepsilon(t))}{dt} + o(1). \tag{2.11}$$

Solving this differential inequality as in (2.5), we find

$$\begin{aligned} E_\varepsilon(u_\varepsilon(t)) &\le \pi n|\log\,\varepsilon| + W_{\mathbf{D}}(p_1^0,\cdots,p_n^0) + n\gamma \\ &\quad + e^{-t|\log\,\varepsilon|/C}\left(E_\varepsilon(u_\varepsilon(0)) - \pi n|\log\,\varepsilon| - W_{\mathbf{D}}(p_1^0,\cdots,p_n^0) - n\gamma + o(1)\right) + o(1) \\ &\le \pi n|\log\,\varepsilon| + W_{\mathbf{D}}(p_1^0,\cdots,p_n^0) + n\gamma + C|\log\,\varepsilon|e^{-t|\log\,\varepsilon|/C} + o(1). \end{aligned} \tag{2.12}$$

We see that choosing $T_\varepsilon = \frac{C\log\,|\log\,\varepsilon|}{|\log\,\varepsilon|}$ with C large enough, we get (2.9). □

In order to prove Theorem 2, there remains to prove that (2.6) holds, i.e. that the vortices do not move in time T_ε. This will follow from a suitable application of the product estimate Theorem 1 (see also (1.23)).

We now assume the hypotheses of Theorem 2 are satisfied. By standard lower bounds (for example the ones that will be proved below in Proposition 4.3 (4.6)), there exists a constant K (depending on the p_i^0's) such that for any u_ε such that (1.32) holds with $D_i = \pm 1$, we have

$$E_\varepsilon(u_\varepsilon) \geq \pi n |\log \varepsilon| - K. \tag{2.13}$$

Let us now assume by contradiction that there exists $T_\varepsilon \leq O\left(\frac{\log\ |\log\ \varepsilon|}{|\log\ \varepsilon|}\right)$ such that

$$|\log \varepsilon| \int_0^{T_\varepsilon} \|f_\varepsilon\|^2_{L^2(\Omega)}(t)\, dt = \frac{1}{|\log\ \varepsilon|} \int_{\Omega\times[0,T_\varepsilon]} |\partial_t u_\varepsilon|^2 = E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(T_\varepsilon)) = \frac{|\log\ \varepsilon|}{\log\ |\log\ \varepsilon|^\beta} + K + 1.$$

Thus,

$$E_\varepsilon(u_\varepsilon(T_\varepsilon)) = E_\varepsilon(u_\varepsilon(0)) - \frac{|\log\ \varepsilon|}{\log\ |\log\ \varepsilon|^\beta} - K - 1 \leq \pi n |\log\ \varepsilon| - K - 1. \tag{2.14}$$

Rescaling in time, and considering $w_\varepsilon(x,t) = u_\varepsilon(x, T_\varepsilon t)$, we have

$$\frac{1}{T_\varepsilon |\log\ \varepsilon|} \int_{\Omega\times[0,1]} |\partial_t w_\varepsilon|^2 = \frac{|\log\ \varepsilon|}{\log\ |\log\ \varepsilon|^\beta} + K + 1. \tag{2.15}$$

Applying Theorem 1, we find that for every test-function f compactly supported in $[0,1]$ such that $|f| \leq 1$, and every test vector field X compactly supported in $\Omega \times [0,1]$, we have

$$\begin{aligned} \left| \int_{\Omega\times[0,1]} V \cdot fX \right|^2 &\leq \varliminf_{\varepsilon\to 0} \frac{1}{|\log\ \varepsilon|^2} \left(\int_{\Omega\times[0,1]} |X\cdot \nabla w_\varepsilon|^2 \int_{\Omega\times[0,1]} f^2 |\partial_t w_\varepsilon|^2 \right) \\ &\leq \varliminf_{\varepsilon\to 0} \left(C T_\varepsilon \frac{|\log\ \varepsilon|}{\log\ |\log\ \varepsilon|^\beta} \right) = 0 \end{aligned} \tag{2.16}$$

where V is the limiting velocity associated to the vortices of w_ε. Here, we have used the upper bound on the energy, giving $\int_\Omega |\nabla w_\varepsilon|^2 \leq C|\log\ \varepsilon|$, and (2.15). But $T_\varepsilon \frac{|\log\ \varepsilon|}{(\log\ |\log\ \varepsilon|)^\beta} \leq o(1)$ because $\beta > 1$ and $T_\varepsilon \leq \frac{\log\ |\log\ \varepsilon|}{|\log\ \varepsilon|}$, hence we deduce $V = 0$, or in other words $\mathrm{curl}\,(iw_\varepsilon, \nabla w_\varepsilon)(t) \rightharpoonup 2\pi \sum_i D_i \delta_{p_i^0}$ for every $t \in [0,1]$. This means that the vortices of u_ε do not move in $[0, T_\varepsilon]$, hence we must have $E_\varepsilon(u_\varepsilon(T_\varepsilon)) \geq \pi n |\log\ \varepsilon| - K$, a contradiction with (2.14). This implies that for every $T_\varepsilon \leq O\left(\frac{\log\ |\log\ \varepsilon|}{|\log\ \varepsilon|}\right)$, we have

$$\begin{aligned} |\log\ \varepsilon| \int_0^{T_\varepsilon} \|f_\varepsilon\|^2_{L^2(\Omega)}(t)\, dt &= \frac{1}{|\log\ \varepsilon|} \int_{\Omega\times[0,T_\varepsilon]} |\partial_t u_\varepsilon|^2 \\ &= E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(T_\varepsilon)) < \frac{|\log\ \varepsilon|}{\log\ |\log\ \varepsilon|^\beta} + K + 1. \end{aligned}$$

Arguing as above, we deduce that $\mathrm{curl}\,(iw_\varepsilon, \nabla w_\varepsilon)(t) \rightharpoonup 2\pi \sum_i D_i \delta_{p_i^0}$ for every $t \in [0,1]$, thus, after rescaling, that

$$\mathrm{curl}\,(iu_\varepsilon, \nabla u_\varepsilon)(t_\varepsilon) \rightharpoonup 2\pi \sum_i D_i \delta_{p_i^0}$$

for every $t_\varepsilon \leq O\left(\frac{\log\ |\log\ \varepsilon|}{|\log\ \varepsilon|}\right)$. Then, Lemma 2.1 applies, and proves Theorem 2.

3 Energy clearing-out during collisions

In this section, we examine how the energy-excess dissipates rapidly during collisions or separation of vortices. Starting with collisions, for simplicity, we consider the generic case of n isolated vortices of degree ± 1, plus a dipole of two vortices of opposite degree ± 1 colliding. We may also assume that this configuration is inherited from a previous evolution and thus that the configuration is "well-prepared" with respect to these vortices, i.e.

$$E_\varepsilon(u_\varepsilon(0)) \leq \pi n |\log\,\varepsilon| + 2\pi \log \frac{l(0)}{\varepsilon} + O(1) \tag{3.1}$$

where $l(0)$ is the initial (small) distance between the two vortices of the dipole.

In a next section, we will show the exact dynamical law of such vortices, Theorem 4, valid as long as $l \geq \frac{1}{|\log\,\varepsilon|^2}$ for example. So we may restrict to the situation where $l(0) \leq \frac{1}{|\log\,\varepsilon|^2}$.

3.1 Motion of the energy-concentration points

We first wish to show that the collision of the two vortices, even though they carry excess-energy which dissipates, does not trigger any motion of the other vortices. This requires examining the evolution of the energy-density space-repartition. This is the only point where the method is not purely energetic, and uses the equation (1.1).

We denote by

$$e_\varepsilon(u) = \frac{1}{2}|\nabla u|^2 + \frac{(1-|u|^2)^2}{4\varepsilon^2}$$

the energy-density. It is a standard result (see for example [Li, JS, BOS1]) that

Lemma 3.1. *Let u_ε be a solution of (1.1) and χ be a C^2 function in Ω, constant in a neighborhood of $\partial\Omega$. Then,*

$$\frac{d}{dt}\int_\Omega \chi e_\varepsilon(u_\varepsilon(t)) = -\int_\Omega \chi \frac{|\partial_t u_\varepsilon|^2}{|\log\,\varepsilon|} - |\log\,\varepsilon| \int_\Omega \sum_{i,j=1,2} (\partial_i\partial_j\chi) T_{ij} \tag{3.2}$$

where T_{ij} denotes the "stress-energy tensor" of coefficients $T_{ij} = e_\varepsilon(u)\delta_{ij} - (\partial_i u, \partial_j u)$ as in [S1].

Proof. A direct calculation yields

$$\partial_t e_\varepsilon(u(x,t)) = \text{div }\, (\partial_t u, \nabla u) - \left(\partial_t u, \left(\Delta u + \frac{u}{\varepsilon^2}(1-|u|^2)\right)\right) = \text{div }\, (\partial_t u, \nabla u) - \frac{|\partial_t u|^2}{|\log\,\varepsilon|}$$

using (1.1). On the other hand, as seen in [S1], eq. (2.3), with another direct computation, we have

$$\sum_i \partial_i T_{ij} = -\left(\partial_j u, \Delta u + \frac{u}{\varepsilon^2}(1-|u|^2)\right) = -\left(\partial_j u, \frac{\partial_t u}{|\log\,\varepsilon|}\right).$$

We also observe that $\left(\partial_t, u\frac{\partial u}{\partial \nu}\right) = 0$ on $\partial\Omega$ in view of the boundary conditions (Dirichlet or Neumann). Combining these relations, and using several integrations by parts, we are led to (3.2). □

We deduce the following lemma, which states that if the energy of a solution concentrates at initial time only at a finite number of isolated points $p_1, \cdots, p_n$, then these points of concentration of energy do not move in time $\leq \left(\frac{1}{|\log\,\varepsilon|}\right)$.

Let $x_1^\varepsilon, \cdots, x_n^\varepsilon$ be points such that there exists $\rho > 0$ independent of ε such that $\min_{i\neq j} |x_i - x_j| > 4\rho$ and $\min_i \text{dist}\,(x_i, \partial\Omega) > 4\rho$. Let us construct a function χ such that

$$\begin{cases} \chi \equiv 1 & \text{in } \Omega\backslash \cup_i B(x_i, 2\rho) \\ \chi = |x - x_i|^2 & \text{in } B(x_i, \rho) \\ \chi \geq \rho^2 & \text{in } \Omega\backslash \cup_i B(x_i, \rho) \\ \chi \in C^2(\Omega) \end{cases} \tag{3.3}$$

Lemma 3.2. *Let u_ε be a solution of (1.1), and let $x_1^\varepsilon, \cdots, x_n^\varepsilon$ and χ be as above, then for any $t \geq 0$,*

$$\int_\Omega \chi e_\varepsilon(u_\varepsilon(t)) \leq e^{ct|\log\,\varepsilon|} \int_\Omega \chi e_\varepsilon(u_\varepsilon(0)), \tag{3.4}$$

where the constant c depends only on ρ.

Proof. We apply Lemma 3.1 with this $\chi \geq 0$. First, we use the property of $|x|^2$ with respect to (3.2), as observed by De Giorgi and used in [So, RS] among others: observing that $\partial_i\partial_j|x - x_0|^2 = 2\delta_{ij}$, we find that in $B(x_k, \rho)$,

$$\sum_{i,j} (\partial_i\partial_j|x - x_k|^2)T_{ij} = 2(T_{11} + T_{22}) = \frac{(1 - |u|^2)^2}{\varepsilon^2} \geq 0$$

Therefore, the contributions in $\cup_i B(x_k, \rho)$ of the right-hand side of (3.2) are nonpositive, and we can write

$$\partial_t \int_\Omega \chi e_\varepsilon(u, t) \leq |\log\,\varepsilon| \int_{\cup_k (B(x_k, 2\rho)\backslash B(x_k, \rho))} \sum_{i,j} \partial_i\partial_j\chi T_{ij}.$$

Observing that $D^2\chi$ is bounded, and $\chi \geq \rho^2$ in $\cup_k (B(x_k, 2\rho)\backslash B(x_k, \rho))$, we may write $|D^2\chi| \leq C_\rho\chi$ where the constant depends on ρ. Using in addition the observation that pointwise, $|T_{ij}| \leq e_\varepsilon(u)$, we are led (changing C_ρ if necessary) to

$$\partial_t \int_\Omega \chi e_\varepsilon(u(t)) \leq |\log\,\varepsilon| \int_{\cup_k (B(x_k, 2\rho)\backslash B(x_k, \rho))} C_\rho \chi e_\varepsilon(u(t)) \leq C_\rho |\log\,\varepsilon| \int_\Omega \chi e_\varepsilon(u(t)).$$

We deduce by Gronwall's lemma that (3.4) holds. □

This allows to deduce

Proposition 3.1. *Let u_ε be a solution of (1.1) such that*

$$\text{curl}\,(iu_\varepsilon^0, \nabla u_\varepsilon^0) \rightharpoonup 2\pi \sum_{i=1}^n D_i \delta_{p_i^0} \tag{3.5}$$

with $D_i = \pm 1$. Assume that there exists $p_\varepsilon \to p_{dip}$ as $\varepsilon \to 0$, with p_{dip} distinct from $\{p_1^0, \cdots, p_n^0\}$ and $l(0) \leq \frac{1}{|\log\,\varepsilon|}$, such that, considering $\overline{u_\varepsilon}(x) = u_\varepsilon^0(p_\varepsilon + l(0)x)$, we have

$$\text{curl}\,(i\overline{u_\varepsilon}, \nabla\overline{u_\varepsilon}) \rightharpoonup 2\pi \left(\delta_{b_+} - \delta_{b_-}\right) \tag{3.6}$$

where b_+ and b_- are two points in $\mathbb{R}^2$ at distance 1 from each other. Assume also that

$$E_\varepsilon(u_\varepsilon^0) \leq \pi n |\log \varepsilon| + 2\pi \log \frac{l(0)}{\varepsilon} + C. \tag{3.7}$$

Then, if $T_\varepsilon = \eta \frac{\log |\log \varepsilon|}{|\log \varepsilon|}$ with η a small enough constant, we have

$$\forall t_\varepsilon \in [0, T_\varepsilon], \quad \mathrm{curl}\,(iu_\varepsilon, \nabla u_\varepsilon)(t_\varepsilon) \rightharpoonup 2\pi \sum_{i=1}^n D_i \delta_{p_i^0} \tag{3.8}$$

and

$$E_\varepsilon(u_\varepsilon(T_\varepsilon)) \leq \pi n |\log \varepsilon| + W_{\mathbf{D}}(p_1^0, \cdots, p_n^0) + n\gamma + o(1). \tag{3.9}$$

We can observe right away that this proposition says that in time $O\left(\frac{\log |\log \varepsilon|}{|\log \varepsilon|}\right)$, the solution above becomes "very-well prepared" with respect to its vortices $p_1^0, \cdots, p_n^0$, thus the dipole and its energy have completely disappeared in that short time, without affecting the other vortices.

Proof. We start by applying the lower bounds obtained through the ball-construction method of Jerrard/Sandier, see for example [SS4], main theorem of Chapter 3. Before we apply the result, we consider a constant $\rho > 0$ small enough such that $\Omega_1 := \cup_i B(p_i^0, \rho)$ and $\Omega_2 := B(p_{dip}, \rho)$ are disjoint. We then apply the main theorem of Chapter 3 of [SS4] in Ω_1 and Ω_2 successively. In Ω_1, we apply it with a final radius $\frac{1}{|\log \varepsilon|}$. It yields the existence of a finite collection $\mathcal{B}$ of disjoint closed balls which cover all the zeroes of u_ε^0 in Ω_1, such that the sum of their radii is smaller than $\frac{1}{|\log \varepsilon|}$ and for every $B \in \mathcal{B}$,

$$\int_B e_\varepsilon(u_\varepsilon^0) \geq \pi |d_B| \left(|\log \varepsilon| - C \log |\log \varepsilon|\right)$$

where $d_B = \deg(u_\varepsilon^0, \partial B)$ if $B \subset \Omega_1$, and 0 otherwise. In view of the hypotheses (3.5) and (3.6), u_ε^0 has at least one zero of nonzero degree converging to each p_i^0. Since the $B \in \mathcal{B}$ cover these zeroes, we can deduce that for each p_i^0, there exists a ball $B \in \mathcal{B}$ whose center converges to p_i^0, and such that $|d_B| \neq 0$, hence $|d_B| \geq 1$. Let us call it B_i and denote x_i its center. We have

$$\int_{B_i} e_\varepsilon(u_\varepsilon^0) \geq \pi |\log \varepsilon| - C \log |\log \varepsilon|. \tag{3.10}$$

Similarly, we apply the method in $\Omega_2 = B(p_{dip}, \rho)$ with final radius $\frac{l(0)}{|\log \varepsilon|}$. Since u_ε^0 has at least two zeroes of nonzero degree converging to p_{dip}, since the radii are $\leq \frac{l(0)}{|\log \varepsilon|} \ll l(0)$, there exists at least two balls with nonzero degree, at distance $\leq l(0)$ from each other, converging to p_{dip} as $\varepsilon \to 0$. They can be included in a larger ball B_{dip} of radius $\leq 2l(0)$, centered at x_{dip}, and such that

$$\int_{B_{dip}} e_\varepsilon(u_\varepsilon^0) \geq 2\pi \log \frac{l(0)}{\varepsilon} - C \log |\log \varepsilon|. \tag{3.11}$$

We keep this set of balls and discard the others. Combining (3.10), (3.11) and (3.7), we find that

$$\int_{\Omega\setminus\left(\cup_{i=1}^n B_i\cup B_{dip}\right)} e_\varepsilon(u_\varepsilon^0)\le C\log\ |\log\varepsilon|, \tag{3.12}$$

and

$$\int_{\Omega_2} e_\varepsilon(u_\varepsilon^0)\le 2\pi\log\frac{l(0)}{\varepsilon}+C\log\ |\log\varepsilon|. \tag{3.13}$$

Moreover, since the radii are bounded by $\max(\frac{l(0)}{|\log\ \varepsilon|},\frac{1}{|\log\ \varepsilon|})\le\frac{1}{|\log\ \varepsilon|}$, we have

$$\int_{B_i}|x-x_i|^2 e_\varepsilon(u_\varepsilon^0)\le\frac{1}{|\log\ \varepsilon|^2}E_\varepsilon(u_\varepsilon^0)\le o(1),$$

and similarly $\int_{B_{dip}}|x-x_{dip}|^2 e_\varepsilon(u_\varepsilon^0)\le o(1)$. Constructing χ associated to the points $x_1,\cdots,x_n,x_{dip}$, as in (3.3), we deduce from this and (3.12) that

$$\int_\Omega \chi e_\varepsilon(u_\varepsilon^0)\le C\log\ |\log\ \varepsilon|.$$

Applying Lemma 3.2, we deduce that for any $t\ge 0$,

$$\int_\Omega \chi e_\varepsilon(u_\varepsilon(t))\le Ce^{ct|\log\ \varepsilon|}\log\ |\log\ \varepsilon|.$$

If $t_\varepsilon\le T_\varepsilon=\frac{\eta\log\ |\log\ \varepsilon|}{|\log\ \varepsilon|}$, with $\eta<\frac{1}{2c}$, we find

$$\int_\Omega \chi e_\varepsilon(u_\varepsilon(t_\varepsilon))\le C|\log\ \varepsilon|^{\frac12}\log\ |\log\ \varepsilon|. \tag{3.14}$$

This suffices to ensure that (3.5) holds. Indeed, if not then, by continuity of the zeroes of u_ε in time, this would imply that for some $t_\varepsilon\le T_\varepsilon$, u_ε has a cluster of zeroes of nonzero total degree, at a distance bounded below away from the x_i's by a constant independent of ε. By the same argument we used above (using lower bounds given by the ball construction), we would get a lower bound contradicting (3.14). Thus (3.5) holds. We shall prove (3.9) after the next proposition. □

By using the same type of arguments as for Proposition 2.1 and Lemma 2.1, i.e. a differential inequality, combined with Theorem 2 of [S1], we now deduce an upper bound on the time of collision of the vortices, characterized by the fact that $|u_\varepsilon|\ge\frac12$ in a neighborhood of the collision point. The fact that u_ε^0 has a dipole at distance $l(0)$ will only be characterized through the hypothesis on the energy.

Proposition 3.2 (Upper bound on the collision-time). *Under the same hypotheses as Proposition 3.1, there exists a time*

$$T'_\varepsilon\le Cl(0)^2+C|\log\ \varepsilon|^4 e^{-2\sqrt{|\log\ \varepsilon|}}$$

such that $u_\varepsilon(T'_\varepsilon)$ has exactly n zeroes (given by Proposition 2.2 in [S1]) of degree D_i.

Proof. Let T_ε be given by Proposition 3.1 and S_ε denote the set of times $\le T_\varepsilon$ at which $\|f_\varepsilon\|^2_{L^2(\Omega)} = \left\|\frac{\partial_t u_\varepsilon}{|\log\,\varepsilon|}\right\|^2_{L^2(\Omega)} \ge \frac{1}{\varepsilon^\beta}$, for some $\beta < 2$. Observe that since

$$|\log\,\varepsilon| \int_0^t \|f_\varepsilon\|^2_{L^2} = E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(t)) \le C|\log\,\varepsilon|, \tag{3.15}$$

we have $meas(S_\varepsilon) \le C\varepsilon^\beta$.

When $t \notin S_\varepsilon$, we have $\|f_\varepsilon\|^2_{L^2} \le \frac{1}{\varepsilon^\beta}$ thus Proposition 2.2 in [S1] applies, yielding vortices (a_i, d_i) for which (1.30) holds, hence

$$\alpha\pi \sum_i d_i^2 \le (n+2)\pi + C|\log\,\varepsilon|^{7/2}\varepsilon^{1-\alpha-\beta/2},$$

and we may choose $\frac{n+2}{n+3} < \alpha < 1$ and $\beta < 2 - 2\alpha$, and thus get

$$\sum_i d_i^2 < n + 3 + o(1).$$

This gives an upper bound on the possible number of zeroes of u_ε : there are fewer than $n+2$. Since there is at least one zero converging to each p_i^0, this means that there are at most 2 extra vortices. Moreover, comparing (3.10) with relation (2.33) in [S1], we have $\sum_{i/a_i \notin \Omega_1} d_i^2 \le o(1)$ hence all extra vortices are at a distance bounded below from the p_i^0's. If there are 0 extra vortices, then what we want is satisfied. If there were only one extra vortex, then, since it would have nonzero degree, (3.5) would be contradicted. We are thus left with the case of 2 vortices, far away from the p_i^0. Therefore, the sum of their degrees must be 0, otherwise they would add an extra contribution in (3.5). We may denote by $l(t)$ their distance, and using lower bounds of Lemma 3.3 in [S1] or arguing as in the proof of Proposition 3.1, (using the lower bounds of [SS4] but with final radii 1 in Ω_1 and $\frac{l(t)}{2}$ in Ω_2), we have

$$E_\varepsilon(u_\varepsilon(t)) \ge \pi n|\log\,\varepsilon| + 2\pi \log \frac{l(t)}{\varepsilon} - C. \tag{3.16}$$

Comparing with (3.7) we must have $\log l(t) \le \log l(0) + C$ hence $l(t) \le Cl(0)$, and thus $l(t) = o(1)$. Therefore, these two vortices form an unbalanced cluster of vortices at scale $l(t)$. If $l(t) \gg \varepsilon\sqrt{|\log\,\varepsilon|}$, then Theorem 2 of [S1] applies and implies that

$$\|f_\varepsilon\|^2_{L^2(\Omega)} \ge \min\left(\frac{C}{l^2(t)|\log\,\varepsilon|}, \frac{C}{l^2(t)\log^2\frac{1}{l(t)}}\right). \tag{3.17}$$

If $l(t) \le O(\varepsilon\sqrt{|\log\,\varepsilon|})$ then the two vortices also form an unbalanced cluster at scale $\varepsilon|\log\,\varepsilon|$ and me may also apply Theorem 2 of [S1] and write that $\|f_\varepsilon\|^2_{L^2(\Omega)} \ge \frac{1}{\varepsilon^2|\log\,\varepsilon|^4}$.

Since we always have $l(t) \ge \varepsilon$, we may always write, for $t \notin S_\varepsilon$,

$$\|f_\varepsilon\|^2_{L^2(\Omega)} \ge \frac{C}{l^2(t)|\log\,\varepsilon|^4}. \tag{3.18}$$

To summarize, in all generality, we can write (3.18), and if $l(t) \ge e^{-\sqrt{|\log\,\varepsilon|}}$, we can write $\|f_\varepsilon\|^2_{L^2(\Omega)} \ge \frac{C}{l^2(t)|\log\,\varepsilon|}$. Consider S'_ε the set of times for which $l(t) \le e^{-\sqrt{|\log\,\varepsilon|}}$, since (3.15) and (3.18) hold, we have

$$|S'_\varepsilon| \le C|\log\,\varepsilon|^4 e^{-2\sqrt{|\log\,\varepsilon|}}.$$

Me way now map $\mathbb{R}_+ \backslash (S_\varepsilon \cup S'_\varepsilon)$ to $\mathbb{R}_+$ by a mapping R_ε which takes out the times in $S_\varepsilon \cup S'_\varepsilon$ and translates otherwise, thus which shifts every time by at most $|S_\varepsilon| + |S'_\varepsilon| \leq C|\log\,\varepsilon|^4 e^{-2\sqrt{|\log\,\varepsilon|}}$. Considering $F(t) = E_\varepsilon(u_\varepsilon(R_\varepsilon^{-1}(t)))$ and $L(t) = l(R_\varepsilon^{-1}(t))$, we find from (3.15) and (3.17) that

$$F(0) - F(t) \geq \int_0^t \frac{C}{L^2(t)}\,dt.$$

On the other hand $F(0) - F(t) = E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(R_\varepsilon^{-1}(t))) \leq 2\pi \left(\log \frac{l(0)}{\varepsilon} - \log \frac{L(t)}{\varepsilon}\right) - C$ from (3.16) and (3.7). Denoting $M(t) = \int_0^t \frac{C}{L^2(t)}\,dt$, we have

$$2\pi \log \frac{l(0)}{L(t)} - C \geq M(t) \tag{3.19}$$

but since $M'(t) = \frac{C}{L^2(t)}$, we may write

$$\pi \log \frac{l^2(0)}{C} + \pi \log M'(t) \geq M(t) \tag{3.20}$$

which transforms into

$$e^{M(t)/\pi} \leq \frac{l^2(0)}{C} M'(t).$$

Integrating, we find

$$e^{-M(t)/\pi} \leq 1 - \frac{t}{Cl^2(0)}, \tag{3.21}$$

for some constant C. If $t \geq Cl^2(0)$, we find $e^{-M(t)/\pi} \leq 0$ hence a contradiction. We deduce that the set of times for which we cannot say that u_ε has exactly n zeroes has a measure less than $T'_\varepsilon = |S_\varepsilon| + |S'_\varepsilon| + Cl^2(0) \leq Cl^2(0) + C|\log\,\varepsilon|^4 e^{-2\sqrt{|\log\,\varepsilon|}}$, which implies the result.

□

Proof of (3.9). From (3.15), the set of times for which $\|f_\varepsilon\|_{L^2(\Omega)} \geq \eta|\log\,\varepsilon|$ is $O(\frac{1}{|\log\,\varepsilon|})$. Hence, with the previous Proposition 3.2, the set of times such that either $\|f_\varepsilon\|_{L^2(\Omega)} \geq \eta|\log\,\varepsilon|$ or u_ε does not have exactly n zeroes of degree D_i has a measure less than $Cl^2(0) + C|\log\,\varepsilon|^4 e^{-2\sqrt{|\log\,\varepsilon|}} + \frac{C}{|\log\,\varepsilon|}$. We deduce that there exists a time $T''_\varepsilon \leq Cl^2(0) + C|\log\,\varepsilon|^4 e^{-2\sqrt{|\log\,\varepsilon|}} + \frac{C}{|\log\,\varepsilon|}$ for which u_ε has exactly n zeroes and $\|f_\varepsilon\|^2_{L^2(\Omega)} \leq \eta|\log\,\varepsilon|$. In view of (3.5) the zeroes of $u_\varepsilon(T''_\varepsilon)$ converge to the p_i^0, and thus we may write with (1.31)

$$E_\varepsilon(u_\varepsilon(T''_\varepsilon)) \leq \pi n|\log\,\varepsilon| + W_{\mathbf{D}}(p_1^0, \cdots, p_n^0) + n\gamma + C\eta|\log\,\varepsilon| + o(1).$$

Choosing η small enough so that $C\eta < \pi$, we find that Lemma 2.1 applies, and thus after a time $\leq T''_\varepsilon + O(\frac{\log\,|\log\,\varepsilon|}{|\log\,\varepsilon|})$, (3.9) holds.

A second possible proof is the following: We claim that, for every $t \leq T_\varepsilon$, we have

$$E_\varepsilon(u_\varepsilon(t)) \leq \pi n|\log\,\varepsilon| + W_{\mathbf{D}}(p_1^0, \cdots, p_n^0) + C\|f_\varepsilon\|^2_{L^2(\Omega)} + o(1). \tag{3.22}$$

If $\|f_\varepsilon\|^2_{L^2} \gg |\log\,\varepsilon|$ then it is trivially true. If not, then $\|f_\varepsilon\|^2_{L^2} \leq O(|\log\,\varepsilon|)$. Returning to the proof of Proposition 3.2, we find that in that case, either $u_\varepsilon(t)$ has exactly n zeroes of degrees

D_i, in which case (3.22) follows from (1.31), or $u_\varepsilon(t)$ has two extra vortices, at distance $l(t)$, with $l(t) \le Cl(0) \ll \frac{1}{|\log \varepsilon|}$, plugging into (3.17) we find $\|f_\varepsilon\|^2_{L^2(\Omega)} \gg |\log \varepsilon|$, a contradiction. Hence, in all cases, we have that (3.22) holds. We may finish as in Lemma 2.1, find that (2.8) holds, from which (3.9) follows. □

To prove Theorem 3, there only remains to prove the lower bounds on the collision time T_1, which will be done in Lemma 5.1, and to see what happens when $l(0) \ge \frac{1}{|\log \varepsilon|^2}$, which will be done in Theorem 6 (see the note after Theorem 6).

3.2 Time of separation of two vortices

Let us see another example of application, this time for the separation of two vortices of degree +1. We consider the simplest case where there are initially two vortices of degree +1 at small distance $l(0)$ from each other, and we assume that initially $E_\varepsilon(u^0_\varepsilon) \le 2\pi \log \frac{1}{l(0)\varepsilon} + C$. The case where there are other well-separated vortices in the sample can be treated as well, as in Theorem 3.

Proposition 3.3. *Let u_ε be a solution of (1.1) with Dirichlet boundary condition of degree 2. Let us assume that at time 0*

$$E_\varepsilon(u^0_\varepsilon) \le 2\pi \log \frac{1}{l(0)\varepsilon} + C, \tag{3.23}$$

with $l_0 > l(0) \ge \varepsilon^\beta$, $\beta < 1$. Then, for every $l \ge 2l(0)$, there exists a time

$$T_\varepsilon \le l^2 \log \frac{l}{l(0)} + |\log \varepsilon|^2 e^{-2\sqrt{|\log \varepsilon|}}$$

for which u_ε has two vortices of degree +1, at distance $\ge l$. If in addition, there exists a point p_ε such that, considering $\overline{u_\varepsilon}(x) = u^0_\varepsilon(p_\varepsilon + l(0)x)$, we have

$$\operatorname{curl}(i\overline{u_\varepsilon}, \nabla \overline{u_\varepsilon}) \rightharpoonup 2\pi(\delta_{b_1} + \delta_{b_2}) \quad \text{as } \varepsilon \to 0, \tag{3.24}$$

where b_1 and b_2 are two points in $\mathbb{R}^2$ at distance 1 from each other; then we must have $T_\varepsilon \ge Cl(0)^2$.

Proof. Let us argue as before, and let S_ε be the set of times for which $\|f_\varepsilon\|^2_{L^2} \ge \frac{1}{\varepsilon^\gamma}$, for some $\gamma < 2$. As previously $|S_\varepsilon| \le \varepsilon^\gamma$. On the other hand, for $t \notin S_\varepsilon$, we have $\|f_\varepsilon\|^2_{L^2} \le \varepsilon^{-\gamma}$, hence Proposition 2.2 in [S1] applies and yields vortices (a_i, d_i), with

$$\alpha\pi \sum_i d_i^2 \le 2\pi(1+\beta) + C|\log \varepsilon|^{7/2} \varepsilon^{1-\alpha-\gamma/2}$$

We may choose γ and α such that $\frac{2+2\beta}{4} < \alpha < 1 - \gamma/2$, and thus get

$$\sum_i d_i^2 < 4$$

for ε small enough. Knowing that $\sum_i d_i = 2$ and $\sum_i d_i^2 \le 3$, the only possibility is to have two vortices of degree +1. Denoting by $l(t)$ their distance, we easily check that

$$E_\varepsilon(u(t)) \ge 2\pi \log \frac{1}{l(t)\varepsilon} - C. \tag{3.25}$$

On the other hand, the 2 vortices form an unbalanced cluster at scale $l(t)$ so Theorem 2 of [S1] yields, if $l(t) \gg \varepsilon\sqrt{|\log \varepsilon|}$,

$$\|f_\varepsilon\|_{L^2}^2 \geq \min\left(\frac{C}{l^2(t)|\log \varepsilon|}, \frac{C}{l^2(t)\log^2 \frac{1}{l(t)}}\right),$$

and if $l(t)$ is smaller, we still have a cluster at scale $\varepsilon|\log \varepsilon|$. In all cases we have

$$|\log \varepsilon|\|f_\varepsilon\|_{L^2}^2 \geq \frac{C}{|\log \varepsilon|^4 l^2(t)}$$

Consider S'_ε the set of times for which $l(t) \leq e^{-\sqrt{|\log \varepsilon|}}$, we have

$$|S'_\varepsilon| \leq C|\log \varepsilon|^4 e^{-2\sqrt{|\log \varepsilon|}}.$$

For $t \notin S'_\varepsilon$, comparing (3.25) and (3.23), we have

$$2\pi \log \frac{l(t)}{l(0)} + C \geq \int_0^t \frac{C}{l^2(t)}. \tag{3.26}$$

Now, assume by contradiction that we have $l(t) \leq l$, then we must have

$$2\pi \log \frac{l}{l(0)} + C \geq \frac{Ct}{l^2}$$

and $t \leq Cl^2\left(\log \frac{l}{l(0)} + 1\right)$. Adding the times when $t \in S_\varepsilon \cup S'_\varepsilon$, we find there must exist some $t \leq Cl^2 \log \frac{l}{l(0)} + C|\log \varepsilon|^4 e^{-2\sqrt{|\log \varepsilon|}}$ for which $l(t) \geq l$. The other assertion can be obtained exactly as in Lemma 5.1.

□

Remark 3.1. *Observe that again we only need to consider small l's here, because otherwise, the dynamics is given by Theorem 6.*

3.3 Exit through the boundary

The situation of vortices exiting through the boundary can only happen for the Neumann boundary condition, and is in fact very similar to the case of colliding vortices, since it can be viewed as the collision of a vortex with its "image vortex", the vortex of opposite degree reflected through the boundary. Assume for example that initially u_ε solution of (1.1) has a vortex converging as $\varepsilon \to 0$ to a point $p \in \partial\Omega$, and that $\partial\Omega$ is locally flat near p. Then, Ω and u_ε can be reflected around this piece of boundary, leading to a double domain with a colliding dipole. The case of a nonflat boundary requires adjustments, but the spirit is the same. Therefore, we shall not treat the exit case in details, but mention that exactly the analogous results to Theorem 3 could be obtained.

4 Applications to lower bounds

4.1 Time-estimates through blow-up

In this subsection, we rescale the "product-estimate" Theorem 1, in order to bound from above the movement of the vortices (or bound from below the time it takes them to collide according to (1.23)). This will allow to retrieve the vortex dynamics after blow-up.

First, we write a blown-up version of Theorem 1.

Proposition 4.1. *Let R be a constant. Let $l \to 0$ as $\varepsilon \to 0$ with $l \geq \varepsilon^\beta$ for some $\beta < 1$, and let $\eta = \frac{\varepsilon}{l}$. Let $u_\varepsilon(x,t)$ be defined over $[0, l^2T] \times B(p_\varepsilon, Rl)$ such that*

$$\forall t \in [0, l^2 t] \qquad E_\varepsilon(u_\varepsilon(t), B(p_\varepsilon, Rl)) \leq C|\log \eta| \tag{4.1}$$

$$\int_{B(p_\varepsilon, Rl)\times[0,l^2T]} |\partial_t u_\varepsilon|^2 \leq C|\log \eta| \tag{4.2}$$

Let us consider $\overline{u_\varepsilon}(x,t) = u_\varepsilon(p_\varepsilon + lx, l^2t)$ defined in $[0,T] \times B(0,R)$.

Then, up to extraction, for every $t \in [0,T]$,

$$\mathrm{curl}\,(i\overline{u_\varepsilon}, \nabla \overline{u_\varepsilon}) \rightharpoonup \mu(t) \quad \textit{in } C_C^{0,\gamma}(B(0,R))^*, \ \forall \gamma > 0,$$

where $\mu(t)$ is of the form

$$2\pi \sum_i D_i(t)\delta_{b_i(t)}, \qquad D_i(t) \in \mathbb{Z}.$$

Moreover, there exists a vector-valued measure V such that $\partial_t \mu + \mathrm{div}\ V = 0$; and, for every $X \in C_C^0([0,T] \times B(0,R), \mathbb{R}^2)$ and $f \in C_C^0([0,T]\times B(0,R))$,

$$\varliminf_{\varepsilon\to 0} \frac{1}{|\log \eta|^2} \int_{B(0,R)\times[0,T]} |X \cdot \nabla \overline{u_\varepsilon}|^2 \int_{B(0,R)\times[0,T]} f^2 |\partial_t \overline{u_\varepsilon}|^2 \geq \frac{1}{4} \left| \int_{B(0,R)\times[0,T]} V \cdot fX \right|^2 \tag{4.3}$$

We deduce the existence of vortex trajectories at that scale, analogous to Proposition 1.1 (as in Proposition 3.2 and 3.3 of [SS2] and Corollary 7 if [SS1]).

Proposition 4.2. *Let u_ε satisfy the same hypotheses as the previous proposition. If $D_i(0) = \pm 1$, the $b_i(0)$ are distinct and $\sum_i |D_i(t)| \leq \sum_i |D_i(0)|$ for every $t \in [0,T]$, then there exists $T^* < T$ and $n = n(0)$ functions $b_i(t) \in H^1((0,T^*), \mathbb{R}^2)$ such that for all $t \in [0, T^*)$, the points $b_i(t)$ are distinct and $\mu(t) = 2\pi \sum_i D_i(0)\delta_{b_i(t)}$. Moreover, if $T^* < T$, as $t \to T^*$, one $b_i(t)$ tends to $\partial B(0,R)$ or there exists $i \neq j$ such that $b_i(t)$ and $b_j(t)$ tend to the same point.*

If in addition, $\int_{B(0,R)} |\nabla \overline{u_\varepsilon}|^2 \leq 2\pi n |\log \eta|(1+o(1))$ for all $t \in [t_1, t_2] \subset [0, T^*)$, then we have

$$\varliminf_{\varepsilon\to 0} \frac{1}{|\log \eta|} \int_{B(0,R)\times[t_1,t_2]} |\partial_t \overline{u_\varepsilon}|^2 \geq \pi \sum_i \int_{t_1}^{t_2} |\partial_t b_i|^2 \, dt. \tag{4.4}$$

4.2 Applications to lower bounds

This section is a little detour out of the question of Ginzburg-Landau dynamics into the question of sharp lower bounds for the Ginzburg-Landau energy. Thanks to the time-dependent approach, we can obtain in a simple manner very general lower bounds for the energy, improving that of Lemma 3.3 in [S1] (which required $\|f_\varepsilon\| \le \frac{C}{\varepsilon^\gamma}$ so that we have vortex small balls given by Proposition 2.2 in [S1]).

The idea is to flow the configuration for a very short time according to (1.1). This decreases the energy and smoothes out small irregularities. It yields an alternative to a discrete parabolic regularisation as done in [AB].

Proposition 4.3. *Assume u_ε is such that (1.2) or (1.3) hold with (1.27) and (1.28). Then, up to extraction, we may assume that there exist distinct points p_j and integers $\mathcal{D}_j$ such that*

$$\mathrm{curl}\,(iu_\varepsilon, \nabla u_\varepsilon) \rightharpoonup 2\pi \sum_{j=1}^n \mathcal{D}_j \delta_{p_j} \quad \text{as } \varepsilon \to 0. \tag{4.5}$$

Moreover we have

$$E_\varepsilon(u_\varepsilon) \ge \pi \sum_j |\mathcal{D}_j||\log\,\varepsilon| + W_{\mathcal{D}}(p_1, \cdots, p_n) + \left(\sum_{i=1}^n |\mathcal{D}_i|\right)\gamma + o(1). \tag{4.6}$$

If there exists a bounded number of points $p_j^\varepsilon \to p_j$ (where the p_j's are the ones above plus possibly some additional ones with $\mathcal{D}_j = 0$) and $l_j = o(1)$ with $|\log l_j| \ll |\log\,\varepsilon|$ such that, denoting $\overline{u_\varepsilon}_j = u_\varepsilon(p_j^\varepsilon + l_j x)$ we have

$$\mathrm{curl}\,(i\overline{u_\varepsilon}_j, \nabla \overline{u_\varepsilon}_j) \rightharpoonup 2\pi \sum_{k=1}^m D_{j,k}\delta_{b_{j,k}} \tag{4.7}$$

with $D_{j,k} \in \mathbb{Z}$, $\sum_{k=1}^m D_{j,k} = \mathcal{D}_j$. Then,

$$E_\varepsilon(u_\varepsilon) \ge \pi \sum_{j,k} |D_{j,k}||\log\,\varepsilon| + W_{\mathcal{D}}(p_1, \cdots, p_n) \\ - \pi \sum_j \sum_{k,k'} D_{j,k} D_{j,k'} \log(l_j|b_{j,k} - b_{j,k'}|) + \sum_{j,k} |D_{j,k}|\gamma + o(1). \tag{4.8}$$

Proof. The fact that we may assume (4.5) comes again from the compactness of the Jacobians $\mathrm{curl}\,(iu_\varepsilon, \nabla u_\varepsilon)$.

Let us write u_ε^0 for u_ε and consider the solution of the Cauchy problem (1.1) with initial data u_ε^0 at time 0, and let us denote it $u_\varepsilon(x,t)$. We have $\int_0^T |\partial_t u_\varepsilon|^2 = |\log\,\varepsilon|(E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(t)) \le C|\log\,\varepsilon|^2$. Therefore, by a mean-value argument, there exists $T_\varepsilon \le \frac{1}{|\log\,\varepsilon|^2}$ such that $\int_\Omega |\partial_t u_\varepsilon|^2(T_\varepsilon) \le C|\log\,\varepsilon|^4$. So $u_\varepsilon(T_\varepsilon)$ solves (1.11) with $\|f_\varepsilon\|^2_{L^2(\Omega)} \le C|\log\,\varepsilon|^2$. On the other hand, from Theorem 1, arguing as in the proof of Lemma 2.1 for example, since $T_\varepsilon \ll \frac{1}{|\log\,\varepsilon|}$, we have

$$\mathrm{curl}\,(iu_\varepsilon, \nabla u_\varepsilon)(T_\varepsilon) \rightharpoonup 2\pi \sum_i \mathcal{D}_j \delta_{p_j} \quad \text{as } \varepsilon \to 0$$

i.e. the limiting vortices p_j have not moved. Moreover, since the parabolic flow decreases the energy, we have

$$E_\varepsilon(u_\varepsilon^0) \ge E_\varepsilon(u_\varepsilon(T_\varepsilon)).$$

Therefore, in order to bound from below $E_\varepsilon(u_\varepsilon^0)$, we may replace it with $u_\varepsilon(T_\varepsilon)$, which has the same vortices in the sense of (4.5) and satisfies (1.11) with $\|f_\varepsilon\|_{L^2(\Omega)} \le C|\log\varepsilon|$. We denote by u_ε again the map obtained after replacement. Since we wish to prove (4.6), we may always assume that

$$E_\varepsilon(u_\varepsilon) \le \pi \sum_j |\mathcal{D}_j||\log\varepsilon| + W_{\mathcal{D}}(p_1,\cdots,p_n) + \sum_{i=1}^n |\mathcal{D}_i|\gamma, \tag{4.9}$$

otherwise the result is true.

Since $\|f_\varepsilon\|_{L^2(\Omega)} \le C|\log\varepsilon|$, u_ε satisfies (1.11), (1.29), and the results of Proposition 2.2 in [S1], in particular this yields the (a_i,d_i)'s, with

$$\sum_i d_i^2 \le \frac{E_\varepsilon(u_\varepsilon)}{|\log\varepsilon|} + o(1),$$

from (1.30). From (4.9), we deduce $\sum_i d_i^2 \le \sum_j |\mathcal{D}_j|$. But, since $\sum_{i/a_i\to p_j} d_i = \mathcal{D}_j$, this implies that the d_i's are all ±1, every a_i converges to one of the p_j's (recall $d_i \ne 0$), and that for each j, the degrees d_i associated to $a_i \to p_j$ all have the sign of $\mathcal{D}_j$, hence $\mathcal{D}_j \ne 0$.

We may find balls $B(p_j,\rho)$ with $\rho \ll 1$ converging to 0 slower than the distance of the a_i's to the p_j's. This ensures that the hypotheses of Lemma 3.4 of [S1] hold for these balls, and (4.6) is a direct consequence of the result of Lemma 3.4 in [S1].

For the second part of the Proposition, we follow the same reasoning. Arguing as above, let us flow u_ε according to (1.1). By a mean value argument, as before, letting $l = \min_j l_j$, we may find $T_\varepsilon \le \frac{1}{|\log\varepsilon|^2}$ such that $u_\varepsilon(T_\varepsilon l^2)$ solves (1.11) with

$$\int_\Omega |f_\varepsilon|^2 \le C\frac{|\log\varepsilon|^4}{l^2} \tag{4.10}$$

and in view of the hypotheses on l_j, this ensures that (1.29) is satisfied at time $l^2 T_\varepsilon$. On the other hand, considering $\overline{u_\varepsilon}_j = u(p_j^\varepsilon + l_j x, l_j^2 t)$, we have $\mathrm{curl}\,(i\overline{u_\varepsilon}_j, \nabla\overline{u_\varepsilon}_j)(T_\varepsilon) \rightharpoonup 2\pi\sum_k D_{j,k}\delta_{b_{j,k}}$ i.e. the $b_{j,k}$'s haven't moved in that time, according to Proposition 4.1 (see (4.3)). Since the energy decreases in time, this means that we can assume that u_ε is such that (1.11)–(1.29) hold. We may also assume that

$$E_\varepsilon(u_\varepsilon) \le \pi\sum_{j,k}|D_{j,k}||\log\varepsilon| + W_{\mathcal{D}}(p_1,\cdots,p_n) - \pi\sum_j\sum_{k,k'} D_{j,k}D_{j,k'}\log(l_j|b_{j,k}-b_{j,k'}|) + \sum_{j,k}|D_{j,k}|\gamma$$

otherwise the desired result is true. Since $|\log l_j| \ll |\log\varepsilon|$, this implies

$$E_\varepsilon(u_\varepsilon) \le \pi\sum_{j,k}|D_{j,k}||\log\varepsilon|(1+o(1)). \tag{4.11}$$

Applying Proposition 2.2 in [S1], we find a bounded collection of (a_i,d_i). Combining (4.11) with (1.30) yields

$$\alpha\sum_i d_i^2 \le \sum_{j,k}|D_{j,k}| + \varepsilon^{1-\alpha}|\log\varepsilon|^{7/2}\|f_\varepsilon\|_{L^2(\Omega)} + o(1) \le \sum_{j,k}|D_{j,k}| + \varepsilon^{1-\alpha}|\log\varepsilon|^{7/2}\frac{|\log\varepsilon|^2}{l} + o(1) \tag{4.12}$$

with (4.10). Using the fact that $|\log l_j| \ll |\log\varepsilon|$ i.e. $l \ge \varepsilon^{c_\varepsilon}$ with $c_\varepsilon \to 0$, and taking α close to 1, we find, since the d_i's and $D_{j,k}$'s are integers, that $\sum_i d_i^2 \le \sum_{j,k}|D_{j,k}|$. This implies that each $d_i = \pm 1$ and has the sign of the corresponding $D_{j,k}$. Moreover, this also implies that (since $d_i \neq 0$) all the a_i's are close to the $p_j^\varepsilon + l_j b_{j,k}$. Let us now consider the $y_{j,k} = p_j^\varepsilon + l_j b_{j,k}$, all the a_i's remain inside the $B(y_{j,k}, \rho l_j)$ for some $\rho \ll 1$. We may also choose ρ large enough so that the hypotheses of Lemma 3.4 in [S1] are satisfied for these balls. The total degree on each ball is $D_{j,k}$ and using that result, we easily deduce (4.8). □

Remark 4.1. *If $|\log l_j| \ll |\log \varepsilon|$ is not satisfied but we still have $l_j \geq \varepsilon^\beta$ for some $\beta < 1$, then we may still get analogue results from Lemma 3.4 in [S1].*

5 Exact dynamical laws - Theorems 4 and 5

5.1 Statement of the result

Given points b_k and integers D_k, we introduce

(5.1)
$$\overline{W}(b_1, \cdots, b_m) = -\pi \sum_{i \neq j} D_i D_j \log |b_i - b_j|.$$

Observe that

(5.2)
$$\nabla_k \overline{W}(b_1, \cdots, b_m) = -\pi \sum_{i \neq k} D_i D_k \frac{b_i - b_k}{|b_i - b_k|^2}.$$

Remark 5.1. *It would be interesting to prove that if $\sum_{i \neq k} D_i D_k \neq 0$ then $\nabla \overline{W}(b_i) \neq 0$.*

Our main result of this section is

Theorem 6. *Assume u_ε is a solution to (1.1), with (1.27) and (1.28). Assume $l = o(1)$ with $\log^4 l \leq C|\log \varepsilon|$, and the points $p_j^\varepsilon \to p_j$, $j \in [1, n]$ are such that, defining $\overline{u_\varepsilon}_j(x,t) = u_\varepsilon(p_j^\varepsilon + lx, l^2 t)$, we have*

(5.3)
$$\operatorname{curl}(i\overline{u_\varepsilon}_j, \nabla \overline{u_\varepsilon}_j)(0) \rightharpoonup 2\pi \sum_{k=1}^m D_{j,k} \delta_{b_{j,k}^0} \quad \text{as } \varepsilon \to 0,$$

with $D_{j,k} = \pm 1$, $\sum_k D_{j,k} = \mathcal{D}_j$, and assume

(5.4)
$$E_\varepsilon(u_\varepsilon^0) \leq \pi \sum_{j,k} |D_{j,k}||\log \varepsilon| + W_{\mathcal{D}}(p_1, \cdots, p_n) - \pi \sum_j \sum_{k \neq k'} D_{j,k} D_{j,k'} \log(l|b_{j,k}^0 - b_{j,k'}^0|) + \sum_{j,k} |D_{j,k}|\gamma + r_\varepsilon$$

with either $r_\varepsilon \leq o(1)$ or $r_\varepsilon \leq \frac{l^2 |\log \varepsilon|}{(\log |\log \varepsilon|)^\beta}$ with $\beta > 1$. Then, there exist $H^1((0,T^*))$ trajectories $b_{j,k}(t)$ such that for every $t \in [0, T^*)$,

$$\operatorname{curl}(i\overline{u_\varepsilon}_j, \nabla \overline{u_\varepsilon}_j)(t) \rightharpoonup 2\pi \sum_k D_{j,k} \delta_{b_{j,k}(t)} \quad \text{as } \varepsilon \to 0$$

where $b_{j,k}$ solves the dynamical law

(5.5)
$$\begin{cases} \dfrac{db_{j,k}}{dt} = -\dfrac{1}{\pi} \displaystyle\sum_{k' \neq k} D_{j,k'} D_{j,k} \frac{b_{j,k'} - b_{j,k}}{|b_{j,k'} - b_{j,k}|^2} = -\frac{1}{\pi} \nabla_k \overline{W}(b_{j,i}) \\ b_{j,k}(0) = b_{j,k}^0 \end{cases}$$

and T^* is the first collision-time under this law. Moreover, for every time $t \in (0, T^*)$,

$$(5.6)\quad E_\varepsilon(u_\varepsilon(l^2t)) \le \pi \sum_{j,k} |D_{j,k}||\log\,\varepsilon| + W_{\mathcal{D}}(p_1, \cdots, p_n) - \pi \sum_j \sum_{k,k'} D_{j,k} D_{j,k'} \log(l|b_{j,k}(t) - b_{j,k'}(t)|) + \sum_{j,k} |D_{j,k}|\gamma + o(1),$$

as $\varepsilon \to 0$.

Remark 5.2. *Observe that this result includes the possibility of only one or several vortices at distance $\ll l$ from p_j^ε, in which case there is only one $b_{j,1}$ equal to the origin, which does not move in this time-scale, according to (5.5). This allows to treat, among others, the case of one dipole colliding while other vortices remain fixed, just as in Theorem 3.*

End of the proof of Theorem 3. To complete the proof of Theorem 3, there remained to consider the case $\frac{1}{|\log\,\varepsilon|^2} \le l = l(0) \le o(1)$, which can be treated by bridging with Theorem 6. To prove that Theorem 3 also holds in this case, it suffices to show the existence of a time $T_\varepsilon \le Cl(0)^2$ at which the hypotheses of Theorem 3 are satisfied (taking the new initial time to be T_ε) with vortices at distance $\frac{1}{|\log\,\varepsilon|^2}$, and that

$$(5.7)\qquad \forall t \in [0, T_\varepsilon], \qquad \mathrm{curl}\,(iu_\varepsilon, \nabla u_\varepsilon)(t) \rightharpoonup 2\pi \sum_{i=1}^n D_i \delta_{p_i^0} \quad \text{as } \varepsilon \to 0.$$

Let us thus start with u_ε satisfying the hypotheses of Theorem 3 at time 0, with $l(0) \ge \frac{1}{|\log\,\varepsilon|^2}$. It also satisfies the hypotheses of Theorem 6, taking $l = l(0)$ and the points p_j to be $p_1^0, \cdots, p_n^0, p_{dip}$ (with the notations of Section 3). We also have $b_{1,1}^0 = \cdots, b_{n,1}^0 = 0$ while $b_{n+1,1} = b_+$ and $b_{n+1,2} = b_-$. Applying Theorem 6, we obtain the dynamical law of the $b_{j,k}(t)$: the $b_{j,1}(t)$ are fixed for $j = 1, \cdots, n$, that is the points p_i^0 do not move in time $O(l(0)^2)$, which will prove that $\forall t \le Cl(0)^2$, we have $\mathrm{curl}\,(iu, \nabla u)(t) \rightharpoonup 2\pi \sum_{i=1}^n D_i \delta_{p_i^0}$, that is (5.7) holds. There remains to prove the existence of T_ε. Examining the dynamical law (5.5) for the dipole after space-time rescaling

$$\frac{db_+(t)}{dt} = \frac{1}{\pi} \frac{b_- - b_+}{|b_- - b_+|^2}$$

(and the symmetric law for b_-) we see that $\frac{d}{dt}|b_+ - b_-|(t) = -\frac{2}{\pi}\frac{1}{|b_+ - b_-|(t)}$, so we easily find that

$$(5.8)\qquad |b_+ - b_-|(t) = \sqrt{|b_+ - b_-|^2(0) - \frac{4t}{\pi}}.$$

Now we saw in Section 3 (see the proof of Proposition 3.2 which still applies here) that for all times except a measure ε^β of times, $u_\varepsilon(t)$ has vortices given by Proposition 2.2 in [S1], and has exactly n of them converging to each p_j^0, plus two (the dipole) at distance $o(l(0))$ from $p_{dip} + l(0)b_\pm(t)$. Therefore, in original space-time, the distance between the vortices of the dipole is $l(0)\sqrt{l(0)^2 - \frac{4t}{\pi l(0)^2}} + o(l(0))$. Thus, in time $t_1 = \frac{3\pi l(0)^2}{16}$, the vortices of the dipole are at a distance $l_1 = l(0)/2 + o(l(0)) < \frac{3}{4}l(0)$. Moreover, at that time t_1, the configuration is well-prepared because (5.6) holds. The hypotheses of Theorem 6 are satisfied again at initial time t_1 with scale l_1. Applying Theorem 6 with this new scale, we find a time $t_2 = t_1 + \frac{3\pi l_1^2}{16}$

at which the distance between the vortices of the dipole is $l_2 = l_1/2 + o(l_1) < \frac{3}{4}l_1$. We may iterate this process and find times

$$t_k = \frac{3\pi}{16}\left(l(0)^2 + l_1^2 \cdots + l_k^2\right) \tag{5.9}$$

at which the distance between the vortices is $< \frac{3}{4}l_k$ with

$$l_k < \frac{3}{4}l_{k-1}. \tag{5.10}$$

This reasoning applies as long as $\log^4 l_k \le C|\log \varepsilon|$, hence we may apply it until final $l_K \le \frac{1}{|\log\ \varepsilon|^2}$. Combining (5.9) and (5.10), we find that $t_K \le Cl(0)^2$. Adding if necessary the times for which Proposition 2.2 in [S1] does not apply, we find that in time $T_\varepsilon \le \varepsilon^\beta + t_K \le C\varepsilon^\beta + Cl(0)^2$, we have a dipole at distance $\le \frac{1}{|\log\ \varepsilon|^2}$ with (5.6) holding. We have seen also that (5.7) holds. Therefore, all the hypotheses of Theorem 3 hold at that new initial time and the proof of Theorem 3 under the restriction $l(0) \le \frac{1}{|\log\ \varepsilon|^2}$ can be used to finish the general proof.

□

5.2 Proof of Theorem 5

The existence of collision times follows from Proposition 1.1. Notice also from the form of W that in the Dirichlet case, no vortex can exit from Ω under the law (1.8). Also with any boundary condition, no pairs of vortices of degree +1 (or −1) can collide under the law (1.8).

Using Theorem 2 (which yields a time T_ε) and then applying the result of [Li, JS, SS2] to the solution $u_\varepsilon(x, t + T_\varepsilon)$ on $\Omega \times \mathbb{R}_+$, we find that

$$\operatorname{curl}(iu_\varepsilon, \nabla u_\varepsilon)(t + T_\varepsilon) \rightharpoonup 2\pi \sum_{i=1}^{n} D_i \delta_{p_i(t)} \quad \text{as } \varepsilon \to 0, \quad \forall t \in [0, T_*)$$

where the p_i's solve (1.8) and T_* is the first collision time under (1.8). Moreover, since (1.34) holds for every $t_\varepsilon \in [0, T_\varepsilon]$ and since the $p_i(t)$ are continuous in time, we deduce that

$$\operatorname{curl}(iu_\varepsilon, \nabla u_\varepsilon)(t) \rightharpoonup \mu(t) = 2\pi \sum_{i=1}^{n} D_i \delta_{p_i(t)} \quad \forall t \in [0, T_*). \tag{5.11}$$

Arguing as in [SS2], the first collision time T_* is also equal to T_1 the first collision time of the trajectories in the sense of Proposition 1.1. At the collision time T_1, there exist one or several pairs of vortices colliding at different places in Ω. Let us assume for simplicity that there is only one pair, say $|p_1(t) - p_2(t)| \to 0$ as $t \to T_1^-$. We must have $D_1 = -D_2$ (otherwise it would contradict the dynamical law after blow up Theorem 6). We deduce

$$\mu(t) \rightharpoonup 2\pi \sum_{i=3}^{n} D_i \delta_{p_i^1} \quad \text{as } t \to T_1^-, \tag{5.12}$$

where $p_i^1 = \lim_{t \to T_1^-} p_i(t)$ for $i = 3, \cdots, n$, are distinct points.

By a mean-value argument combining (1.25) and (1.27), we may find a positive $\tau_\varepsilon \to 0$ such that at the time $T_1 - \tau_\varepsilon$ we have $\|f_\varepsilon\|^2_{L^2(\Omega)} \le O(1)$ (where f_ε denotes $\frac{\partial_t u_\varepsilon}{|\log\ \varepsilon|}$). Applying then (1.30), we find that the vortices (a_i, d_i) of $u_\varepsilon(T_1 - \tau_\varepsilon)$ given by Proposition 2.2 of [S1] satisfy $\sum_i d_i^2 \le n$. It is then easy to check that there is one vortex a_i of degree D_i converging to each p_i^1 respectively as $\varepsilon \to 0$ for $i = 3, \cdots, n$; and two vortices of opposite degrees a_1, a_2, at distances $o(1)$ respectively to $p_1(T_1 - \tau_\varepsilon)$ and $p_2(T_2 - \tau_\varepsilon)$, hence at a distance $l_\varepsilon = o(1)$ from each other. Denoting by $v_\varepsilon(t) = u_\varepsilon(t + T_1 - \tau_\varepsilon)$, we deduce that v_ε satisfies the hypotheses (1.36)–(1.37) of Theorem 3. Moreover, from (1.31) (see Theorem 1 in [S1]) and the bound on $\|f_\varepsilon\|_{L^2}$, in view of the expression of W, we have $E_\varepsilon(v_\varepsilon(0)) \le \pi(n-2)|\log \varepsilon| + 2\pi \log \frac{l_\varepsilon}{\varepsilon} + O(1)$. Therefore, we may apply Theorem 3 to v_ε, and we deduce the existence of a time $\tau'_\varepsilon = o(1)$ such that

$$\operatorname{curl}(iv_\varepsilon, \nabla v_\varepsilon)(t_\varepsilon) \rightharpoonup 2\pi \sum_{i=3}^n D_i \delta_{p_i^1} \quad \forall t_\varepsilon \in [0, \tau'_\varepsilon)$$

and

$$E_\varepsilon(v_\varepsilon(\tau'_\varepsilon)) \le \pi(n-2)|\log \varepsilon| + W_{\mathbf{D}}(p_3^1, \cdots, p_n^1) + (n-2)\gamma + o(1).$$

We deduce

$$\operatorname{curl}(iu_\varepsilon, \nabla u_\varepsilon)(t_\varepsilon) \rightharpoonup 2\pi \sum_{i=3}^n D_i \delta_{p_i} \quad \forall t_\varepsilon \in (T_1 - \tau_\varepsilon, T_1 - \tau_\varepsilon + \tau'_\varepsilon) \quad \text{as } \varepsilon \to 0. \tag{5.13}$$

We may now apply the result of [Li, JS, SS2] to v_ε starting at time τ'_ε and find that

$$\forall t \in [0, T_*) \qquad \operatorname{curl}(iv_\varepsilon, \nabla v_\varepsilon)(t + \tau'_\varepsilon) \rightharpoonup 2\pi \sum_{i=3}^n D_i \delta_{p_i(t)}$$

where $p_i(t)$ solves

$$\begin{cases} \dfrac{dp_i}{dt} = -\dfrac{1}{\pi} \nabla_i W_{\mathbf{D}}(p_3, \cdots, p_n)(t) \\ p_i(0) = p_i^1 \quad i = 3, \cdots, n, \end{cases} \tag{5.14}$$

until the first collision time T_* under this law. Combining with (5.13) and using the continuity of the p_i's, we find

$$\operatorname{curl}(iu_\varepsilon, \nabla u_\varepsilon)(t) \rightharpoonup 2\pi \sum_{i=3}^n D_i \delta_{p_i^1(t)}$$

for every $t \in [T_1, T_2)$ where $T_2 = T_1 + T_*$ is the second collision time. The relation (1.45) follows easily from the analysis of [SS2] for example.

The case of more than one collision pair can be treated similarly, observing that just like for Theorems 3 and 6 (applying the method of Proposition 3.1) collisions centered at distinct points in Ω do not interfere with one another. Moreover the number of vortices decreases by at least 2 during each collision. The proof can then be iterated at the next collision time T_2, under the assumption of simple collisions. This completes the proof of Theorem 5.

5.3 Proof of Theorem 6

Before we prove this theorem, we will state a few propositions.

Proposition 5.1. *Let u_ε be such that (1.27) holds. Assume for each j, the points p_j^ε are such that, defining $\overline{u_\varepsilon}_j(x) = u_\varepsilon(p_j^\varepsilon + lx)$, we have*

$$\operatorname{curl}(i\overline{u}_j, \nabla \overline{u}_j) \rightharpoonup 2\pi \sum_k D_{j,k}\delta_{b_{j,k}} \quad \text{as } \varepsilon \to 0 \tag{5.15}$$

with $D_{j,k} = \pm 1$, $\sum_k D_{j,k} = \mathcal{D}_j$ and $\log^4 l \le C|\log \varepsilon|$, and for every constant R,

$$E_\varepsilon(u_\varepsilon, B(p_j^\varepsilon, lR)) \le \pi \sum_k |D_{j,k}||\log \varepsilon|(1+o(1)) \tag{5.16}$$

and

$$\log^4 l \le O(|\log \varepsilon|). \tag{5.17}$$

Then, we have

$$\varliminf_{\varepsilon\to 0}\left(|\log \varepsilon| l^2 \int_\Omega |\Delta u + \frac{u}{\varepsilon^2}(1-|u|^2)|^2\right) \ge \frac{1}{\pi}\sum_j \|\nabla \overline{W}(b_{j,k})\|^2 + o_R(1), \tag{5.18}$$

where $o_R(1) \to 0$ as $R \to \infty$.

This proposition will be proved further below. We also need a result which is an analogue of Theorem 2 after blow-up.

Proposition 5.2. *Under the hypotheses of Theorem 6 without the hypothesis (5.17), there exists $T_\varepsilon \le \frac{\log(l^2|\log \varepsilon|)}{l^2|\log \varepsilon|}$ with $T_\varepsilon \le o(1)$, and such that for every $t_\varepsilon \in [0, T_\varepsilon]$,*

$$\operatorname{curl}(i\overline{u}_j, \nabla\overline{u}_j)(t_\varepsilon) \rightharpoonup 2\pi \sum_k D_{j,k}\delta_{b^0_{j,k}}, \tag{5.19}$$

and

$$E_\varepsilon(u_\varepsilon(l^2T_\varepsilon)) \le \sum_{j,k}|D_{j,k}||\log \varepsilon| + W_{\mathcal{D}}(p_1,\cdots,p_n) - \pi\sum_j\sum_{k\neq k'} D_{j,k}D_{j,k'}\log(l|b^0_{j,k}-b^0_{j,k'}|) + \sum_{j,k}|D_{j,k}|\gamma + o(1). \tag{5.20}$$

Moreover, there exists T_0 and C independent of ε such that for every R,

$$\frac{1}{|\log \eta|}\int_{B(0,R)\times[T_\varepsilon,T_0]} |\partial_t \overline{u}_j|^2 \le C \tag{5.21}$$

and thus the results of Propositions 4.1 and 4.2 apply, giving H^1 trajectories $b_{j,k}(t)$ for $t \in [0, T_0)$ (before collision time).

Proof. Let us start with the first assertion, the existence of T_ε. If $\frac{l^2|\log \varepsilon|}{(\log |\log \varepsilon|)^\beta} \le o(1)$ then one should take $r_\varepsilon = o(1)$ and $T_\varepsilon = 0$ and there is nothing to prove. We can thus focus on $l^2 \ge C\frac{(\log |\log \varepsilon|)^\beta}{|\log \varepsilon|}$, which implies that $T_\varepsilon = o(1)$ in all cases, and $|\log l| \ll |\log \varepsilon|$.

We can easily show an analogue of Lemma 2.1: there exists a time $T_\varepsilon = \frac{\log(l^2|\log \varepsilon|)}{l^2|\log \varepsilon|}$ such that if $\forall t \in [0, T_\varepsilon]$, and for all j, $\operatorname{curl}(i\overline{u_\varepsilon}_j, \nabla\overline{u_\varepsilon}_j)(t) \rightharpoonup 2\pi\sum_k D_{j,k}\delta_{b^0_{j,k}}$, then (5.20) holds. The proof is exactly along the same lines as Lemma 2.1. We show that the hypothesis (5.4) combined with (1.30) implies that for most times $t_\varepsilon \le l^2 T_\varepsilon$, $u_\varepsilon(t_\varepsilon)$ has exactly 1 vortex of degree $D_{j,k}$ converging (after rescaling) to each $b^0_{j,k}$. Then, (1.31) i.e. Theorem 1 of [S1], and a differential inequality lead to

$$E_\varepsilon(u_\varepsilon(t)) \le \pi \sum_j \sum_k |D_{j,k}||\log\,\varepsilon| + W_{\mathcal{D}}(p_1,\cdots,p_n) + \pi \sum_j \sum_{k\neq k'} D_{j,k}D_{j,k'} \log(l|b^0_{j,k}-b^0_{j,k'}|) \\ + \sum_{j,k} |D_{j,k}|\gamma + Ce^{-t|\log\,\varepsilon|} r_\varepsilon.$$

Taking $T_\varepsilon = \frac{\log(l^2|\log\,\varepsilon|)}{l^2|\log\,\varepsilon|}$, in view of the bound on r_ε, we find that (5.20) holds, provided the vortices have not moved.

To prove that the vortices $b_{j,k}$ have not moved in that time, we argue as in the proof of Theorem 2, and use the product estimate as given in Proposition 4.1.

Assuming that there exists $T_\varepsilon \le \frac{\log(l^2|\log\,\varepsilon|)}{l^2|\log\,\varepsilon|}$ for which

$$E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(l^2T_\varepsilon)) = r_\varepsilon + K. \tag{5.22}$$

Defining for each j, $w(x,t) = \overline{u_\varepsilon}_j\,(x, T_\varepsilon t) = u_\varepsilon(p_j^\varepsilon + lx, l^2 T_\varepsilon t)$. Since

$$\frac{1}{|\log\,\varepsilon|}\int_{B(0,R)\times[0,1]} |\partial_t w|^2 = \frac{T_\varepsilon}{|\log\,\varepsilon|}\int_{B(p_j^\varepsilon, lR)\times[0,l^2T_\varepsilon]} |\partial_t u_\varepsilon|^2 \le T_\varepsilon\left(E_\varepsilon(u^0_\varepsilon) - E_\varepsilon(u_\varepsilon(l^2T_\varepsilon))\right).$$

Letting V be the vortex-velocity associated to w, we deduce that for every compactly supported X and $|f|\le 1$,

$$\left|\int_{\mathbb{R}^2\times[0,1]} V\cdot fX\right|^2 \le 4\varliminf_{\varepsilon\to 0}\frac{1}{|\log\,\varepsilon|^2}\int_{\mathbb{R}^2\times[0,1]}|X\cdot\nabla w|^2\int_{\mathbb{R}^2\times[0,1]} f^2|\partial_t w|^2 \tag{5.23}$$
$$\le \varliminf_{\varepsilon\to 0} CT_\varepsilon(r_\varepsilon + K)$$
$$\le C\frac{\log(l^2|\log\,\varepsilon|)}{l^2|\log\,\varepsilon|}\left(\frac{l^2|\log\,\varepsilon|}{(\log\,|\log\,\varepsilon|)^\beta} + K\right) \le \frac{\log\,|\log\,\varepsilon|}{(\log\,|\log\,\varepsilon|)^\beta} = o(1).$$

We deduce $V=0$, and thus, the vortices of w do not move in time 1 i.e. the vortices of $\overline{u_\varepsilon}_j$ do not move in time T_ε, which implies, from Proposition 4.3 (whose hypotheses are satisfied) the lower bound

$$E_\varepsilon(u_\varepsilon(l^2T_\varepsilon)) \ge \pi\sum_{j,k}|D_{j,k}||\log\varepsilon| + W_{\mathcal{D}}(p_1,\cdots,p_n) + \pi\sum_j\sum_{k\neq k'} D_{j,k}D_{j,k'}\log(l|b^0_{j,k}-b^0_{j,k'}|) + \sum_{j,k}|D_{j,k}|\gamma + o(1) \tag{5.24}$$

a contradiction with (5.22) if K is chosen large enough. We deduce that for every $t \le \frac{\log(l^2|\log\,\varepsilon|)}{|\log\,\varepsilon|}$ we have $E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(t)) \le r_\varepsilon + K$, and following the same reasoning, that the vortices of $\overline{u_\varepsilon}_j$ do not move in time $\frac{t}{l^2}$. This proves the first part of the proposition.

For the second part, the reasoning is the same. First, we may start from the new initial time $l^2 T_\varepsilon$ and assume that the solution is very-well prepared originally i.e. that (5.20) holds. Assume by contradiction that there exists $\tau_\varepsilon \ll l^2$ such that

$$E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(\tau_\varepsilon)) = 1, \tag{5.25}$$

arguing as above (replacing T_ε by τ_ε and r_ε by 1) we find that $V=0$ in (5.23), and thus the vortices of $\overline{u_\varepsilon}_j$ have not moved in time $\frac{\tau_\varepsilon}{l^2}$, a contradiction between (5.25), (5.20) holding at time 0 and the lower bound (4.8). Thus, we deduce that there exists a constant T_0 independent of ε such that $E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(l^2T_0)) \le 1$, from which (5.21) follows. □

We now show the lower bound on the collision time, which was left to prove from Section 3 to complete the proof of Theorem 3.

Lemma 5.1 (Lower bound on collision time). *Under the hypotheses of Theorem 3, letting T_1 be the first time such that $u_\varepsilon(x,t)$ has only n zeroes, we have $T_1 \ge C_1 l^2$, for some constant $C_1 > 0$.*

Proof. We may consider $\overline{u_\varepsilon}(x,t) = u_\varepsilon(p_\varepsilon + lx, l^2 t)$. Arguing exactly as in the previous proof, we can show that there exists a constant $T_0 > 0$ independent of ε such that

$$C \ge E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(l^2T_0)) = \frac{1}{|\log\varepsilon|}\int_0^{l^2T_0}\int_\Omega |\partial_t u_\varepsilon|^2 \ge \frac{1}{|\log\varepsilon|}\int_0^{T_0}\int_{B(0,R)} |\partial_t \overline{u_\varepsilon}|^2$$

and thus Proposition 4.2 applies, giving H^1 trajectories $b_+(t)$ and $b_-(t)$ for the vortices of $\overline{u_\varepsilon}$. Since $|b_+(0) - b_-(0)| = 1$, by continuity, $|b_+(t) - b_-(t)| \ge \frac{1}{2}$ in some time-interval $[0, C_1]$ which implies that u_ε does have 2 zeroes near p_ε in the time interval $[0, C_1 l^2]$, hence $n+2$ zeroes total, implying $T_1 \ge C_1 l^2$. □

Proof of Theorem 6. Under the hypotheses of Theorem 6, Proposition 5.2 applies. It first proves that we can reduce to the case of very-well prepared data, i.e. the case where (5.20) holds, since $T_\varepsilon = o(1)$. It also proves that Propositions 4.1 and 4.2 apply on some interval $[0, T_0]$ (or $[0, l^2 T_0]$ in original time), giving that for each j, the $b_{j,k}(t)$'s move continuously and remain distinct until collision, while the p_j's do not move in that time-scale. Moreover, we can check through lower bounds that at each time $t \ge 0$, the hypothesis (5.16) of Proposition 5.1 holds and we may apply it to $u_\varepsilon(t)$.

We then follow the scheme of [SS2], as presented in the introduction (see for example (1.24)) and write

$$\begin{aligned} E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(l^2t)) &= \frac{1}{|\log\varepsilon|}\int_{\Omega\times[0,l^2t]} |\partial_t u_\varepsilon|^2 \\ &= \frac{1}{2}\int_{\Omega\times[0,l^2t]} \frac{1}{|\log\varepsilon|}|\partial_t u_\varepsilon|^2 + \frac{1}{2}\int_{\Omega\times[0,l^2t]} |\log\varepsilon| |\Delta u_\varepsilon + \frac{u_\varepsilon}{\varepsilon^2}(1-|u_\varepsilon|^2)|^2 \end{aligned}$$

Now given R, since $l = o(1)$, the $B(p_j^\varepsilon, Rl)$ are disjoint balls for ε small enough, hence, after a change of scales, we may write

$$\begin{aligned} E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(l^2t)) \ \ge\ & \sum_j \Big(\frac{1}{|\log\varepsilon|}\int_{B(0,R)\times[0,t]} \frac{1}{2}|\partial_t \overline{u_\varepsilon}_j|^2 \\ & + \frac{1}{2}\int_{B(p_j^\varepsilon, Rl)\times[0,t]} l^2 |\log\varepsilon| |\Delta u_\varepsilon + \frac{u_\varepsilon}{\varepsilon^2}(1-|u_\varepsilon|^2)|^2(l^2t)\Big) \end{aligned}$$

Using the fact that $|\log \varepsilon| \sim |\log \eta|$ and plugging in (4.4) for the first part and (5.18) for the second, we are led to

(5.26)

$$E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(l^2t)) \geq \sum_j \left(\int_0^t \left(\frac{\pi}{2} \sum_k |d_t b_{j,k}|^2 + \frac{1}{2\pi} \|\nabla \overline{W}(b_{j,k}(t)\|^2 + o_R(1) \right) ds + o(1) \right)$$

Using the crucial Cauchy-Schwarz argument of [SS2], this becomes

$$E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(l^2t)) \geq -\sum_j \int_0^t \sum_k d_t b_{j,k} \cdot \nabla_k \overline{W}(b_{j,k}(t)) + o_R(1) + o(1)$$

hence (taking the limit $R \to \infty$)

$$\varliminf_{\varepsilon \to 0} \left(E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(l^2t)) \right) \geq \sum_j \overline{W}(b_{j,k}(0)) - \overline{W}(b_{j,k}(t)) \tag{5.27}$$

with equality if and only if for every j, $\partial_t b_{j,k} = -\frac{1}{\pi} \nabla_k \overline{W}(b_{j,i})$ for every k. But, in view of Proposition 4.3, we must have

$$E_\varepsilon(u_\varepsilon(l^2t)) \geq \pi \sum_j \sum_k |D_{j,k}||\log \varepsilon| - \pi \sum_j \sum_{k,k'} D_{j,k} D_{j,k'} \log l + \sum_j \overline{W}(b_{j,k}(t)) + C + o(1)$$

where the constant C depends only on the p_j's and the set of degrees $D_{j,k}$ both constant during the motion. Examining the hypothesis at initial time $t = 0$, we see that

$$E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(l^2t)) \leq \sum_j \overline{W}(b_{j,k}(0)) - \overline{W}(b_{j,k}(t)) + o(1) \tag{5.28}$$

Hence there has to be equality in (5.27) and we conclude that (5.5) holds. □

Remark 5.3. *Of course, this can be generalized to multiple scales. Here we have zoomed up at the scale l, but one should first zoom up at the smallest scale when we see distinct vortices, and rescale time by that l^2. In that timescale, the other vortices do not move, just like the p_j's do not move, only the vortices at small distances from the others move, etc...*

5.4 Proof of Proposition 5.1

We assume for simplicity that p_j^ε is the origin. We recall that $\eta = \frac{\varepsilon}{l}$ and that $|\log l| \ll |\log \varepsilon|$ so that $|\log \eta| \sim |\log \varepsilon|$. First, we may assume that

$$\|f_\varepsilon\|^2_{L^2(\Omega)} \leq \frac{C}{l^2 |\log \varepsilon|} \tag{5.29}$$

otherwise, the result stated is true.

Then, Proposition 2.2 in [S1] applies and gives vortex points a_i. For each j, let us consider the a_i's which are at distance $O(l)$ from p_j^ε, and consider their blown-up points $\overline{a_i} = \frac{a_i - p_j^\varepsilon}{l}$. We may find a constant K such that, for R arbitrarily large, $B(p_j^\varepsilon, 2Rl) \backslash B(p_j^\varepsilon, Kl)$ does not contain any of these $\overline{a_i}$'s. Moreover, we claim that the $\overline{a_i}$'s converge, up to extraction, to some points, which are the $b_{j,k}$'s of (5.15). Indeed, if not, then there would be some subset of them converging to another point, with total degree 0 (otherwise it would appear in the right-hand side of (5.15)). But they would then form an unbalanced cluster of vortices at original scale $\ll l$. From Theorem 2 of [S1] we would deduce $\|f_\varepsilon\|^2_{L^2(\Omega)} \gg \frac{1}{l^2|\log\varepsilon|}$, contradicting (5.29).

We may thus find a radius $\rho > 0$ such that the $B(b_{j,k},\rho)$ are disjoint, and for small ε, the $B(\overline{a_i}, \frac{R_\varepsilon \varepsilon}{l})$ we consider are included in the $B(b_{j,k},\rho)$, and we recall $\sum_{i/\overline{a_i}\to b_{j,k}} d_i = D_{j,k} = \pm 1$.

Let us define

$$\begin{cases} -\Delta\Phi_0 = 2\pi\sum_k D_{j,k}\delta_{b_{j,k}} & \text{in } B(0,R) \\ \Phi_0 = h & \text{on } \partial B(0,R), \end{cases} \tag{5.30}$$

where h will be specified later, and G by

$$\begin{cases} -\Delta_x G(x,y) = \delta_y & \text{in } B(0,R) \\ G(x,y) = \frac{h(x)}{2\pi\sum_k D_{j,k}} & \text{on } \partial B(0,R), \end{cases} \tag{5.31}$$

and $S(x,y) = 2\pi G(x,y) + \log|x-y|$, we have

$$\Phi_0(x) = 2\pi\sum_k D_{j,k}G(x,b_{j,k}) = \sum_k -D_{j,k}\log|x-b_{j,k}| + D_{j,k}S(x,b_{j,k}).$$

We introduce the renormalized energy relative to the ball $B(0,R)$:

$$\overline{W}_R(b_1,\cdots,b_m) = -\pi\sum_{k\neq k'} D_{j,k}D_{j,k'}\log|b_{j,k}-b_{j,k'}| + 2\pi\sum_{k,k'} D_{j,k}D_{j,k'}S(b_{j,k},b_{j,k'}) + \int_{\partial B(0,R)} h(x)\frac{\partial\Phi_0}{\partial\nu} \tag{5.32}$$

It is a direct calculation identical to the one done for Lemma 3.1 of [S1] to show that

$$\frac{1}{2}\int_{B(0,R)\setminus\cup_k B(b_{j,k},r)} |\nabla\Phi_0|^2 = \pi\sum_k D_{j,k}^2\log\frac{1}{r} + \overline{W}_R(b_{j,k}) + o_r(1). \tag{5.33}$$

Let us now gather a few intermediate results.

Lemma 5.2. *We have*

$$\int_{B(0,R)\setminus\cup_k B(b_{j,k},\rho)} |\nabla\overline{\Phi} - \nabla\Phi_0|^2 \to 0 \quad \text{as } \varepsilon\to 0 \tag{5.34}$$

where Φ_0 is defined as the solution of (5.30), $\overline{\Phi}$ as $\Phi(p_j^\varepsilon + lx)$ (where Φ solves (3.5) in [S1]), and h is taken to be the uniform limit of $\overline{\Phi} - \frac{1}{2\pi R}\int_{\partial B(0,R)}\overline{\Phi}$ on $\partial B(0,R)$. Moreover,

$$\lim_{R\to\infty} \|\nabla\overline{W}_R(b_{j,1},\cdots,b_{j,m}) - \nabla\overline{W}(b_{j,1},\cdots,b_{j,m})\|_{L^\infty(B(0,K+1)^m)} = 0 \tag{5.35}$$

Lemma 5.3. *Let the $b_{j,k}, D_{j,k}$ be as before, with $D_{j,k} = \pm 1$ and (5.16), then for any ρ such that the $B(b_{j,k},\rho)$ are disjoint and do not intersect $\partial B(0,R)$, we have*

$$\frac{1}{2}\int_{B(b_{j,k},\rho)} |\nabla\overline{u_\varepsilon}_j|^2 = \pi|\log\eta|(1+o(1)) \tag{5.36}$$

$$\int_{B(0,R)\setminus\cup_k B(b_{j,k},\rho)} |\nabla|\overline{u_\varepsilon}_j||^2 + \frac{1}{2\eta^2}(1-|\overline{u_\varepsilon}_j|^2)^2 \leq o\left(l^2\log^2 l\|f_\varepsilon\|^2_{L^2(\Omega)} + 1\right) \tag{5.37}$$

$$\int_{B(0,R)\setminus\cup_k B(b_{j,k},\rho)} |\nabla\overline{u_\varepsilon} - i\overline{u_\varepsilon}\nabla^\perp\Phi_0|^2 \leq o\left(l^2\log^4 l\|f_\varepsilon\|^2_{L^2} + 1\right). \tag{5.38}$$

Once we have these results, we can follow closely the proof of [SS2], Proposition 3.5. For simplicity, we drop the subscripts j.

Through the change of scales we have

(5.39)
$$\overline{E}(\overline{u_\varepsilon}, B(0,R)) := \frac{1}{2}\int_{B(0,R)} |\nabla \overline{u_\varepsilon}|^2 + \frac{(1-|\overline{u_\varepsilon}|^2)^2}{2\eta^2} = \frac{1}{2}\int_{B(p_j^\varepsilon, lR)} |\nabla u|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} \le C|\log\varepsilon| \le C|\log\eta|.$$

By scaling, from (5.29), for every $R>0$,

$$\int_{B(0,R)} |\Delta \overline{u_\varepsilon} + \frac{\overline{u_\varepsilon}}{\eta^2}(1-|\overline{u_\varepsilon}|^2)|^2 \le \frac{C}{|\log\,\varepsilon|}, \tag{5.40}$$

and also $o(1) l^2 \|f_\varepsilon\|_{L^2}^2 \log^4 l \le o(1) \frac{\log^4 l}{|\log\,\varepsilon|} \le o(1)$ from the assumption (5.17). Hence all the right-hand sides in (5.37) and (5.38) tend to 0 and we can rewrite this

$$\int_{B(0,R)\setminus \cup_k B(b_{j,k},\rho)} |\nabla |\overline{u_\varepsilon}||^2 + \frac{1}{2\eta^2}(1-|\overline{u_\varepsilon}|^2)^2 \le o(1) \tag{5.41}$$

$$\int_{B(0,R)\setminus \cup_k B(b_{j,k},\rho)} |\nabla \overline{u_\varepsilon} - i\overline{u_\varepsilon}\nabla^\perp \Phi_0|^2 \le o(1). \tag{5.42}$$

The relation (5.40) is used to obtain as in [SS2], $(i\overline{u_\varepsilon}, \nabla\overline{u_\varepsilon}) \rightharpoonup \nabla^\perp\Phi_0 + cst$, and in view of (5.42), the constant vector is 0, that is

$$(i\overline{u_\varepsilon}, \nabla\overline{u_\varepsilon}) \rightharpoonup \nabla^\perp\Phi_0. \tag{5.43}$$

As in [SS2], we consider a set of vectors $(V_1,\cdots,V_m)\in\mathbb{R}^2$, and χ_t a family of diffeomorphisms of $B(0,R)$ preserving $\partial B(0,R)$ and such that $\chi_t(x) = x + tV_k$ in each $B(b_{j,k},\rho)$. We also define $b_{j,k}(t) = b_{j,k} + tV_{j,k}$, and

$$\begin{cases} -\Delta\Phi_t = 2\pi \sum_{j,k} D_{j,k}\delta_{b_{j,k}(t)} & \text{in } \ B(0,R) \\ \dfrac{\partial\Phi_t}{\partial\nu} = \dfrac{\partial\Phi_0}{\partial\nu} & \text{on } \partial B(0,R). \end{cases} \tag{5.44}$$

From Φ_t we define ψ_t exactly as in [SS2], eq. (3.24), vanishing on $\partial B(0,R)$, and $v_\varepsilon(\chi_t(x),t) = \overline{u_\varepsilon}(x)e^{i\psi_t}$. Reproducing the proof of [SS2] yields that under the previous conditions (5.41), (5.42) and (5.43), we have

$$\frac{d}{dt}_{|t=0} \overline{E}(v_\varepsilon(x,t)) = \lim_{r\to 0} \frac{d}{dt}_{|t=0} \frac{1}{2}\int_{B(0,R)\setminus\cup_i B(b_{j,k}(t),r)} |\nabla\Phi_t|^2. \tag{5.45}$$

while

$$\frac{1}{|\log\eta|}\int_{B(0,R)} |\partial_t v_\varepsilon|(0)^2 = \pi\sum_k |V_k|^2 + o(1). \tag{5.46}$$

As in [SS2], this follows from (5.36) and [SS1], Corollary 4.

Next, we claim we have

$$\lim_{r\to 0}\frac{d}{dt}_{|t=0}\frac{1}{2}\int_{B(0,R)\setminus\cup_k B(b_{j,k}(t),r)} |\nabla\Phi_t|^2 = \frac{d}{dt}_{|t=0}\overline{W}_R(b_{j,k}(t)) = \sum_k \nabla_k \overline{W}_R(b_{j,k})\cdot V_k \tag{5.47}$$

(the proof can be reproduced from (3.39) of [SS2].) This will suffice to conclude that

$$|\log\eta|\int_{B(0,R)}|\Delta\overline{u_\varepsilon}+\frac{\overline{u_\varepsilon}}{\eta^2}(1-|\overline{u_\varepsilon}|^2)|^2\geq\frac{1}{\pi}\|\nabla\overline{W}_R(b_{j,k})\|^2+o(1), \tag{5.48}$$

that is

$$|\log\varepsilon|l^2\int_{B(p_j^\varepsilon,Rl)}|\Delta u_\varepsilon+\frac{u_\varepsilon}{\varepsilon^2}(1-|u_\varepsilon|^2)|^2\geq\frac{1}{\pi}\|\nabla\overline{W}_R(b_{j,k})\|^2+o(1). \tag{5.49}$$

Indeed, it follows as in [SS2] by a simple Cauchy-Schwarz inequality: choosing $V_k=\nabla_k\overline{W}_R(b_i)$; since $v_\varepsilon(x,t)=\overline{u_\varepsilon}(x)$ on $\partial B(0,R)$ for each t, we have

$$\begin{aligned}\frac{d}{dt}_{|t=0}\overline{E}(v_\varepsilon(x,t))&=\int_{B(0,R)}\partial_t v_\varepsilon(0)\left(\Delta\overline{u_\varepsilon}+\frac{\overline{u_\varepsilon}}{\eta^2}(1-|\overline{u_\varepsilon}|^2)\right)\\&\leq\left(\frac{1}{|\log\eta|}\int_{B(0,R)}|\partial_t v_\varepsilon|^2(0)\right)^{\frac12}\left(|\log\eta|\int_{B(0,R)}|-\Delta\overline{u_\varepsilon}+\frac{\overline{u_\varepsilon}}{\eta^2}(1-|\overline{u_\varepsilon}|^2)|^2\right)^{\frac12}\\&\leq\left(\pi\sum_k|\nabla_k\overline{W}_R(b_{j,k})|^2+o(1)\right)^{\frac12}\left(|\log\varepsilon|l^2\int_{B(p_j^\varepsilon,Rl)}|f_\varepsilon|^2\right)^{\frac12},\end{aligned} \tag{5.50}$$

using (5.46) and the choice of V_k. Inserting this and (5.47) into (5.45), we are led to (5.49). Adding up the relations (5.49) over j and using (5.35), we conclude that (5.18) holds.

Remark 5.4. *This proof is the only place where the assumption (5.17) is needed.*

5.5 Proof of the additional lemmas

Proof of Lemma 5.2. First, we recall that $\overline{\Phi}$ verifies $-\Delta\overline{\Phi}=2\pi\sum_i d_i\delta_{\overline{a_i}}$ with $\sum_i d_i\delta_{\overline{a_i}}\rightharpoonup\sum_k D_{j,k}\delta_{b_{j,k}}$ in the weak sense of measures in $B(0,R)$. Moreover, since $B(p_j^\varepsilon,2Rl)\backslash B(p_j^\varepsilon,Kl)$ does not contain any vortex, thus examining (3.5) of [S1], as for (3.9) of [S1], we have

$$|\nabla\Phi|\leq\tfrac{C}{Rl}\qquad\text{on }\partial B(p_j^\varepsilon,Rl) \tag{5.51}$$

$$|D^2\Phi|\leq\tfrac{C}{R^2l^2}\qquad\text{on }\partial B(p_j^\varepsilon,Rl) \tag{5.52}$$

so $|\nabla\overline{\Phi}|\leq\frac{C}{R}$ and $|D^2\overline{\Phi}|\leq\frac{C}{R^2}$ on $\partial B(0,R)$. Thus, $\overline{\Phi}-\frac{1}{2\pi R}\int_{\partial B(0,R)}\overline{\Phi}$ is uniformly bounded and equicontinuous on $\partial B(0,R)$ and we may assume it converges uniformly to some h, as $\varepsilon\to0$. Moreover, returning to (3.5) of [S1], and since $B(p_j^\varepsilon,2Rl)\backslash B(p_j^\varepsilon,Kl)$ contains no a_i, Φ behaves more and more like a constant on $\partial B(p_j^\varepsilon,Rl)$ as R becomes large. That is $h\to0$ uniformly on $\partial B(0,R)$ as $R\to\infty$.

On the other hand, $\overline{\Phi}-\frac{1}{2\pi R}\int_{\partial B(0,R)}\overline{\Phi}-\Phi_0$ tends to 0 uniformly on $\partial B(0,R)$ and its Laplacian tends to 0 in the weak sense of measures on $B(0,R)$, we may conclude that the function converges to 0 uniformly on $B(0,R)$ as $\varepsilon\to0$. Then, using an integration by parts (and assuming $\frac{1}{2\pi R}\int_{\partial B(0,R)}\overline{\Phi}=0$ for simplicity), we have

$$\begin{aligned}&\int_{B(0,R)\backslash\cup_k B(b_{j,k},\rho)}|\nabla\overline{\Phi}-\nabla\Phi_0|^2\\&\qquad=\left|\int_{\partial B(0,R)}(\overline{\Phi}-\Phi_0)\frac{\partial}{\partial\nu}(\overline{\Phi}-\Phi_0)-\sum_k\int_{\partial B(b_{j,k},\rho)}(\overline{\Phi}-\Phi_0)\frac{\partial}{\partial\nu}(\overline{\Phi}-\Phi_0)\right|\to0\end{aligned}$$

in view of the bounds on $|\nabla\Phi|$ and $|\nabla\Phi_0|$. This proves (5.34).

Let us now prove (5.35). We observe that $S(x,y) = 2\pi(G_0(x,y) + G_h(x)) + \log|x-y|$ where G is written as $G_0 + G_h$, with

$$\begin{cases} -\Delta_x G_0(x,y) = \delta_y & \text{in } B(0,R) \\ G_0(x,y) = 0 & \text{on } \partial B(0,R), \end{cases} \tag{5.53}$$

and

$$\begin{cases} -\Delta G_h = 0 & \text{in } B(0,R) \\ G_h = h(x) & \text{on } \partial B(0,R). \end{cases} \tag{5.54}$$

It is a standard fact that G_0 is symmetric i.e. $G_0(x,y) = G_0(y,x)$. In fact there is an explicit expression (in complex coordinates)

$$G_0(x,y) = \frac{1}{2\pi}\log\left|\frac{R(x-y)}{R^2 - x\overline{y}}\right|. \tag{5.55}$$

Thus $S(x,y)$ is the sum of a symmetric function and a function that depends only on x. Now,

$$\overline{W}_R(b_{j,k}) - \overline{W}(b_{j,k}) = 2\pi\sum_{k,k'} D_{j,k}D_{j,k'}S(b_{j,k},b_{j,k'}) + \int_{\partial B(0,R)} h(x)\frac{\partial}{\partial\nu}\left(2\pi\sum_k D_{j,k}G(x,b_{j,k})\right),$$

hence

$$\nabla(\overline{W}_R(b_{j,k}) - \overline{W}(b_{j,k})) = 2\pi\sum_{k,k'} D_{j,k}D_{j,k'}\left(\nabla_x S(b_{j,k},b_{j,k'}) + \nabla_y S(b_{j,k},b_{j,k'})\right)$$
$$+ \int_{\partial B(0,R)} h(x)\frac{\partial}{\partial\nu}\left(2\pi\sum_k D_{j,k}\nabla_y G_0(x,b_{j,k})\right)$$

Thus, to conclude that (5.35) holds, it suffices to check that $\max_{x,y\in B(0,K+1)}|\nabla_x S|$ and $\max_{x,y\in B(0,K+1)}|\nabla_y S|$ tend to 0 as $R\to\infty$, and that $|\frac{\partial}{\partial\nu}\nabla_y G_0(x,b_{j,k})| \le \frac{C}{R^2}$. The second fact follows from the formula (5.55). For the first fact, use that $S(x,y) = 2\pi\left(G_0(x,y) + G_h(x,y)\right) + \log|x-y|$. G_h is harmonic, with values $h\to 0$ on $\partial B(0,R)$, hence tends to 0 in $C^1(B(0,K+1))$ as $R\to\infty$, by elliptic estimates. The remaining part of S is easy to handle.

□

Proof of Lemma 5.3. By an application of the standard lower bounds, since all the vortices converge to the $b_{j,k}$'s with total degree $D_{j,k} = \pm 1$, we have

$$\frac{1}{2}\int_{B(b_{j,k},\rho)} |\nabla\overline{u_\varepsilon}|^2 \ge \pi\log\frac{1}{\eta}(1 - o(1))$$

and since (5.16) holds, we must have (5.36).

On the other hand Theorem 1 of [S1] yields

$$\int_{B(0,R)} |\nabla|\overline{u_\varepsilon}||^2 + \frac{(1-|\overline{u_\varepsilon}|^2)^2}{\eta^2} \le o\left(l^2\log^2 l\|f_\varepsilon\|^2_{L^2(\Omega)} + 1\right)$$

and

$$\int_{B(0,R)} |\nabla\overline{\psi}|^2 \leq o\left(l^2 \log^4 l \|f_\varepsilon\|^2_{L^2(\Omega)} + 1\right), \tag{5.56}$$

where $\overline{\psi}$ is the blown-up of ψ. This proves (5.37). In addition, we can easily check that $|\nabla\overline{u_\varepsilon} - i\overline{u_\varepsilon}\nabla^\perp\Phi_0|^2 = |\nabla|\overline{u_\varepsilon}||^2 + |\overline{u_\varepsilon}|^2|\nabla\overline{\psi} + \nabla^\perp\overline{\Phi} - \nabla^\perp\Phi_0|^2$, hence

$$\int_{B(0,R)\setminus\cup_k B(b_{j,k},\rho)} |\nabla\overline{u_\varepsilon} - i\overline{u_\varepsilon}\nabla^\perp\Phi_0|^2 \leq \int_{B(0,R)\setminus\cup_k B(b_{j,k},\rho)} |\nabla|\overline{u_\varepsilon}||^2 + 2|\nabla\overline{\psi}|^2 + 2|\nabla(\overline{\Phi} - \Phi_0)|^2$$

and in view of (5.34) and (5.56), (5.38) follows.

□

References

[Al] G. Alberti, A variational convergence result for Ginzburg-Landau functionals in any dimension, *Boll. Unione Mat. Ital. Ser. B*, (2001), no. 2, 289–310.

[AB] L. Almeida and F. Bethuel, Topological methods for the Ginzburg-Landau Equations. *J. Math. Pures Appl. (9)* **77** (1998), no. 1, 1–49.

[BBH] F. Bethuel, H. Brezis and F. Hélein, *Ginzburg-Landau vortices,* Birkhaüser, (1994).

[BCPS] P. Bauman, C.N. Chen, D. Phillips and P. Sternberg, Vortex annihilation in nonlinear heat flow for Ginzburg-Landau systems, *European J. Appl. Math.* **6** *(1995), no. 2, 115–126.*

[BOS1] F. Bethuel, G. Orlandi and D. Smets, Collisions and phase-vortex interactions in dissipative Ginzburg-Landau dynamics, to appear in *Duke Math. J.*

[BOS2] F. Bethuel, G. Orlandi, and D. Smets, Quantization and motion law for Ginzburg-Landau vortices, preprint.

[JS] R.L. Jerrard and H.M. Soner, Dynamics of Ginzburg-Landau vortices, *Arch. Rational Mech. Anal.* **142** (1998), no. 2, 99–125.

[JS2] R.L. Jerrard and H.M. Soner, The Jacobian and the Ginzburg-Landau functional, *Calc. Var. PDE.* **14** (2002), no. 2, 151-191.

[Li] F.H. Lin, Some Dynamical Properties of Ginzburg-Landau Vortices, *Comm. Pure Appl. Math.* **49** (1996), 323-359.

[M] P. Mironescu, Les minimiseurs locaux pour l'équation de Ginzburg-Landau sont à symétrie radiale. *C. R. Acad. Sci. Paris, Ser. I* **323** (1996), no 6, 593–598.

[RS] J. Rubinstein and P. Sternberg, On the slow motion of vortices in the Ginzburg-Landau heat-flow, *SIAM J. Appl. Math* **26** (1995), 1452-1466.

[OS] Y. N. Ovchinnikov and I. M. Sigal, Symmetry-breaking solutions to the Ginzburg-Landau equation. *J. Exp. Theor. Phys. 99*, (2004), no. 5, 1090-1107.

[Sa1] E. Sandier, Lower bounds for the energy of unit vector fields and applications. *J. Funct. Anal.* **152** (1998), no. 2, 379–403; Erratum, Ibid, **171** (2000), no. 1, 233.

[Sa2] E. Sandier, Locally minimising solutions of $-\Delta u = u(1-|u|^2)$ in R^2. *Proc. Roy. Soc. Edinburgh Sect. A* **128** (1998), no. 2, 349–358.

[So] H. M. Soner, Ginzburg-Landau equation and motion by mean curvature. I. Convergence II. Development of the initial interface, *J. Geom. Anal.* **7** (1997), no. 3, 437-491.

[SS1] E. Sandier and S. Serfaty, A product estimate for Ginzburg-Landau and corollaries, *J. Funct. Anal.* **211** (2004), no. 1, 219–244.

[SS2] E. Sandier and S. Serfaty, Gamma-convergence of gradient flows and application to Ginzburg-Landau, *Comm. Pure Appl. Math.* **57** (2004), no. 12, 1627–1672.

[SS3] E. Sandier and S. Serfaty, A Rigorous Derivation of a Free-Boundary Problem Arising in Superconductivity, *Annales Sci. Ecole Normale Sup., 4e série,* **33** (2000), 561-592.

[SS4] E. Sandier and S. Serfaty, *Vortices in the Magnetic Ginzburg-Landau Model*, monograph to appear.

[S1] S. Serfaty, Vortex-collisions and energy-dissipation rates in the Ginzburg-Landau heat flow, Part I: Study of the perturbed Ginzburg-Landau equation, to appear in *J. Eur. Math. Soc.*

[S2] S. Serfaty, Stability in 2D Ginzburg-Landau Passes to the Limit, *Indiana Univ. Math. J* **54** (2005), no. 1, 199–222.

[Sp] D. Spirn, Vortex dynamics for the full time-dependent Ginzburg-Landau equations, *Comm Pure Appl Math* **55** (2002), no. 5, 537-581.