SYSTEMS OF POINTS WITH COULOMB INTERACTIONS

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ABSTRACT. Large ensembles of points with Coulomb interactions arise in various settings of condensed matter physics, classical and quantum mechanics, and even approximation theory, and give rise to a variety of questions pertaining to calculus of variations, partial differential equations, and probability. We will review motivations from these fields as well as "mean-field limit" results that allow us to derive effective models and equations describing those systems at the macroscopic scale. We then explain how to analyze the next-order beyond the mean-field limit, obtaining information about systems at the microscopic level. In the setting of statistical mechanics this allows, for instance, to observe the effect of temperature and to connect with crystallization questions.

1. General setup

The 18th century physicist Charles-Augustin de Coulomb was the first to postulate that electrically charged particles interact with one another by a force proportional to the inverse square of their distances, in a way similar to Newton's gravitational force. In this paper we are interested in large systems of points (or particles) interacting by such forces, having as motivation, besides the case of classical mechanics, numerous other situations that we will detail below.

Recalling that the force is the gradient of the energy, we consider a system of N particles with energy of the form

$$(1.1) \qquad \mathcal{H}_N(x_1, \ldots, x_N) = \frac{1}{2} \sum_{1 \leq i \neq j \leq N} \mathsf{g}(x_i - x_j) + N \sum_{i=1}^{N} V(x_i).$$

Here the points x_i belong to the Euclidean space $\mathbb{R}^\mathsf{d}$, although it is also interesting to consider points on manifolds. The interaction kernel $\mathsf{g}(x)$ is taken to be

$$(1.2) \qquad \mathsf{g}(x) = -\log|x|, \quad \text{in dimension } \mathsf{d} = 2,$$

$$(1.3) \qquad \mathsf{g}(x) = \frac{1}{|x|^{\mathsf{d}-2}}, \quad \text{in dimension } \mathsf{d} \geq 3.$$

Up to a multiplicative constant, this is the Coulomb kernel in dimension $\mathsf{d} \geq 2$, i.e. the fundamental solution to the Laplace operator, solving

$$(1.4) \qquad -\Delta \mathsf{g} = \mathsf{c}_\mathsf{d} \delta_0$$

where δ_0 is the Dirac mass at the origin, and c_d is an explicit constant depending only on the dimension.

It is also interesting to broaden the study to the one-dimensional logarithmic case

(1.5) $$\mathsf{g}(x) = -\log|x|, \quad \text{in dimension } \mathsf{d} = 1,$$

which is not Coulombian, and to more general Riesz interaction kernels of the form

(1.6) $$\mathsf{g}(x) = \frac{1}{|x|^{\mathsf{s}}} \quad \mathsf{s} > 0.$$

The one-dimensional Coulomb interaction with kernel $-|x|$ is also of interest, but has been extensively studied and is well understood.

We also include a possible external field or confining potential V, which is assumed to be sufficiently smooth and tending to infinity fast enough at infinity. The factor N in front of V makes the total confinement energy of the same order as the total repulsion energy, effectively balancing them and confining the system to a subset of $\mathbb{R}^d$ of fixed size.

The Coulomb interaction and the Laplace operator are obviously extremely important and ubiquitous in physics as the fundamental interactions of nature (gravitational and electromagnetic) are Coulombic. Below we will further review the reasons for studying this type of systems.

There are several mathematical problems that are interesting to study, all in the asymptotic limit of $N \to \infty$:

(1) understand the minimizers and possibly critical points of (1.1) ;

(2) understand the statistical mechanics of systems with energy $\mathcal{H}_N$ and inverse temperature $\beta > 0$, governed by the so-called Gibbs measure

(1.7) $$d\mathbb{P}_{N,\beta}(x_1, \ldots, x_N) = \frac{1}{Z_{N,\beta}} e^{-\beta \mathcal{H}_N(x_1, \ldots, x_N)} dx_1 \ldots dx_N.$$

Here $\mathbb{P}_{N,\beta}$ is the probability density of observing the system in the configuration $(x_1, \ldots, x_N)$ if the inverse of the temperature is $1/\beta$. The constant $Z_{N,\beta}$, which is called the "partition function" in physics, is the normalization constant that makes $\mathbb{P}_{N,\beta}$ a probability measure[1] i.e.

(1.8) $$Z_{N,\beta} = \int_{(\mathbb{R}^{\mathsf{d}})^N} e^{-\beta \mathcal{H}_N(x_1, \ldots, x_N)} dx_1 \ldots dx_N;$$

(3) understand the dynamic evolutions associated to (1.1), such as the gradient flow of $\mathcal{H}_N$ given by the system of coupled ODEs

(1.9) $$\dot{x}_i = -\frac{1}{N} \nabla_i \mathcal{H}_N(x_1, \ldots, x_N),$$

the conservative dynamics given by the systems of ODEs

(1.10) $$\dot{x}_i = \frac{1}{N} \mathbb{J} \nabla_i \mathcal{H}_N(x_1, \ldots, x_N),$$

where $\mathbb{J}$ is an antisymmetric matrix, or the Hamiltonian dynamics given by Newton's law

(1.11) $$\ddot{x}_i = -\frac{1}{N} \nabla_i \mathcal{H}_N(x_1, \ldots, x_N);$$

and we can also be interested in these dynamics with an added noise.

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[1]One does not know how to explicitly compute the integrals (1.8) except in the particular case of (1.5) for specific cases of V where they are called Selberg integrals (cf. [Fo])

From a mathematical point of view, the study of such systems touches on several fields of mathematical analysis (partial differential equations and calculus of variations, approximation theory), probability theory, mathematical physics, and even geometry (when one considers such systems on manifolds). Some of the crystallization questions they lead to also overlap with number theory, as we will see below.

2. Motivations

There is a large number of motivations for the study of the above questions. We briefly describe some of them:

(1) In superconductors, superfluids, and Bose-Einstein condensates, one observes the occurrence of quantized "vortices" which behave mathematically like interacting particles with two-dimensional Coulomb interactions. In these systems the vortices repel each other logarithmically, while being confined together by the effect of the magnetic field or rotation, and the result of the competition between these two effects is that, as predicted by Abrikosov, the vortices arrange themselves in a perfect triangular lattice pattern, called *Abrikosov lattice*, cf. Figure 1 (for more pictures, see `www.fys.uio.no/super/vortex/`).
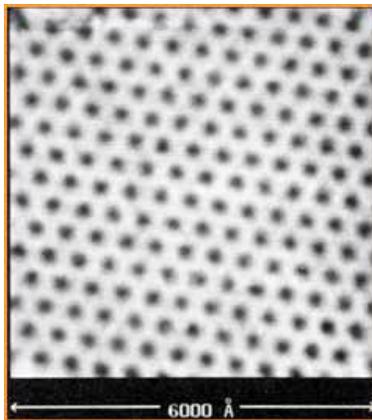

FIGURE 1. Vortices (in black) forming an Abrikosov lattice. H. F. Hess et al. Bell Labs *Phys. Rev. Lett.* 62, 214 (1989).

These systems are, in fact, described by an energy (the Ginzburg-Landau energy) and the associated PDEs, but we can show rigorously (in a study started by Bethuel-Brezis-Hélein and continued by [SS], see also [Se1]) that, in the case (1.2), the analysis of the vortices is reduced to the discrete problems described above.

Another motivation is the analysis of vortices in classical fluids, such as initiated by Onsager, see [MP], or in plasma physics.

(2) Fekete points in approximation theory: these points arise in interpolation theory as the points minimizing interpolation errors for numerical integration. They are defined as those points maximizing the quantity

$$\prod_{i \neq j} |x_i - x_j|$$

or, equivalently, minimizing

$$-\sum_{i \neq j} \log |x_i - x_j|.$$

They are often studied on the sphere, or on other manifolds. In approximation theory [SK] we are also interested in the minimization of Riesz energies

(2.1) $$\sum_{i \neq j} \frac{1}{|x_i - x_j|^{\mathsf{s}}}$$

for all values of s. One can show that, by letting $\mathsf{s} \to 0$, the minimizers of Riesz energies converge to those of the logarithmic energy, whereas when $\mathsf{s} \to \infty$ they converge to the minimizers of the optimal sphere packing problem (whose solution in dimension 2 is known, from Fejes Tóth, to be also the triangular lattice represented in Figure 2) It has been proved by
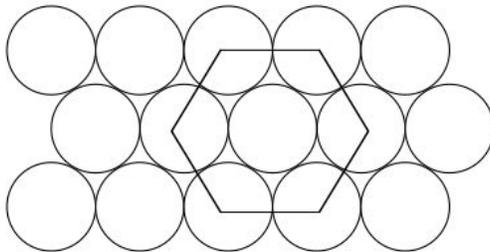

FIGURE 2. Solution of the sphere packing problem in dimension 2.

Hales that the solution of the same packing problem in dimension 3 is an FCC (face-centered cubic) lattice, as was conjectured by Kepler. In higher dimensions, the solution is only known in dimensions 8 and 24, due to a recent breakthrough by Viazovska (see the presentation in [Coh] and the review [Sl]). In high dimensions, where the problem is important for error correcting codes, the solution is expected *not* to be a lattice.

(3) Statistical mechanics and quantum mechanics: In physics the ensemble given by (1.7) in the Coulomb case is called a two-dimensional Coulomb gas or one-component plasma and is a classical ensemble of statistical mechanics whose analysis is considered difficult due to the long range of the interactions. The study of the two-dimensional Coulomb gas, as well as the one-dimensional log gas, is also motivated by the analysis of certain quantum wave-functions (fractional quantum Hall effect, free fermions in a magnetic field, . . .), and well as by several stochastic models in Probability, cf. [Fo]. The variant of the two-dimensional Coulomb case with coexisting positive and negative charges is interesting in certain theoretical physics

models (XY-model, sine-Gordon) which exhibit a Kosterlitz-Thouless phase transition (see [Spe]).

(4) Random matrices (see [Fo]): in the particular cases (1.5) and (1.2), the Gibbs measure (1.7) corresponds, in certain instances, to the distribution law of the eigenvalues of certain well known ensembles:

- the law of the complex eigenvalues of an $N \times N$ matrix where the entries are Gaussian i.i.d. is (1.7) with (1.2), $\beta = 2$, and $V(x) = |x|^2$. This is called the Ginibre ensemble.
- the law of the real eigenvalues of an $N \times N$ Hermitean matrix with complex Gaussian i.i.d. entries is (1.7) with (1.5), $\beta = 2$ and $V(x) = x^2/2$. This is called the GUE (unitary Gaussian) ensemble.
- the law of the real eigenvalues of an $N \times N$ symmetric matrix with Gaussian i.i.d. entries is (1.7) with (1.5), $\beta = 1$ and $V(x) = x^2/2$. This is called the GOE (orthogonal Gaussian) ensemble.

(5) Complex geometry provides other examples of motivations. See, for instance, the works of Robert Berman and co-authors.

3. The mean-field limit and theoretical physics

3.1. **Questions.** The first question that naturally arises is to understand the limit as $N \to \infty$ of the *empirical measure* defined by

$$(3.1) \qquad \mu_N := \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}$$

for configurations of points that minimize the energy (1.1), critical points, solutions of the evolution problems presented above, or typical configurations under the Gibbs measure (1.7), thus hoping to derive effective equations or minimization problems that describe the average or mean-field behavior of the system. The term mean-field refers to the fact that, from the physics point of view, each particle feels the collective (mean) field $\mathsf{g} * \mu_N$ generated by all other particles. Convergence in the mean-field sense is, thus, equivalent, in some sense, to the "propagation of molecular chaos" (see [Go]). From the statistical mechanics point of view, we also try to understand the temperature dependence of the behavior of the system and the eventual occurrence of phase transitions.

3.2. **The equilibrium measure.** The energy (1.1) can be written as

$$\mathcal{H}_N(x_1, \ldots, x_N) = N^2 \left(\frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus \triangle} \mathsf{g}(x - y) d\mu_N(x) d\mu_N(y) + \int_{\mathbb{R}^d} V(x) d\mu_N(x) \right)$$

where $\triangle$ denotes the diagonal of $\mathbb{R}^d \times \mathbb{R}^d$. Thus, it is natural to consider the "continuum version" of the energy, namely:

$$\mathcal{I}_V(\mu) := \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \mathsf{g}(x - y) d\mu(x) d\mu(y) + \int_{\mathbb{R}^d} V(x) d\mu(x).$$

It is well known from potential theory that, in the space of probability measures, $\mathcal{I}_V$ admits a unique minimizer, μ_V, which is called the *equilibrium measure*, and is characterized by the fact that there exists a constant c such that

$$(3.2) \qquad \begin{cases} h^{\mu_V} + V \geq c & \text{in } \mathbb{R}^d \\ h^{\mu_V} + V = c & \text{in the support of } \mu_V \end{cases}$$

where

$$(3.3) \qquad h^{\mu_V}(x) := \int_{\mathbb{R}^{\mathsf{d}}} \mathsf{g}(x - y) d\mu_V(y)$$

is the (electric) potential generated by μ_V. This is true for Coulomb and for Riesz interactions, as well as for more general kernels. In the Coulomb case, the equilibrium measure can be interpreted with the help of an obstacle problem (in the Riesz case with a fractional obstacle problem), cf. [Se1, Chap. 2]. An example is provided by Coulomb interaction (in any dimension) with confinement potential $V = c|x|^2$: in this case we can verify that the equilibrium measure is always (a multiple of the) characteristic function of a ball. In the context of the Ginibre ensemble in Random Matrix Theory, this is known as the "circle law". Another important example is that of the logarithm interaction in dimension 1 with quadratic potential V: the equilibrium measure has density $\sqrt{x^2 - a^2}\mathbf{1}_{|x|<a}$, known, in Random Matrix Theory, as the (Wigner) *semi-circle law* for the ensembles GOE and GUE.

The energy $\mathcal{I}_V$ is the "mean-field limit" of the energy $\mathcal{H}_N$, and one can show without much difficulty that, for the minimizers of $\mathcal{H}_N$, the empirical measure converges to μ_V, and $\frac{1}{N^2} \min \mathcal{H}_N$ converges to $\mathcal{I}_V(\mu_V)$.

We can interpret $\nabla(h^\mu + V)$ as the total mean-force felt by a distribution with density μ. Therefore, in view of (3.2) it is null for the minimizers. More generally, we expect that the critical points of $\mathcal{H}_N$ have a limiting empirical distribution satisfying

$$(3.4) \qquad \nabla(h^\mu + V)\mu = 0.$$

For the dynamics **(3)**, the formal limit of (1.9) or (1.10) is

$$(3.5) \qquad \partial_t \mu = -\mathrm{div}\left(\nabla(h^\mu + V)\mu\right),$$

or

$$(3.6) \qquad \partial_t \mu = -\mathrm{div}\left(\mathbb{J}\nabla(h^\mu + V)\mu\right),$$

again with $h^\mu = \mathsf{g} * \mu$. In the case (1.2), (3.6) with $V = 0$ is also well known as the vorticity form of Euler's equation.

The difficulty in rigorously proving the convergence towards solutions of these equations (whose well posedness also needs to be proved) consists in passing to the limit in the products of the type $\nabla h^\mu \mu$, which are nonlinear and, a priori, ill defined in the energy space. In the case of (1.2) we can overcome these difficulties by the reformulation of these terms introduced by Delort in the context of his works in fluid mechanics, but this approach does not work in higher dimensions.

Until recently, all convergence results were limited to sub-Coulomb singularities ($\mathsf{s} < \mathsf{d} - 2$) or to dimension 1. Recently, a modulated energy method developed in [Se2] for the mean-field limit of the Ginzburg-Landau equations, based on the stability of solutions of the limiting equations for the "Coulomb norm" (or "Riesz norm")

$$\|\mu\|^2 = \iint \mathsf{g}(x - y) d\mu(x) d\mu(y),$$

allowed for the treatment of Coulomb interactions and for more singular Riesz cases:

Theorem 1 ([Se3]). *For the dynamics (1.9) and (1.10), for all* d*, and all* $\mathsf{s} \in [\mathsf{d}-2, \mathsf{d})$ *in (1.6), or (1.5) or (1.2), the empirical measures converge to the solutions*

of (3.5) or (3.6), when $N \to +\infty$, provided these are sufficiently smooth and the initial data energies converge to those of their limits.

This result was preceded by one by Duerinckx in dimension 1 and 2 for $\mathsf{s} < 1$.

As far as (1.11) is concerned, the limiting equation is formally found to be the Vlasov-Poisson equation

$$(3.7) \qquad \partial_t \rho + v \cdot \nabla_x \rho + \nabla(h^\mu + V) \cdot \nabla_v \rho = 0,$$

where $\rho(t, x, v)$ is the density of particles at time t with position x and velocity v, and $\mu(t, x) = \int \rho(t, x, v) dv$ is the density of particles. Notwithstanding recent progresses, we do not yet know how to prove convergence of (1.11) to (3.7) when the interaction is Coulomb or have an stronger singularity. About this topic, one can consult the reviews [Jab, Go].

3.3. **With temperature: statistical mechanics.** Let us now consider (1.8) and turn our attention to problem (2). It is known that even with temperature the behavior of the system is still governed by the equilibrium measure. The result can be phrased using the language of Large Deviations Principles, and states, essentially, that if E is a subset of the space of probability measures, after identifying the configurations $(x_1, \ldots, x_N)$ in $(\mathbb{R}^\mathsf{d})^N$ with their empirical measures, we have

$$(3.8) \qquad \mathbb{P}_{N,\beta}(E) \approx e^{-\beta N^2 (\min_E \mathcal{I}_V - \min \mathcal{I}_V)},$$

which implies, due to the uniqueness of the minimizer μ_V of $\mathcal{I}_V$, that the configurations for which the empirical measure do not converge to μ_V have a very small probability. For example, in the case of matrices in GOE or GUE, for which the equilibrium measure is the semi-circle law, we deduce as an application a corollary of a result by Ben Arous and Guionnet: the probability that a GOE or GUE matrix is definite positive (and thus, that all their eigenvalues are positive, which is incompatible with the semi-circle law which is symmetric relative to 0) decreases like e^{-cN^2}.

In other words, at this leading order, temperature does not affect the mean-field behavior of the system. (This is not what happens if we replace β by β/N: in this case we have a modified equilibrium measure which spreads out with the temperature, minimizing $\beta \mathcal{I}_V(\mu) + \int \mu \log \mu$).

4. Beyond mean-field

In order to observe, for example, the effect of temperature (see Figure 3) it is interesting to go beyond the mean-field limit: expanding the energy $\mathcal{H}_N$ to next order we have, at the same time, access to information about the typical *microscopic* behavior of the configurations. Observe that, at the microscopic scale, the typical distance between nearest neighbors is $N^{-1/\mathsf{d}}$.

4.1. **Rigidity and Gaussian fluctuations.** For minimizers of the energy $\mathcal{H}_N$ or of typical configurations under (1.7), since one already knows that $\sum_{i=1}^N \delta_{x_i} - N\mu_V$ is small, one knows, for instance, that the so-called discrepancy in balls $B_r(x)$, defined by

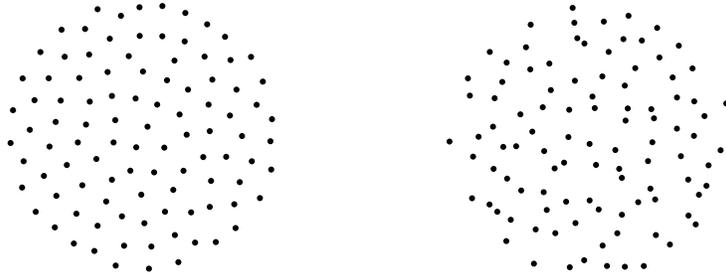$$D(x, r) := \int_{B_r(x)} \sum_{i=1}^N \delta_{x_i} - N \, d\mu_V$$

FIGURE 3. Case (1.2) with $N = 100$ and $V(x) = |x|^2$, for $\beta = 400$ (left) and $\beta = 5$ (right).

is of order $o(r^{\mathsf{d}} N)$, for fixed $r > 0$. It can be asked whether this estimation can be refined, and if it remains true at mesoscopic scales, i.e., for r of order $N^{-\alpha}$ with $\alpha < 1/\mathsf{d}$, and for all temperatures. This would correspond to a *rigidity result*. We do get such a result for the energy minimizers. For configurations with temperature, in the context of bi-dimensional Coulomb interactions, we can prove a slightly different form of such a result: it is true when we integrate $\sum_{i=1}^{N} \delta_{x_i} - N\mu_V$ not over a ball, but against a sufficiently smooth test function. In this way we get an even more precise result, since we can prove that these quantities converge to a Gaussian with explicitly known mean and variance:

Theorem 2 ([LS2]). *In case (1.2), let us assume that $V \in C^4$ and μ_V has connected support Σ with a regular boundary. Let $f \in C_c^3(\Sigma)$. Then*

$$\sum_{i=1}^{N} f(x_i) - N \int_{\Sigma} f \, d\mu_V$$

converges in law to a Gaussian with

$$mean = \frac{1}{2\pi}\left(\frac{1}{\beta} - \frac{1}{4}\right)\int_{\mathbb{R}^2} \Delta f \log \Delta V \qquad variance = \frac{1}{2\pi\beta}\int_{\mathbb{R}^2} |\nabla f|^2.$$

This result can be localized with test-functions f supported on any mesoscale $N^{-\alpha}$, $\alpha < \frac{1}{2}$. It is also true for energy minimizers, taking formally $\beta = \infty$.

For an idea of the proof we suggest the lecture notes [Se4].

This result can be interpreted in terms of the convergence to a suitable *Gaussian free field*, a sort of two-dimensional analogue of Brownian motion. Note that a similar result was obtained by Bauerschmidt-Bourgade-Nikula-Yau, and it was previously known for $\beta = 2$, and in the uni-dimensional logarithm case for all values of β.

If f is sufficiently smooth, the associated fluctuations are typically of order 1, i.e. much smaller than we could expect, for example comparing with the standard Central Limit Theorem where the fluctuation of the sum of N i.i.d. random variables is typically of order $\sqrt{N}$. Proving this result in higher dimension or for more general interactions remains an open problem.

4.2. **Next order in the energy.** As we pointed out above, the approach we employ (initiated with Etienne Sandier, and continued with Nicolas Rougerie, Mircea Petrache, and Thomas Leblé) consists in studying the next order of the expansion of the energy about the measure $N\mu_V$, which is formaly the minimizer. Expanding and using the characterization (3.2), the "order 1" terms in $\sum_{i=1}^{N} \delta_{x_i} - N\mu_V$ vanish and we obtain

$$(4.1) \qquad \mathcal{H}_N(x_1, \ldots, x_N) = N^2 \mathcal{I}_V(\mu_V) + F_N^{\mu_V}(x_1, \ldots, x_N)$$

where
(4.2)
$$F_N^{\mu_V}(x_1, \ldots, x_N) = \frac{1}{2} \iint_{\triangle^c} \mathsf{g}(x-y) d\Big(\sum_{i=1}^{N} \delta_{x_i} - N\mu_V\Big)(x) d\Big(\sum_{i=1}^{N} \delta_{x_i} - N\mu_V\Big)(y),$$

and again $\triangle$ denote the diagonal $\mathbb{R}^{\mathsf{d}} \times \mathbb{R}^{\mathsf{d}}$. This is a next-order expansion of $\mathcal{H}_N$ valid for arbitrary configurations.

The "next-order energy" $F_N^{\mu_V}$ can be seen as the total Coulomb energy of the neutral system formed by the N positive point charges at the points x_i and the diffuse negative charge $-N\mu_V$ with the same mass. The goal is now to define a limit of this energy when $N \to \infty$, that will be the total Coulomb energy (per unit volume) of an infinite system of positive charges and a (let us say) uniformly distributed negative charge. In physics such a system is called a *jellium*. The precise definition of this limiting energy is a bit complex, but it uses, in a crucial way, the Coulomb nature of the interaction. In fact, since g is the kernel of the Laplacian, we observe that if $h^\mu = \mathsf{g} * \mu$ is the electrostatic potential generated by a charge distribution μ (with zero integral), then h^μ solves Poisson equation

$$-\Delta h^\mu = \mathsf{c_d}\mu,$$

which is a local elliptic PDE, and, aditionally, using the Gauss-Green formula, we can write

$$\iint_{\mathbb{R}^{\mathsf{d}} \times \mathbb{R}^{\mathsf{d}}} \mathsf{g}(x-y) d\mu(x) d\mu(y) = -\frac{1}{\mathsf{c_d}} \int_{\mathbb{R}^{\mathsf{d}}} h^\mu \Delta h^\mu = \frac{1}{\mathsf{c_d}} \int_{\mathbb{R}^{\mathsf{d}}} |\nabla h^\mu|^2.$$

In another way, we can rewrite the interaction energy (which involves a double integral) in the form of a single integral of a local function of the electrostatic (or Coulomb) potential generated by this distribution, itself a solution of a local equation. In Riesz's case, these manipulations can be replaced by similar ones using the fact that g is the kernel of an elliptic operator in divergence form, which is still local.

With the help of this observation we succeed in defining an infinite volume energy for an infinite configuration of points $\mathcal{C}$ neutralized by a distributed charge (let us say -1), via the solutions of

$$-\Delta H = \mathsf{c_d} \left(\sum_{p \in \mathcal{C}} \delta_p - 1 \right).$$

We shall denote this energy by $\mathbb{W}(\mathcal{C})$. When the configuration of points $\mathcal{C}$ is periodic with respect to a lattice Λ, the energy $\mathbb{W}(\mathcal{C})$ has an explicit form: if there are M

points a_i in the fundamental cell we have (up to constants)

$$\mathbb{W}(\mathcal{C}) = \sum_{1 \leq i \neq j \leq M} G_{\mathbb{T}}(a_i, a_j)$$

where $G_{\mathbb{T}}$ is the Green function of the torus $\mathbb{T} := \mathbb{R}^{\mathsf{d}}/\Lambda$.

We can show that $\mathbb{W}$ can be obtained as the limit (in a certain sense) of the functional $F_N^{\mu_V}$ in (4.1). It also follows from an expansion to the next order of the minimum of the energy $\mathcal{H}_N$ and from the fact that, after dilation, the minimizers of $\mathcal{H}_N$ must converge (almost everywhere with respect to the origin of the dilation) to a minimizer of $\mathbb{W}$ (see, for example, [Se1]).

We are therefore led to try to determine the minimizers of $\mathbb{W}$. This problem is extremely difficult, with the exception of the one dimensional case, where we can prove that the minimum of $\mathbb{W}$ is attained by the lattice $\mathbb{Z}$. In dimension 2 and higher the problem remains open and the only positive result is the following:

Theorem 3. *The minimum of $\mathbb{W}$ over lattices of volume 1 in dimension 2 is achieved uniquely by the triangular lattice.*

Here the triangular lattice means $\mathbb{Z} + \mathbb{Z}e^{i\pi/3}$, properly scaled, i.e., what is called the Abrikosov lattice in the context of superconductivity. This partial result is, in fact, a result from Number Theory, known since the 1950s, about the minimization of Epstein's zeta function (cf. [Mont] and references therein). It corresponds to minimizing the height of a flat torus in Arakelov geometry.

Since the triangular lattice is observed in experiments with superconductors, and since we have proved that the minimization of the Ginzburg-Landau energy of the superconductor reduces to that of $\mathbb{W}$ [SS], it is natural to conjecture that the triangular lattice is a global minimizer of the energy.

According to a conjecture of Cohn-Kumar, the triangular lattice should be a universal minimizer in dimension 2 (i.e., should minimize a large class of interaction energies). An analogous role is played in dimensions 8 by the lattice E_8, and in dimension 24 by the Leech lattice, which are the solutions of the optimal packing problem, as was recently proved [Coh]. In these dimensions the prove of the universal minimization is near at hand.

In dimensions $\mathsf{d} \geq 3$ (except for $\mathsf{d} = 8$ and $\mathsf{d} = 24$), the minimization of $\mathbb{W}$, even among lattices, is an open problem. As before, we can think that this relative minimum is global, but we expect this to be true only in low dimensions since computer simulations provide clear indications that in dimensions $\mathsf{d} \geq 9$ the minimizers are not lattices.

These questions belong to the more general family of crystallization problems for which very few positive results are known once the dimension is larger than or equal to 2 (cf. the review [BLe]).

4.3. **Next order with temperature.** In order to observe interesting temperature effects, as well as for applications to random matrices, we must consider $\beta_N = \beta N^{\frac{2}{\mathsf{d}}-1}$.

As we saw above, the macroscopic (or "mean-field") behavior of the system does not depend upon the temperature and is given by the equilibrium measure. In the other hand, one shows that the microscopic behavior depends on the temperature and is governed by a weighted sum of the energy $\mathbb{W}$ in the previous paragraph and of a relative entropy. To formulate the result, one needs to dilate the configurations
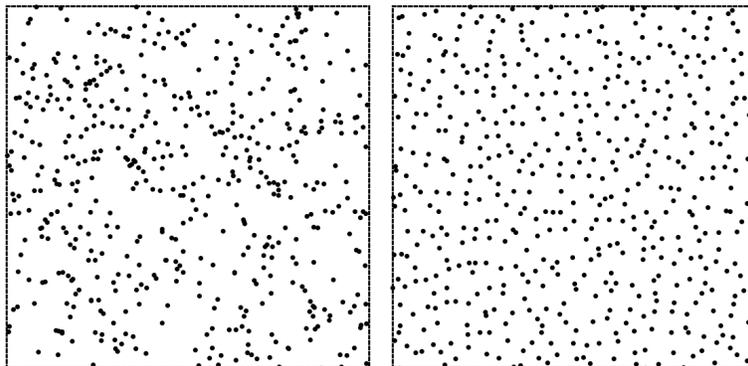
FIGURE 4. Simulation of the Poisson point process with intensity 1 (left), and of the Ginibre process (right)

by $N^{-1/\mathsf{d}}$, as in the previous paragraph, and consider the limiting point process P^x obtained by averaging near each point x. Here a point process is a law on infinite configurations of points. For instance, Ginibre process is obtained by passing to the limit $N \to \infty$ (after dilation) of the Ginibre ensemble; Poisson process Π with intensity 1 corresponds to points thrown independently of each other in such a way that the probability of having $N(B)$ points in a set B is

$$\Pi\left(N(B) = n\right) = \frac{|B|^n}{n!} e^{-|B|}.$$

Thus, one defines a "specific relative entropy" with respect to the Poisson process, denoted by $\mathsf{ent}[\cdot|\Pi]$, that we can think of as measuring how much the process P is close to Poisson.

For all $\beta > 0$, we define the functional $\mathcal{F}_\beta$

(4.3) $$\mathcal{F}_\beta(P) := \int_\Sigma \frac{\beta}{2} \mathbb{W}(P^x) + \mathsf{ent}[P^x|\Pi] dx,$$

with $P = \int_\Sigma P^x dx$. We can now formulate a large deviations result.

Theorem 4 ([LS1]). *For all cases* (1.5), (1.2) *and* (1.3) *with* $\mathsf{d} - 2 \le \mathsf{s} < \mathsf{d}$, *with smooth assumptions on V and μ_V, and for all $\beta > 0$, we have a Large Deviations Principle at speed N with rate function $\mathcal{F}_\beta - \inf \mathcal{F}_\beta$, in the sense that*

$$\mathbb{P}_{N,\beta}(P_N \simeq P) \simeq e^{-N(\mathcal{F}_\beta(P) - \inf \mathcal{F}_\beta)}.$$

In this way, the Gibbs measure $\mathbb{P}_{N,\beta}$ concentrates on microscopic point processes which minimize $\mathcal{F}_\beta$. This minimization is due to a competition between energy and entropy. When $\beta \to 0$ the entropy dominates and we can prove that the limit processes converge to a Poisson process. When $\beta \to \infty$, the energy $\mathbb{W}$ dominates, which, heuristically, forces the configurations to be more "ordered" and to converge to the minimizers of $\mathbb{W}$. Between these two extremes we have intermediate situations and to know if there is a critical β corresponding to a crystalization, or to a liquid-solid phase transition (which is conjectured to take place for (1.2) in some physics papers), is a problem that remains open. In dimension 1, on the other

hand, due to the fact that we can identify the minimizers of $\mathbb{W}$, it can be concluded that a true crystallization result holds when temperature tends to 0.

One consequence of this result is to provide a variational interpretation for the few known limiting processes: the so-called "sine-β" process, limit in the uni-dimensional case (1.5), and the Ginibre's process: they must minimize $\beta\mathbb{W} + \mathsf{ent}$.

We would like to obtain more information about the limiting point processes, namely the behavior and decay of the "two-points correlation functions", which would shed light on the existence of phase transitions and crystallization. Unfortunately, this theorem does not seem to provide much help for those problems.

As we have seen, many questions remain open, notably those of crystallization, of identification of the minimizers and of the minima of $\mathbb{W}$ and $\mathcal{F}_\beta$, and of the generalization of Theorem 2 to dimensions $\mathsf{d} \geq 3$, to Riesz interactions, and even to more general interactions.

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