The Γ-limit of the two-dimensional Ohta-Kawasaki energy. I. Droplet density.

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Abstract

This is the first in a series of two papers in which we derive a Γ-expansion for a two-dimensional non-local Ginzburg-Landau energy with Coulomb repulsion, also known as the Ohta-Kawasaki model in connection with diblock copolymer systems. In that model, two phases appear, which interact via a nonlocal Coulomb type energy. We focus on the regime where one of the phases has very small volume fraction, thus creating small "droplets" of the minority phase in a "sea" of the majority phase. In this paper we show that an appropriate setting for Γ-convergence in the considered parameter regime is via weak convergence of the suitably normalized charge density in the sense of measures. We prove that, after a suitable rescaling, the Ohta-Kawasaki energy functional Γ-converges to a quadratic energy functional of the limit charge density generated by the screened Coulomb kernel. A consequence of our results is that minimizers (or almost minimizers) of the energy have droplets which are almost all asymptotically round, have the same radius and are uniformly distributed in the domain. The proof relies mainly on the analysis of the sharp interface version of the energy, with the connection to the original diffuse interface model obtained via matching upper and lower bounds for the energy. We thus also obtain an asymptotic characterization of the energy minimizers in the diffuse interface model.

1 Introduction

In the studies of energy-driven pattern formation, one often encounters variational problems with competing terms operating on different spatial scales [25,26,32,39,50,53,55]. Despite the fundamental importance of these problems to a multitude of physical systems, their detailed mathematical studies are fairly recent (see e.g. [7–11,19,29,48]). To a great extent this fact is related to the emerging multiscale structure of the energy minimizing patterns and the associated difficulty of their description [8,10,16,30,35]. In particular, the popular approach of Γ-convergence [4] is rendered difficult due to the emergence of more than two well-separated spatial scales in suitable asymptotic limits (see e.g. [8–11,16,30,35,49]).

These issues can be readily seen in the case of the Ohta-Kawasaki model, a canonical mathematical model in the studies of energy-driven pattern forming systems. This model, originally proposed in [42] to describe different morphologies observed in diblock copolymer melts (see e.g. [3]) is defined (up to a choice of scales) by the energy functional

$$\mathcal{E}[u] = \int_\Omega \left(\frac{\varepsilon^2}{2} |\nabla u|^2 + W(u) \right) dx + \frac{1}{2} \int_\Omega \int_\Omega (u(x) - \bar{u}) G_0(x,y)(u(y) - \bar{u})\, dx\, dy, \tag{1.1}$$

where Ω is the domain occupied by the material, $u : \Omega \to \mathbb{R}$ is the scalar order parameter, $W(u)$ is a symmetric double-well potential with minima at $u = \pm 1$, such as the usual Ginzburg-Landau potential $W(u) = \frac{1}{4}(1-u^2)^2$, $\varepsilon > 0$ is a parameter characterizing interfacial thickness, $\bar{u} \in (-1, 1)$ is the background charge density, and G_0 is the Neumann Green's function of the Laplacian, i.e., G_0 solves

$$-\Delta G_0(x,y) = \delta(x-y) - \frac{1}{|\Omega|}, \qquad \int_\Omega G_0(x,y)\, dx = 0, \tag{1.2}$$

where Δ is the Laplacian in x and $\delta(x)$ is the Dirac delta-function, with Neumann boundary conditions. Note that u is also assumed to satisfy the "charge neutrality" condition

$$\frac{1}{|\Omega|} \int_\Omega u\, dx = \bar{u}. \tag{1.3}$$

Let us point out that in addition to a number of polymer systems [15, 41, 52], this model is also applicable to many other physical systems due to the fundamental nature of the Coulombic non-local term in (1.1) [6,17,22,32,37,40]. Because of this Coulomb interaction, we also like to think of u as a density of "charge".

The Ohta-Kawasaki functional admits the following "sharp-interface" version:

$$E[u] = \frac{\varepsilon}{2} \int_\Omega |\nabla u|\, dx + \frac{1}{2} \int_\Omega \int_\Omega (u(x) - \bar{u}) G(x,y)(u(y) - \bar{u})\, dx\, dy, \tag{1.4}$$

where now $u : \Omega \to \{-1, +1\}$ and $G(x,y)$ is the *screened* Green's function of the Laplacian, i.e., it solves the Neumann problem for the equation (distinguish from (1.2))

$$-\Delta G + \kappa^2 G = \delta(x-y), \tag{1.5}$$

where $\kappa := 1/\sqrt{W''(1)} > 0$. Note also that in contrast to the diffuse interface energy in (1.1), for the sharp interface energy in (1.4) the charge neutrality constraint in (1.3) is no longer imposed. This is due to the fact that in a minimizer of the diffuse interface energy, the charge of the minority phase is expected to partially redistribute into the majority phase to ensure screening of the induced non-local field (see a more detailed discussion in the following section).

The two terms in the energy (1.4) are competing: the second term favors u to be constant and equal to its average $\bar{u}$, but since u is valued in $\{+1, -1\}$ this means in effect that it is advantageous for u to oscillate rapidly between the two phases $u = +1$ and

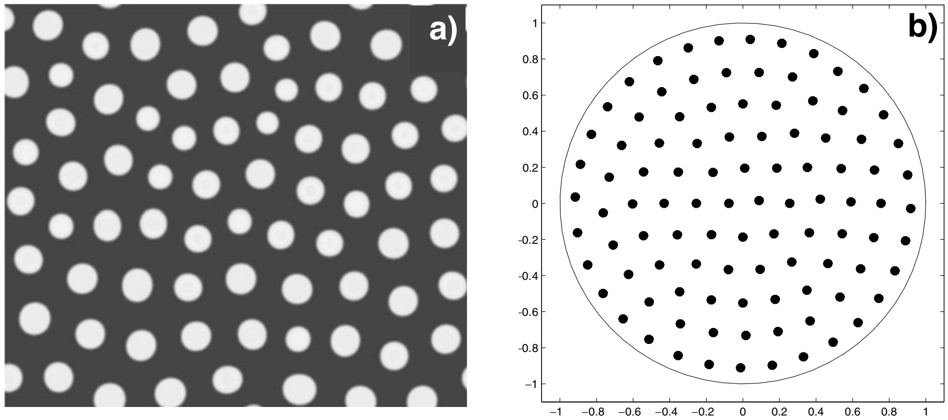

Figure 1: Two-dimensional multi-droplet patterns in systems with Coulombic repulsion: a local minimizer of the Ohta-Kawasaki energy on a rectangle with periodic boundary conditions; a local minimizer of the sum of two-point Coulombic potentials on a disk with Neumann boundary conditions. Taken from [37, 45].

$u = -1$; the first term penalizes the perimeter of the interface between the two phases, and thus opposes too much spreading and oscillation. The competition between these two selects a length scale, which is a function of ε. In the diffuse interface version (1.1), the sharp transitions between $\{u = +1\}$ and $\{u = -1\}$ are replaced by smooth transitions at the scale $\varepsilon > 0$ as soon as $\varepsilon \ll 1$.

In one space dimension and in the particular case $\bar{u} = 0$ (symmetric phases) the behavior of the energy can be understood from the work of Müller [35]: the minimizer u is periodic and alternates between $u = +1$ and $u = -1$ at scale $\varepsilon^{1/3}$ (for other one-dimensional results, see also [44, 46, 58]). In higher dimensions the patterns of minimizers are much more complex and are not well understood. The behavior depends on the volume fraction between the phases, i.e. on the constant $\bar{u}$ chosen, and also on the dimension. When $\bar{u} < 0$, we call $u = -1$ the majority phase and $u = +1$ the minority phase, and conversely when $\bar{u} > 0$. In two dimensions, numerical simulations lead to expecting round "droplets" of the minority phase surrounded by a "sea" of the majority phase (see Fig. 1) for sufficient asymmetries between the majority and the minority phases (i.e., for $\bar{u}$ sufficiently far away from zero) [36, 37, 42, 45]. The situation is less clear for $\bar{u}$ close to zero, although it is commonly believed that in this case the minimizers are one-dimensional stripe patterns [12, 36, 37, 42].

In all cases, minimizers are intuitively expected to be periodic. However, at the moment this seems to be very difficult to prove. The only general result in that direction to date is that of Alberti, Choksi and Otto [1], which proves that the energy of minimizers of the sharp interface energy from (1.4) with no screening (with $\kappa = 0$ and the neutrality condition from (1.3)) is uniformly distributed in the limit where the size of the domain Ω goes to infinity (see also [7, 51]). Their results, however, do not provide any further information about the structure of the energy-minimizing patterns. Note in passing that the question of proving any periodicity of minimizers for multi-dimensional energies is unsolved even for systems of point particles forming simple crystals (see e.g. [30, 49]), with a notable exception of certain two-dimensional particle systems with short-range interactions which somehow reduce to packing problems [43, 54, 56]. Naturally, the situation can be expected to be more complicated for pattern forming systems in which the constitutive elements are "soft" objects, such as, e.g., droplets of the minority phase in the matrix of the majority phase in the Ohta-Kawasaki model.

Here we are going to focus on the two-dimensional case and the situation where one phase is in strong majority with respect to the other, which is imposed by taking $\bar{u}$ very close to -1 as $\varepsilon \to 0$. Thus we can expect a distribution of small droplets of $u = +1$ surrounded by a sea of $u = -1$. In this regime, Choksi and Peletier analyzed the asymptotic properties of a suitably rescaled version of the sharp interface energy (1.4) with no screening in [13], as well as (1.1) in [14]. They work in the setting of a fixed domain Ω, and in a regime where the number of droplets remains finite as $\varepsilon \to 0$. They showed that the energy minimizing patterns concentrate to a finite number of point masses, whose magnitudes and locations are determined via a Γ-expansion of the energy [5]. Here, in contrast, we work in a regime where the number of droplets is divergent as $\varepsilon \to 0$. We note that Γ-convergence of (1.1) to the functional (1.4) with no screening and for fixed volume fractions was established by Ren and Wei in [46], who also analyzed local minimizers of the sharp interface energy in the strong asymmetry regime in two space dimensions [45].

All these works are in the finite domain Ω setting, while we are generally interested in the *large volume* (macroscopic) limit, i.e., the regime when the number of droplets tends to infinity. A rather detailed study of the behavior of the minimizers for the Ohta-Kawasaki energy in macroscopically large domains was recently performed in [38], still in the regime of $\bar{u}$ close to -1. There the two-dimensional Ohta-Kawasaki energy was considered in the case when Ω is a unit square with periodic boundary conditions. The interesting regime corresponds to the parameters $\varepsilon \ll 1$ and $1 + \bar{u} = O(\varepsilon^{2/3} |\ln \varepsilon|^{1/3}) \ll 1$. It is shown in [38] that under these assumptions on the parameters and some technical assumptions on W, (1.4) gives the correct asymptotic limit of the minimal energy in (1.1). Moreover, it is shown that when $\bar{\delta} := \varepsilon^{-2/3} |\ln \varepsilon|^{-1/3} (1 + \bar{u})$ becomes greater than a certain critical constant $\bar{\delta}_c$, the minimizers of E in (1.4) consist of $O(|\ln \varepsilon|)$ simply connected, nearly round droplets of radius $\simeq 3^{1/3} \varepsilon^{1/3} |\ln \varepsilon|^{-1/3}$, and uniformly distributed throughout the domain [38]. Thus, the following hierarchy of length scales is established in the considered regime:

$$\varepsilon \ll \varepsilon^{1/3} |\ln \varepsilon|^{-1/3} \ll |\ln \varepsilon|^{-1/2} \ll 1, \tag{1.6}$$

where the scales above correspond to the width of the interface, the radius of the droplets, the average distance between the droplets, and the screening length, respectively. The multiscale nature of the energy minimizing pattern is readily apparent from (1.6).

The analysis of [38] makes heavy use of the minimality condition for (1.4) and, in particular, the Euler-Lagrange equation associated with the energy. One is thus naturally led to asking whether the qualitative properties of the minimizers established in [38] (roundness of the droplets, identical radii, uniform distribution) carry over to, e.g., almost minimizers of E, for which no Euler-Lagrange equation is available. More broadly, it is natural to ask how robust the properties of the energy minimizing patterns are with respect to various perturbations of the energy, for example, how the picture presented above is affected when the charge density $\bar{u}$ is spatially modulated. A natural way to approach these questions is via Γ-convergence. However, for a multiscale problem such as the one we are considering the proper setting for studying Γ-limits of the functionals in (1.1) or (1.4) is presently lacking. The purpose of this paper is to formulate such a setting and extract the leading order term in the Γ-expansion of the energy in (1.1). In our forthcoming paper [23], we obtain the next order term in the Γ-expansion, using the method of "lower bounds for 2-scale energies" via Γ-convergence introduced in [49].

The main question for setting up the Γ-limit in the present context is to choose a suitable metric for Γ-convergence. This metric turns out to be similar to the one used for the analysis of vortices in the two-dimensional magnetic Ginzburg-Landau model from the theory of superconductivity [48]. In fact, the problem under consideration and its mathematical treatment (here as well as in [23]) share several important features with the latter [48]. In the theory of superconductivity the role of droplets is played by the Ginzburg-Landau vortices, which in the appropriate limits also become uniformly distributed throughout the domain [47]. We note, however, that the approach developed in [47, 48] cannot be carried over directly to the problem under consideration, since the vortices are more rigid than their droplet counterparts: the topological degrees of the vortices are quantized and can only take integer values, while the droplet volumes are not. Thus we also have to consider the possibility of many very small droplets. Developing a control on the droplet volumes from above and below is one of the key ingredient of the proofs presented below, and relies on the control of their perimeter via the energy.

For simplicity, as in [38] we consider the energy defined on a flat torus (a square with periodic boundary conditions). The metric we consider is the weak convergence of measures for a suitably rescaled sequence of characteristic functions associated with droplets (see the next section for precise definitions and statements of theorems). Then, up to a rescaling, we show that both the energy $\mathcal{E}$ from (1.1) and E from (1.4) Γ-converge to a quadratic functional in terms of the limit measure, with the quadratic term generated by the screened Coulomb kernel from (1.5) and the linear term depending explicitly on $\bar{\delta}$ and κ. To be more precise, we will see that in the regime we study, there are two contributions to the energy which operate at leading order: one contribution is linear in the density of the droplets and corresponds to the "self-interaction energy" of each droplet coming from both the perimeter term and self-interaction part of the double integral in (1.4), and the other is a quadratic term corresponding to the interaction between the droplets, i.e. the rest of the contribution of the double-integral term in (1.4). This setting, where both terms are of the same order of magnitude is very similar to the regime of [47] and [48, Chap. 7] in the context of the magnetic Ginzburg-Landau energy.

We note that the obtained limit variational problem is strictly convex and its unique minimizer is a measure with constant density across the domain Ω. In particular, this implies equidistribution of mass and energy for the minimizers of the diffuse interface energy $\mathcal{E}$ in (1.1) in the considered regime. In our companion paper [23], we further address the mutual arrangement of the droplets in the energy minimizing patterns, using the formalism developed recently for Ginzburg-Landau vortices [49]. We also obtain a characterization of the droplet shapes for almost minimizers of the sharp interface energy E, which, in turn, allows us to make the same conclusions about minimizers of the diffuse interface energy $\mathcal{E}$ for $\varepsilon \ll 1$, which is a new result. The reason we can characterize the droplets at the diffuse interface level is because the difference between the zero superlevel set of the minimizers at the diffuse interface level and the jump set of almost minimizers at the sharp interface level occurs essentially on the length scale ε (interfacial thickness), which is much smaller than the characteristic length scale $\varepsilon^{1/3}|\ln \varepsilon|^{-1/3}$ of the droplets.

Let us mention other closely related systems from the studies of ferromagnetism and superconductivity, where the role of droplets is played by the slender needle-like domains of opposite magnetization in a three-dimensional ferromagnetic slab at the onset of magnetization reversal [27], or superconducting tunnels in a slab of type-I superconducting material near the critical field [8, 11]. It may be possible to obtain similar Γ-convergence results with respect to convergence of measures in the plane for those problems. At the same time, we point out that extending our results to higher dimensions meets with serious difficulties, since in the suitable limit the droplets in higher-dimensional problems are expected to solve a non-local isoperimetric problem whose solution is not well characterized at present [28].

Our paper is organized as follows. In Sec. 2, we introduce the considered scaling regime and state our main results; in Sec. 4 we prove the Γ-convergence result in the sharp interface setting; in Sec. 5 we prove the results on the characterization of almost minimizers of sharp interface energy; and in Sec. 6 we treat the Γ-limit for the case of the diffuse interface energy.

Some notations. We use the notation $(u^\varepsilon) \in \mathcal{A}$ to denote sequences of functions $u^\varepsilon \in \mathcal{A}$ as $\varepsilon = \varepsilon_n \to 0$, where $\mathcal{A}$ is an admissible class. For a measurable set E, we use $|E|$ to denote its Lebesgue measure and $|\partial E|$ to denote its perimeter (in the sense of De Giorgi). We also use the notation $\mu \in \mathcal{M}^+(\Omega)$ to denote a non-negative Radon measure μ on the domain Ω. With a slight abuse, we will often speak of μ as the "density" on Ω. The symbols $H^1(\Omega)$, $BV(\Omega)$, $C(\Omega)$ and $H^{-1}(\Omega)$ denote the usual Sobolev space, space of functions of bounded variation, space of continuous functions, and the dual of $H^1(\Omega)$, respectively.

2 Statement of results

Throughout the rest of the paper the parameters $\kappa > 0$, $\bar{\delta} > 0$ and $\ell > 0$ are assumed to be fixed, and the domain Ω is assumed to be a flat two-dimensional torus of side length ℓ, i.e., $\Omega = \mathbb{T}^2_\ell = [0,\ell)^2$, with periodic boundary conditions. For every $\varepsilon > 0$ we define

$$\bar{u}^\varepsilon := -1 + \varepsilon^{2/3} |\ln \varepsilon|^{1/3} \bar{\delta}. \tag{2.1}$$

Under this scaling assumption the sharp interface version of the Ohta-Kawasaki energy (cf. (1.4)) can be written as

$$E^\varepsilon[u] = \frac{\varepsilon}{2} \int_{\mathbb{T}^2_\ell} |\nabla u| \, dx + \frac{1}{2} \int_{\mathbb{T}^2_\ell} (u - \bar{u}^\varepsilon)(-\Delta + \kappa^2)^{-1} (u - \bar{u}^\varepsilon) \, dx, \tag{2.2}$$

for all $u \in \mathcal{A}$, where

$$\mathcal{A} := BV(\mathbb{T}^2_\ell; \{-1, 1\}). \tag{2.3}$$

We wish to understand the asymptotic properties of the energy E^ε in (2.2) as $\varepsilon \to 0$ when all other parameters are fixed. We then relate our conclusions based on the study of this energy to its diffuse interface version, which under the same scaling assumptions takes the form

$$\mathcal{E}^\varepsilon[u] = \int_{\mathbb{T}^2_\ell} \left(\frac{\varepsilon^2}{2} |\nabla u|^2 + W(u) \ + \frac{1}{2} (u - \bar{u}^\varepsilon)(-\Delta)^{-1} (u - \bar{u}^\varepsilon) \right) dx, \tag{2.4}$$

with $u \in \mathcal{A}^\varepsilon$, where

$$\mathcal{A}^\varepsilon := \left\{ u \in H^1(\mathbb{T}^2_\ell) : \frac{1}{\ell^2} \int_{\mathbb{T}^2_\ell} u \, dx = \bar{u}^\varepsilon \right\}. \tag{2.5}$$

Here the symmetric double-well potential $W \geq 0$ needs to satisfy

$$W(1) = 0, \qquad W''(1) = \frac{1}{\kappa^2}, \qquad \int_{-1}^{1} \sqrt{2W(u)} \, du = 1, \tag{2.6}$$

in order for E^ε to be compatible with $\mathcal{E}^\varepsilon$ (see further discussion at the beginning of Sec. 3 and [38, Sec. 4] for precise assumptions on W). We note that the relation between E^ε and $\mathcal{E}^\varepsilon$ does not amount to a straightforward application of the standard Modica-Mortola argument [33, 34], as will be explained in more detail in Sec. 2.2. A formal application of the latter to (2.4) would result in an energy of the type in (2.2), but with the same (i.e., unscreened) Coulomb kernel as in (2.4), which is *not* Γ-equivalent to $\mathcal{E}^\varepsilon$. We also note that at the level of the energy minimizers the relation between the two functionals was established in [38].

2.1 Sharp interface energy

The sharp interface energy in (2.2) is most conveniently expressed in terms of *droplets*, i.e., the connected components Ω_i^+ of the set $\Omega^+ := \{u = +1\}$ (see Lemma 3.1 for technical details). Inserting

$$u = -1 + 2 \sum_i \chi_{\Omega_i^+}, \tag{2.7}$$

into (2.2), where $\chi_{\Omega_i^+}$ are the characteristic functions of Ω_i^+, expressing the result via G that solves

$$-\Delta G(x) + \kappa^2 G(x) = \delta(x) \qquad \text{in} \quad \mathbb{T}_\ell^2, \tag{2.8}$$

expanding all the terms and using the fact that $\int_{\mathbb{T}_\ell^2} G(x)dx = \kappa^{-2}$, we arrive at (see also [38])

$$\begin{aligned} E^\varepsilon[u] = {} & \frac{\ell^2(1+\bar{u}^\varepsilon)^2}{2\kappa^2} \\ & + \sum_i \left\{ \varepsilon|\partial\Omega_i^+| - 2\kappa^{-2}(1+\bar{u}^\varepsilon)|\Omega_i^+| \right\} + 2\sum_{i,j} \int_{\Omega_i^+} \int_{\Omega_j^+} G(x-y)\,dx\,dy, \end{aligned} \tag{2.9}$$

where we took into account the translational symmetry of the problem in $\mathbb{T}_\ell^2$. Moreover, since the optimal configurations for Ω_i^+ are expected to consist of droplets of size of order $\varepsilon^{1/3}|\ln\varepsilon|^{-1/3}$ (see (1.6) and the discussion around), it is convenient to introduce the rescaled area and perimeter of each droplet:

$$A_i := \varepsilon^{-2/3}|\ln\varepsilon|^{2/3}|\Omega_i^+|, \qquad P_i := \varepsilon^{-1/3}|\ln\varepsilon|^{1/3}|\partial\Omega_i^+|. \tag{2.10}$$

Similarly, let us introduce the suitably rescaled measure μ associated with the droplets:

$$d\mu(x) := \varepsilon^{-2/3}|\ln\varepsilon|^{-1/3} \sum_i \chi_{\Omega_i^+}(x)dx = \frac{1}{2}\varepsilon^{-2/3}|\ln\varepsilon|^{-1/3}(1+u)\,dx. \tag{2.11}$$

Note that by the definitions in (2.10) and (2.11) we have

$$\frac{1}{|\ln\varepsilon|} \sum_i A_i = \int_{\mathbb{T}_\ell^2} d\mu, \tag{2.12}$$

and the energy $E^\varepsilon[u]$ may be rewritten as

$$E^\varepsilon[u] = \varepsilon^{4/3}|\ln\varepsilon|^{2/3} \left(\frac{\bar{\delta}^2\ell^2}{2\kappa^2} + \bar{E}^\varepsilon[u] \right), \tag{2.13}$$

where

$$\bar{E}^\varepsilon[u] := \frac{1}{|\ln\varepsilon|}\sum_i \left(P_i - \frac{2\bar{\delta}}{\kappa^2}A_i\right) + 2\int_{\mathbb{T}^2_\ell}\int_{\mathbb{T}^2_\ell} G(x-y)d\mu(x)d\mu(y). \tag{2.14}$$

We now state our Γ-convergence result, which is obtained for configurations (u^ε) that obey the optimal energy scaling, i.e. when $\bar{E}^\varepsilon[u^\varepsilon]$ remains bounded as $\varepsilon \to 0$. The result is obtained with the help of the framework established in [47], where an analogous result for the Ginzburg-Landau functional of superconductivity was obtained. What we show is that the limit functional E^0 depends only on the limit density μ of the droplets (more precisely, on a limit measure $\mu \in \mathcal{M}^+(\mathbb{T}^2_\ell) \cap H^{-1}(\mathbb{T}^2_\ell)$, see Lemma 3.2 for technical details about such measures). In passing to the limit the second term in (2.14) remains unchanged, while the first term is converted into a term proportional to the integral of the measure. The proportionality constant is non-trivially determined by the optimal droplet profile that will be discussed later on. We give the statement of the result in terms of the original screened sharp interface energy E^ε, which is defined in terms of $u \in \mathcal{A}$. In the proof, we work instead with the equivalent energy $\bar{E}^\varepsilon$, which is defined through $\{A_i^\varepsilon\}$, $\{P_i^\varepsilon\}$ and μ^ε corresponding to $u = u^\varepsilon$ (cf. (2.13) and (2.14)).

Theorem 1. (Γ-convergence of E^ε) *Let E^ε be defined by (2.2) with $\bar{u}^\varepsilon$ given by* (2.1)*. Then, as $\varepsilon \to 0$ we have that*

$$\varepsilon^{-4/3}|\ln\varepsilon|^{-2/3}E^\varepsilon \xrightarrow{\Gamma} E^0[\mu] := \frac{\bar{\delta}^2\ell^2}{2\kappa^2} + \left(3^{2/3} - \frac{2\bar{\delta}}{\kappa^2}\right)\int_{\mathbb{T}^2_\ell} d\mu + 2\int_{\mathbb{T}^2_\ell}\int_{\mathbb{T}^2_\ell} G(x-y)d\mu(x)d\mu(y),$$

where $\mu \in \mathcal{M}^+(\mathbb{T}^2_\ell) \cap H^{-1}(\mathbb{T}^2_\ell)$. More precisely, we have

i) (Lower Bound) Let $(u^\varepsilon) \in \mathcal{A}$ be such that

$$\limsup_{\varepsilon\to 0} \varepsilon^{-4/3}|\ln\varepsilon|^{-2/3}E^\varepsilon[u^\varepsilon] < +\infty, \tag{2.15}$$

let

$$d\mu^\varepsilon(x) := \tfrac{1}{2}\varepsilon^{-2/3}|\ln\varepsilon|^{-1/3}(1+u^\varepsilon(x))dx, \tag{2.16}$$

and let v^ε satisfy

$$-\Delta v^\varepsilon + \kappa^2 v^\varepsilon = \mu^\varepsilon \qquad in \quad \mathbb{T}^2_\ell. \tag{2.17}$$

Then, up to extraction of a subsequence, we have

$$\mu^\varepsilon \rightharpoonup \mu \text{ in } (C(\mathbb{T}^2_\ell))^*, \quad v^\varepsilon \rightharpoonup v \text{ in } H^1(\mathbb{T}^2_\ell),$$

as $\varepsilon \to 0$, where $\mu \in \mathcal{M}^+(\mathbb{T}^2_\ell) \cap H^{-1}(\mathbb{T}^2_\ell)$ and $v \in H^1(\mathbb{T}^2_\ell)$ satisfy

$$-\Delta v + \kappa^2 v = \mu \qquad in \quad \mathbb{T}^2_\ell. \tag{2.18}$$

Moreover, we have

$$\liminf_{\varepsilon\to 0} \varepsilon^{-4/3}|\ln\varepsilon|^{-2/3}E^\varepsilon[u^\varepsilon] \ge E^0[\mu].$$

ii) (Upper Bound) Conversely, given $\mu \in \mathcal{M}^+(\mathbb{T}_\ell^2) \cap H^{-1}(\mathbb{T}_\ell^2)$ and $v \in H^1(\mathbb{T}_\ell^2)$ solving (2.18)*, there exist $(u^\varepsilon) \in \mathcal{A}$ such that for the corresponding μ^ε, v^ε as in* (2.16) *and (2.17) we have*

$$\mu^\varepsilon \rightharpoonup \mu \text{ in } (C(\mathbb{T}_\ell^2))^*, \quad v^\varepsilon \rightharpoonup v \text{ in } H^1(\mathbb{T}_\ell^2),$$

as $\varepsilon \to 0$, and

$$\limsup_{\varepsilon \to 0} \varepsilon^{-4/3} |\ln \varepsilon|^{-2/3} E^\varepsilon[u^\varepsilon] \leq E^0[\mu].$$

We note that the limit energy E^0 obtained in Theorem 1 may be viewed as the *homogenized* (or mean-field) version of the non-local part of the energy in the definition of E^ε associated with the limit charge density μ of the droplets, plus a term associated with the self-energy of the droplets. The functional E^0 is strictly convex, so there exists a unique minimizer $\bar{\mu} \in \mathcal{M}^+(\mathbb{T}_\ell^2) \cap H^{-1}(\mathbb{T}_\ell^2)$ of E^0, which is easily seen to be either $\bar{\mu} = 0$ for $\bar{\delta} \leq \frac{1}{2} 3^{2/3} \kappa^2$ or $\bar{\mu} = \frac{1}{2}(\bar{\delta} - \frac{1}{2} 3^{2/3} \kappa^2)$ otherwise. The latter can also be seen immediately from Remark 2.1 below, which gives a local characterization of the limit energy E^0 (see Lemma 3.2).

Remark 2.1. *The limit energy E^0 in Theorem 1 becomes local when written in terms of the limit potential v defined in* (2.18)*:*

$$E^0[\mu] = \frac{\bar{\delta}^2 \ell^2}{2\kappa^2} + \left(3^{2/3}\kappa^2 - 2\bar{\delta}\right) \int_{\mathbb{T}_\ell^2} v \, dx + 2 \int_{\mathbb{T}_\ell^2} \left(|\nabla v|^2 + \kappa^2 v^2\right) dx. \tag{2.19}$$

Also, by the usual properties of Γ-convergence [4], the optimal density $\bar{\mu}$ above is exhibited by the minimizers of E^ε in the limit $\varepsilon \to 0$, in agreement with [38, Theorem 2.2]:

Corollary 2.2. *Let $\bar{u}^\varepsilon$ be given by* (2.1) *and let $(u^\varepsilon) \in \mathcal{A}$ be minimizers of E^ε defined in* (2.2)*. Then, letting $\bar{\delta}_c := \frac{1}{2} 3^{2/3} \kappa^2$, if μ^ε is given by* (2.16)*, as $\varepsilon \to 0$ we have*

(i) If $\bar{\delta} \leq \bar{\delta}_c$, then

$$\mu^\varepsilon \rightharpoonup 0 \text{ in } (C(\mathbb{T}_\ell^2))^* \quad \text{and} \quad \varepsilon^{-4/3} |\ln \varepsilon|^{-2/3} \ell^{-2} \min E^\varepsilon \to \frac{\bar{\delta}^2}{2\kappa^2}. \tag{2.20}$$

(ii) If $\bar{\delta} > \bar{\delta}_c$, then

$$\mu^\varepsilon \rightharpoonup \tfrac{1}{2}(\bar{\delta} - \bar{\delta}_c) \text{ in } (C(\mathbb{T}_\ell^2))^* \quad \text{and} \quad \varepsilon^{-4/3} |\ln \varepsilon|^{-2/3} \ell^{-2} \min E^\varepsilon \to \tfrac{\bar{\delta}_c}{2\kappa^2}(2\bar{\delta} - \bar{\delta}_c). \tag{2.21}$$

In particular, since the minimal energy scales with the area of $\mathbb{T}_\ell^2$, it is an *extensive quantity.*

We next give the definition of almost minimizers with prescribed limit density, for which a number of further results may be obtained. These can be viewed, e.g., as almost minimizers of E^ε in the presence of an external potential. We note that in view of the strict convexity of E^0, minimizing $E^0[\mu] + \int_{\mathbb{T}_\ell^2} \varphi(x) d\mu(x)$ for a given $\varphi \in H^1(\mathbb{T}_\ell^2)$ one obtains a one-to-one correspondence between the minimizing density μ and the potential φ. It then makes sense to talk about almost minimizers of the energy E^ε with prescribed limit density μ by viewing them as almost minimizers of $E^\varepsilon + \int_{\mathbb{T}_\ell^2} \varphi^\varepsilon d\mu^\varepsilon$, where $\varphi^\varepsilon = \varepsilon^{2/3} |\ln \varepsilon|^{1/3} \varphi$. Also, observe that almost minimizers with the particular prescribed density $\bar{\mu}$ from Corollary 2.2 are simply almost minimizers of E^ε. Below we give a precise definition.

Definition 2.3. *For a given $\mu \in \mathcal{M}^+(\mathbb{T}_\ell^2) \cap H^{-1}(\mathbb{T}_\ell^2)$, we will call every recovery sequence $(u^\varepsilon) \in \mathcal{A}$ in Theorem 1(ii) almost minimizers of E^ε with prescribed limit density μ.*

For almost minimizers with prescribed limit density, we show that in the limit $\varepsilon \to 0$ most of the droplets, with the exception of possibly many tiny droplets comprising a vanishing fraction of the total droplet area, converge to disks of radius $r = 3^{1/3} \varepsilon^{1/3} |\ln \varepsilon|^{-1/3}$. More precisely, we have the following result.

Theorem 2. *Let $(u^\varepsilon) \in \mathcal{A}$ be a sequence of almost minimizers of E^ε with prescribed limit density μ. For every $\gamma \in (0,1)$ define the set $I_\gamma^\varepsilon := \{ i \in \mathbb{N} : 3^{2/3}\pi\gamma \le A_i^\varepsilon \le 3^{2/3}\pi\gamma^{-1} \}$. Then*

$$\lim_{\varepsilon \to 0} \frac{1}{|\ln \varepsilon|} \sum_i \left(P_i^\varepsilon - \sqrt{4\pi A_i^\varepsilon} \right) = 0, \tag{2.22}$$

$$\lim_{\varepsilon \to 0} \frac{1}{|\ln \varepsilon|} \sum_{i \in I_\gamma^\varepsilon} \left(A_i^\varepsilon - 3^{2/3}\pi \right)^2 = 0, \tag{2.23}$$

$$\lim_{\varepsilon \to 0} \frac{1}{|\ln \varepsilon|} \sum_{i \notin I_\gamma^\varepsilon} A_i^\varepsilon = 0, \tag{2.24}$$

where $\{A_i^\varepsilon\}$ and $\{P_i^\varepsilon\}$ are given by (2.10) with $u = u^\varepsilon$.

Note that we may use the isoperimetric deficit terms present in (2.22) to control the Fraenkel asymmetry of the droplets. The Fraenkel asymmetry measures the deviation of the set E from the ball of the same area that best approximates E and is defined for any Borel set $E \subset \mathbb{R}^2$ by

$$\alpha(E) = \min \frac{|E \triangle B|}{|E|}, \tag{2.25}$$

where the minimum is taken over all balls $B \subset \mathbb{R}^2$ with $|B| = |E|$, and $\triangle$ denotes the symmetric difference between sets. Note that the following sharp quantitative isoperimetric inequality holds for $\alpha(E)$ [20]:

$$|\partial E| - \sqrt{4\pi |E|} \ge C \alpha^2(E) \sqrt{|E|}, \tag{2.26}$$

with some universal constant $C > 0$. As a direct consequence of Theorem 2 and (2.12), we then have the following result.

Corollary 2.4. *Under the assumptions of Theorem 2, when $\int_{\mathbb{T}_\ell^2} d\mu > 0$ we have*

$$\lim_{\varepsilon\to 0} \frac{3^{2/3}\pi |I_\gamma^\varepsilon|}{|\ln \varepsilon|} = \int_{\mathbb{T}_\ell^2} d\mu, \qquad \lim_{\varepsilon\to 0} \frac{1}{|I_\gamma^\varepsilon|} \sum_{i\in I_\gamma^\varepsilon} \alpha(\Omega_i^+) = 0, \tag{2.27}$$

where $|I_\gamma^\varepsilon|$ denotes the cardinality of I_γ^ε.

This result generalizes the one in [38], where it was found that in the case of the minimizers *all* the droplets are uniformly close to disks of the optimal radius $r = 3^{1/3}\varepsilon^{1/3}|\ln\varepsilon|^{-1/3}$. What we showed here is that this result holds for almost all droplets in the case of almost minimizers, in the sense that in the limit almost all the mass concentrates in the droplets of optimal area and vanishing isoperimetric deficit. We note that the density μ is also the limit of the number density of the droplets, up to a normalization constant, once the droplets of vanishing area have been discarded.

The result that almost all droplets in almost minimizers with prescribed limit density have asymptotically the *same* size, even if the limit density is not constant in $\mathbb{T}_\ell^2$ appears to be quite surprising, since in this regime the self-interaction energy, which governs the droplet shapes and partly their sizes is exactly of the same order as the droplet mutual interaction energy, as was already mentioned at the end of Sec. 1. In addition, the other terms governing the droplets extracted in (2.14) (the perimeter and interaction with the background uniform charge) are equally strong. This result would hold, for example, for minimizers of the energy in the presence of a non-uniform potential, i.e., with a term $\frac{1}{2}\varepsilon^{2/3}|\ln\varepsilon|^{1/3}\int_{\mathbb{T}_\ell^2}\varphi(x)u(x)\,dx$ added to E^ε in (2.2) (see also the paragraph before Definition 2.3). It means that while the density of the energy minimizing droplets would be dependent on φ, their radii would not. We note that this observation is consistent with the expectation that quantum mechanical charged particle systems form Wigner crystals at low particle densities [24,32,57]. Let us point out that the Ohta-Kawasaki energy $\mathcal{E}^\varepsilon$ bears resemblance with the classical Thomas-Fermi-Dirac-Von Weizsäcker model arising in the context of density functional theory of quantum systems (see e.g. [30–32]).

2.2 Diffuse interface energy

We now turn to relating the results obtained so far for the screened sharp interface energy E^ε to the original diffuse interface energy $\mathcal{E}^\varepsilon$. On the level of the minimal energy, the asymptotic equivalence of the energies in the considered regime, namely, that for every $\delta > 0$

$$(1-\delta)\min E^\varepsilon \le \min \mathcal{E}^\varepsilon \le (1+\delta)\min E^\varepsilon \tag{2.28}$$

for $\varepsilon \ll 1$ was established in [38, Theorem 2.3]. The main idea of the proof in [38] is for a given function $u^\varepsilon \in \mathcal{A}^\varepsilon$ to establish an approximate lower bound for $\mathcal{E}^\varepsilon[u^\varepsilon]$ in terms of $(1-\delta)E^\varepsilon[\tilde{u}^\varepsilon]$ for some $\tilde{u}^\varepsilon \in \mathcal{A}$, with $\delta > 0$ which can be chosen arbitrarily small for

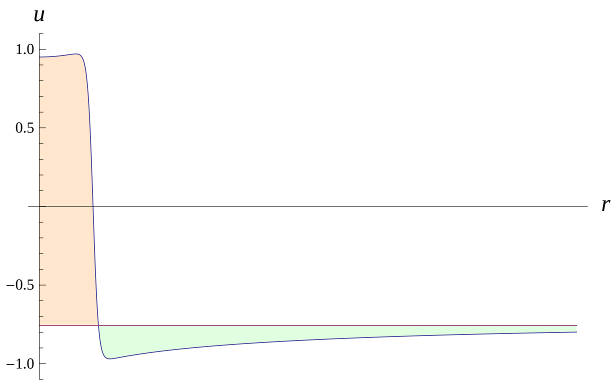

Figure 2: A qualitative form of the u-profile for a single droplet from the Euler-Lagrange equation associated with $\mathcal{E}$. The horizontal line shows the level corresponding to $\bar{u}$. Charge is transferred from the region where $u < \bar{u}$ (depletion shown in green) to the region where $u > \bar{u}$ (excess shown in orange). At the sharp interface level the corresponding profile is given by $\mathrm{sgn}(u)$, whose average charge is *not* equal to $\bar{u}$.

$\varepsilon \ll 1$. The matching approximate upper bound is then obtained by a suitable lifting of the minimizer $u^\varepsilon \in \mathcal{A}$ of E^ε into $\mathcal{A}^\varepsilon$.

Here we show that the procedure outlined above may also be applied to almost minimizers of $\mathcal{E}^\varepsilon$ in a suitably modified version of Definition 2.3 involving $\mathcal{E}^\varepsilon$, using almost minimizers of E^ε for comparisons. We note right away, however, that it is not possible to simply replace E^ε with $\mathcal{E}^\varepsilon$ in Definition 2.3. The reason for this is the presence of the mass constraint in the definition of the admissible class $\mathcal{A}^\varepsilon$ for $\mathcal{E}^\varepsilon$. This implies, for example, that any sequence of almost minimizers $(u^\varepsilon) \in \mathcal{A}^\varepsilon$ of $\mathcal{E}^\varepsilon$ must satisfy $\ell^{-2} \int_{\mathbb{T}^2_\ell} d\mu^\varepsilon = \frac{1}{2}\bar{\delta}$, while, according to Corollary 2.2, for sequences of almost minimizers $(u^\varepsilon) \in \mathcal{A}$ of E^ε we have $\ell^{-2} \int_{\mathbb{T}^2_\ell} d\mu^\varepsilon \to \bar{\mu} \neq \frac{1}{2}\bar{\delta}$. This phenomenon is intimately related to the effect of screening of the Coulombic potential from the droplets by the compensating charges that move into their vicinity [37]. For a single radially symmetric droplet the solution of the Euler-Lagrange equation associated with $\mathcal{E}^\varepsilon$ has the form shown in Fig. 2, which illustrates the gap between the "prescribed" total charge at the diffuse interface level and the total charge at the sharp interface level.

In order to be able to extract the limit behavior of the energy, we need to take into consideration the redistribution of charge discussed above and define almost minimizers with prescribed limit density that belong to $\mathcal{A}^\varepsilon$ and for which the screening charges are removed from the consideration of convergence to the limit density. Hence, given a candidate function $u^\varepsilon \in \mathcal{A}^\varepsilon$, we define a new function

$$u_0^\varepsilon(x) := \begin{cases} +1, & u^\varepsilon(x) > 0, \\ -1, & u^\varepsilon(x) \le 0, \end{cases} \tag{2.29}$$

whose jump set coincides with the zero level set of u^ε. This introduces a nonlinear filtering operation that eliminates the effect of the small deviations of u^ε from ± 1 in almost minimizers on the limit density (compare also with [27]). The measure μ_0^ε associated with the droplets is now defined via

$$d\mu_0^\varepsilon := \tfrac{1}{2}\varepsilon^{-2/3}|\ln\varepsilon|^{-1/3}(1+u_0^\varepsilon(x))dx. \tag{2.30}$$

We can follow the ideas of [38] to establish an analog of Theorem 1 for the diffuse interface energy. To avoid many technical assumptions, we formulate the result for a specific choice of $W(u) = \frac{9}{32}(1-u^2)^2$ and $\kappa = 1/\sqrt{W''(1)} = \frac{2}{3}$ (see the discussion at the beginning of Sec. 3). A general result may easily be reconstructed. Also, we make a technical assumption to avoid dealing with the case $\limsup_{\varepsilon\to 0}\|u^\varepsilon\|_{L^\infty(\mathbb{T}_\ell^2)} > 1$, when spiky configurations in which $|u^\varepsilon|$ significantly exceeds 1 in regions of vanishing size may appear. We note that this condition is satisfied by the minimizers of $\mathcal{E}^\varepsilon$ [38, Proposition 4.1].

Theorem 3. (Γ-convergence of $\mathcal{E}^\varepsilon$) *Let $\mathcal{E}^\varepsilon$ be defined by (2.4) with $W(u) = \frac{9}{32}(1-u^2)^2$ and $\bar{u}^\varepsilon$ given by* (2.1)*. Then, as $\varepsilon \to 0$ we have that*

$$\varepsilon^{-4/3}|\ln\varepsilon|^{-2/3}\mathcal{E}^\varepsilon \xrightarrow{\Gamma} E^0[\mu] := \frac{\bar{\delta}^2\ell^2}{2\kappa^2} + \left(3^{2/3} - \frac{2\bar{\delta}}{\kappa^2}\right)\int_{\mathbb{T}_\ell^2} d\mu + 2\int_{\mathbb{T}_\ell^2}\int_{\mathbb{T}_\ell^2} G(x-y)d\mu(x)d\mu(y),$$

where $\mu \in \mathcal{M}^+(\mathbb{T}_\ell^2) \cap H^{-1}(\mathbb{T}_\ell^2)$ and $\kappa = \frac{2}{3}$. More precisely, we have

i) (Lower Bound) Let $(u^\varepsilon) \in \mathcal{A}^\varepsilon$ be such that $\limsup_{\varepsilon\to 0}\|u^\varepsilon\|_{L^\infty(\mathbb{T}_\ell^2)} \le 1$ *and*

$$\limsup_{\varepsilon\to 0}\varepsilon^{-4/3}|\ln\varepsilon|^{-2/3}\mathcal{E}^\varepsilon[u^\varepsilon] < +\infty, \tag{2.31}$$

and let $\mu_0^\varepsilon(x)$ be defined by (2.29) *and* (2.30)*.*

Then, up to extraction of subsequences, we have

$$\mu_0^\varepsilon \rightharpoonup \mu \text{ in } (C(\mathbb{T}_\ell^2))^*,$$

as $\varepsilon \to 0$, where $\mu \in \mathcal{M}^+(\mathbb{T}_\ell^2)\cap H^{-1}(\mathbb{T}_\ell^2)$. Moreover, we have $\limsup_{\varepsilon\to 0}\|u^\varepsilon\|_{L^\infty(\mathbb{T}_\ell^2)} = 1$ *and*

$$\liminf_{\varepsilon\to 0}\varepsilon^{-4/3}|\ln\varepsilon|^{-2/3}\mathcal{E}^\varepsilon[u^\varepsilon] \ge E^0[\mu].$$

ii) (Upper Bound) Conversely, given $\mu \in \mathcal{M}^+(\mathbb{T}_\ell^2)\cap H^{-1}(\mathbb{T}_\ell^2)$, there exist $(u^\varepsilon) \in \mathcal{A}^\varepsilon$ such that $\limsup_{\varepsilon\to 0}\|u^\varepsilon\|_{L^\infty(\mathbb{T}_\ell^2)} = 1$ *and for μ_0^ε defined by* (2.29) *and* (2.30) *we have*

$$\mu_0^\varepsilon \rightharpoonup \mu \text{ in } (C(\mathbb{T}_\ell^2))^*,$$

as $\varepsilon \to 0$, and

$$\limsup_{\varepsilon\to 0}\varepsilon^{-4/3}|\ln\varepsilon|^{-2/3}\mathcal{E}^\varepsilon[u^\varepsilon] \le E^0[\mu].$$

Based on the result of Theorem 3, we have the following analog of Corollary 2.2 for the diffuse interface energy $\mathcal{E}^\varepsilon$.

Corollary 2.5. *Let $\bar{u}^\varepsilon$ be given by (2.1) and let $(u^\varepsilon) \in \mathcal{A}^\varepsilon$ be minimizers of $\mathcal{E}^\varepsilon$ defined in (2.4) with $W(u) = \frac{9}{32}(1-u^2)^2$. Then, letting $\kappa = \frac{2}{3}$ and $\bar{\delta}_c := \frac{1}{2}3^{2/3}\kappa^2$, if u_0^ε and μ_0^ε are defined via (2.29) and (2.30), respectively, as $\varepsilon \to 0$ we have*

(i) If $\bar{\delta} \le \bar{\delta}_c$, then

$$\mu_0^\varepsilon \rightharpoonup 0 \text{ in } (C(\mathbb{T}_\ell^2))^*, \quad \text{and} \quad \varepsilon^{-4/3}|\ln\varepsilon|^{-2/3}\ell^{-2}\min E^\varepsilon \to \frac{\bar{\delta}^2}{2\kappa^2}. \tag{2.32}$$

(ii) If $\bar{\delta} > \bar{\delta}_c$, then

$$\mu^\varepsilon \rightharpoonup \tfrac{1}{2}(\bar{\delta} - \bar{\delta}_c) \text{ in } (C(\mathbb{T}_\ell^2))^*, \quad \text{and} \quad \varepsilon^{-4/3}|\ln\varepsilon|^{-2/3}\ell^{-2}\min \mathcal{E}^\varepsilon \to \tfrac{\bar{\delta}_c}{2\kappa^2}(2\bar{\delta} - \bar{\delta}_c). \tag{2.33}$$

In addition, we have the following analog of Theorem 2, which, in particular, applies to minimizers of the diffuse interface energy $\mathcal{E}^\varepsilon$.

Theorem 4. *Let $(u^\varepsilon) \in \mathcal{A}^\varepsilon$ be a recovery sequence as in Theorem 3(ii) and let $\int_{\mathbb{T}_\ell^2} d\mu > 0$. Then there exists a set of finite perimeter Ω^+ such that if Ω_i^+ are its connected components, then the conclusion of Theorem 2 holds with $\{A_i^\varepsilon\}$ and $\{P_i^\varepsilon\}$ given by (2.10), and*

$$\lim_{\varepsilon\to 0} \frac{|\Omega^+ \triangle \{u^\varepsilon > 0\}|}{|\Omega^+|} = 0. \tag{2.34}$$

Theorem 4 essentially says that the zero superlevel set of u^ε from every recovery sequence of Theorem 3 may be well approximated in L^1 sense by a union of of droplets that are, in turn, close to disks of radius $r = 3^{1/3}\varepsilon^{1/3}|\ln\varepsilon|^{-1/3}$ for $\varepsilon \ll 1$. The L^1 error arises, since we do not have control on the perimeter of every superlevel set of u^ε. At the same time, the choice of the zero superlevel set of u^ε in the definition of the truncated version u_0^ε of u^ε in (2.29) was arbitrary. We could equivalently use the superlevel set $\{u^\varepsilon > c\}$ for any $c \in (-1, 1)$ fixed. Also, we point out that the conclusions of Corollary 2.4 remain true for Ω^+ in Theorem 4 under the assumptions of Theorem 3.

3 Some auxiliary lemmas

In this section we collect some technical results that are needed in the proofs of our theorems. Before proceeding to those results, however, let us first show that the assumption in (2.6) that needs to be imposed on W in order to have Γ-equivalence between E^ε and

$\mathcal{E}^\varepsilon$ defined in (2.2) and (2.4), respectively, and, hence, the conclusion of Theorem 3 (see also [38]), is not restrictive. Indeed, given the definition of $\mathcal{E}^\varepsilon$ in (2.4), introduce a rescaling:

$$W = \lambda^2 \widetilde{W}, \qquad \ell = \lambda \tilde{\ell}, \qquad \varepsilon = \lambda^2 \tilde{\varepsilon}. \tag{3.1}$$

Then it is easy to see that if $\tilde{u}(x) := u(\lambda x)$, then $\mathcal{E}^\varepsilon[u] = \lambda^4 \tilde{\mathcal{E}}^{\tilde{\varepsilon}}[\tilde{u}]$, where $\tilde{\mathcal{E}}^{\tilde{\varepsilon}}$ is obtained from (2.4) by replacing all the quantities with their tilde equivalents. In particular, choosing $\lambda = 3/(2\sqrt{2})$ we can relate the original Ohta-Kawasaki energy $\tilde{\mathcal{E}}^{\tilde{\varepsilon}}$, which has $\widetilde{W}(u) = \frac{1}{4}(1-u^2)^2$ [42], to the energy appearing in the statement of Theorem 3. The choice of W satisfying (2.6) simply avoids many extra constants appearing in the statements of results.

As was already mentioned, the energy E^ε may be alternatively written in terms of the level sets of u. Indeed, when $E^\varepsilon[u] < +\infty$, the set $\Omega^+ := \{u = +1\}$ is a set of finite perimeter (for precise definitions and the terminology used below, see [2]). We then have the following result about decomposing Ω^+ into measure theoretic connected components Ω_i^+, which in view of the scaling of the upper bound on energy will be shown to hold for all sufficiently small $\varepsilon > 0$. Note that the latter assumption implies that each connected component on the torus has the same geometric structure as connected components of sets of finite perimeter in the whole plane, thus excluding a possibility of stripe-like components winding around the torus and, hence, justifying the use of the word "droplet". We will also make repeated use of the basic fact that the diameter of a connected component is essentially controlled by its perimeter (i.e., modulo a set of measure zero).

Lemma 3.1. *Let $\Omega^+ \subset \mathbb{T}_\ell^2$ be a set of finite perimeter, and assume that $|\Omega^+| \leq \frac{1}{160}\ell^2$ and $|\partial\Omega^+| \leq \frac{1}{10}\ell$. Then Ω^+ may be uniquely decomposed (up to negligible sets) into an at most countable union of connected sets Ω_i^+ of positive measure, which, after a suitable translation and extension to $\mathbb{R}^2$, are essentially bounded and whose essential boundaries $\partial^M\Omega_i^+$ are (up to negligible sets) at most countable unions of Jordan curves that are essentially disjoint. Furthermore, we have*

$$\operatorname{ess\,diam}\ \Omega_i^+ \ \leq \frac{1}{2}|\partial\Omega_i^+|. \tag{3.2}$$

Proof. Let $\Omega_\#^+$ be the periodic extension of Ω^+ from $\mathbb{T}_\ell^2$ to $\mathbb{R}^2$, and let $K_R := (-R,R)^2$. Then for every $R \in (\ell, \frac{3}{2}\ell)$ the set $\Omega_\#^+ \cap K_R \subset \mathbb{R}^2$ is a set of finite perimeter, and we have

$$|\partial(\Omega_\#^+ \cap K_R)| \leq 9|\partial\Omega^+| + \mathcal{H}^1(\mathring{\Omega}_\#^+ \cap \partial K_R). \tag{3.3}$$

On the other hand, by the co-area formula we have

$$\int_\ell^{\frac{3}{2}\ell} \mathcal{H}^1(\mathring{\Omega}_\#^+ \cap \partial K_t)dt = |\Omega_\#^+ \cap K_{\frac{3}{2}\ell} \backslash K_\ell| \leq 8|\Omega^+|. \tag{3.4}$$

Therefore, there exists $R \in (\ell, \frac{3}{2}\ell)$ such that $\mathcal{H}^1(\mathring{\Omega}_\#^+ \cap \partial K_R) \leq 16\ell^{-1}|\Omega^+|$. Using the assumptions of the Lemma, we then conclude that $\mathcal{H}^1(\mathring{\Omega}_\#^+ \cap \partial K_R) \leq \frac{1}{10}\ell$ and by (3.3) we have $|\partial(\Omega_\#^+ \cap K_R)| \leq \ell$.

We now apply the results of [2, Corollary 1 and Theorem 8] to the set $\Omega_{\#}^{+} \cap K_R$ to obtain its decomposition into connected components and denote by Ω_i^+ those components for which $|\Omega_i^+ \cap K_{\frac{1}{2}\ell}| > 0$. In turn, by [2, Theorem 7 and Lemma 4] and noting that in view of [2, Proposition 6(ii)] it is sufficient to consider only simple sets (see [2, Definition 3]), we have that Ω_i^+ satisfy (3.2). Therefore, from our estimate on $|\partial(\Omega_{\#}^{+} \cap K_R)|$ we conclude that $|\Omega_i^+ \cap K_{\frac{3}{2}\ell} \backslash K_\ell| = 0$, and so $|\partial \Omega_i^+|$ does not have contributions from ∂K_R. Together with the assumptions of the Lemma, this then implies that each Ω_i^+ is essentially contained, after a suitable translation, in $K_{\frac{1}{4}\ell}$. Finally, identifying all translates of Ω_i^+ by $\pm\ell$ in either coordinate direction with the connected components of Ω^+ in $\mathbb{T}_\ell^2$, we obtain the desired decomposition of $\Omega^+ \subset \mathbb{T}_\ell^2$ for which (3.2) also holds in the case of the perimeter relative to $\mathbb{T}_\ell^2$. $\square$

In the context of Γ-convergence the sets Ω_i^+ may be viewed as a suitable generalization of the droplets introduced earlier in the studies of energy minimizing patterns [38]. Note, however, that the sets Ω_i^+ lack the regularity properties of the energy minimizers in [38] and may in general be fairly ill-behaved (in particular, they do not have to be simply connected). Nevertheless, they are fundamental for the description of the low energy states associated with E^ε and, in particular, will be shown to be close, in some average sense, to disks of prescribed radii for almost minimizers of energy.

We now discuss the precise nature of the limit measures appearing in our analysis. We say that $\mu \in \mathcal{M}^+(\mathbb{T}_\ell^2) \cap H^{-1}(\mathbb{T}_\ell^2)$, if the non-negative Radon measure μ has bounded Coulombic energy, i.e., if

$$\int_{\mathbb{T}_\ell^2} \int_{\mathbb{T}_\ell^2} G(x-y)\, d\mu(x)\, d\mu(y) < \infty. \tag{3.5}$$

Our notation is justified by the following fundamental properties of such measures.

Lemma 3.2. *Let $\mu \in \mathcal{M}^+(\mathbb{T}_\ell^2)$ and let (3.5) hold. Then*

(i) μ can be extended to a bounded linear functional over $H^1(\mathbb{T}_\ell^2)$.

(ii) If

$$v(x) := \int_{\mathbb{T}_\ell^2} G(x-y)\, d\mu(y), \tag{3.6}$$

then $v \in H^1(\mathbb{T}_\ell^2)$. Furthermore, v solves

$$-\Delta v + \kappa^2 v = \mu, \tag{3.7}$$

weakly in $H^1(\mathbb{T}_\ell^2)$, and

$$\nabla v(x) = \int_{\mathbb{T}_\ell^2} \nabla G(x-y)\, d\mu(y), \tag{3.8}$$

in the sense of distributions.

(iii) If v is as in (ii), we have $\kappa^2 \int_{\mathbb{T}_\ell^2} v\,dx = \int_{\mathbb{T}_\ell^2} d\mu$ and

$$\int_{\mathbb{T}_\ell^2}\int_{\mathbb{T}_\ell^2} G(x-y)\,d\mu(x)\,d\mu(y) = \int_{\mathbb{T}_\ell^2} \left(|\nabla v|^2 + \kappa^2 v^2\right) dx. \tag{3.9}$$

Proof. We first show that v defined in (3.6) has distributional first derivatives in $L^2(\mathbb{T}_\ell^2)$. Introduce $H(x) > 0$ defined for all $x \in \mathbb{T}_\ell^2$ by

$$H(x) := \frac{1}{2\pi} \sum_{\mathbf{n}\in\mathbb{Z}^2} \frac{e^{-\kappa|x-\mathbf{n}\ell|}}{|x-\mathbf{n}\ell|}, \tag{3.10}$$

whose Fourier coefficients are easily seen to be

$$\widehat{H}(k) := \int_{\mathbb{T}_\ell^2} e^{ik\cdot x} H(x)\,dx = \frac{1}{\sqrt{\kappa^2+|k|^2}}, \qquad k \in 2\pi\ell^{-1}\mathbb{Z}^2. \tag{3.11}$$

Indeed, $H(x)$ may be viewed as the trace $\widetilde{H}(x,0)$ of the solution of

$$-\Delta\widetilde{H}(x) + \kappa^2\widetilde{H}(x) = 2\sum_{\mathbf{n}\in\mathbb{Z}^2\times\{0\}} \delta(x-\mathbf{n}\ell), \qquad x\in\mathbb{R}^3, \tag{3.12}$$

which is given by the same formula as in (3.10). Denoting by $\widetilde{H}_k(z)$ the Fourier coefficients of $\widetilde{H}(x,z)$ in $x \in \mathbb{T}_\ell^2$, from (3.12) one obtains that $\widetilde{H}_k(z)$ solves

$$-\widetilde{H}_k''(z) + (\kappa^2+|k|^2)\widetilde{H}_k(z) = 2\delta(z), \tag{3.13}$$

whose explicit solution is $\widetilde{H}_k(z) = e^{-z\sqrt{\kappa^2+|k|^2}}/\sqrt{\kappa^2+|k|^2}$.

From (3.11) and the equation satisfied by G one immediately concludes that

$$G(x) = \int_{\mathbb{T}_\ell^2} H(x-y)H(y)\,dy. \tag{3.14}$$

Furthermore, by direct inspection one can see that

$$|\nabla G(x)| \leq CH(x) \qquad \forall x\in\mathbb{T}_\ell^2, \tag{3.15}$$

for some $C > 0$. In addition, defining

$$b(x) := \int_{\mathbb{T}_\ell^2} H(x-y)\,d\mu(y), \tag{3.16}$$

by Tonelli's theorem and (3.14) we have

$$\int_{\mathbb{T}_\ell^2} b^2 dx = \int_{\mathbb{T}_\ell^2}\int_{\mathbb{T}_\ell^2}\int_{\mathbb{T}_\ell^2} H(x-z)H(y-z)\,d\mu(x)\,d\mu(y)\,dz$$
$$= \int_{\mathbb{T}_\ell^2}\int_{\mathbb{T}_\ell^2} G(x-y)\,d\mu(x)\,d\mu(y), \tag{3.17}$$

and, hence, by (3.5) we have $b \in L^2(\mathbb{T}_\ell^2)$. Therefore, if

$$h(x) := \int_{\mathbb{T}_\ell^2} \nabla G(x-y)\,d\mu(y), \tag{3.18}$$

then by (3.15) and (3.16) it is well defined, and we have $h \in L^2(\mathbb{T}_\ell^2;\mathbb{R}^2)$ as well.

Now, testing (3.6) with $\nabla\varphi$, where $\varphi \in C^\infty(\mathbb{T}_\ell^2)$, yields

$$-\int_{\mathbb{T}_\ell^2} \nabla\varphi(x)v(x)\,dx = -\int_{\mathbb{T}_\ell^2}\int_{\mathbb{T}_\ell^2} \nabla\varphi(x)G(x-y)d\mu(y) = \int_{\mathbb{T}_\ell^2} \varphi(x)h(x)\,dx, \tag{3.19}$$

which is justified by Fubini's theorem, in view of the fact that $h \in L^2(\mathbb{T}_\ell^2;\mathbb{R}^2)$. Hence $\nabla v = h \in L^2(\mathbb{T}_\ell^2;\mathbb{R}^2)$ distributionally, proving (3.8). To prove that $v \in H^1(\mathbb{T}_\ell^2)$, observe that by Tonelli's theorem

$$\int_{\mathbb{T}_\ell^2} v^2 dx = \int_{\mathbb{T}_\ell^2}\int_{\mathbb{T}_\ell^2}\int_{\mathbb{T}_\ell^2} G(x-z)G(y-z)\,d\mu(x)\,d\mu(y)\,dz \le C\left(\int_{\mathbb{T}_\ell^2} d\mu\right)^2, \tag{3.20}$$

for some $C > 0$. On the other hand, since by maximum principle $G(x) \ge c > 0$ for all $x \in \mathbb{T}_\ell^2$, we conclude that

$$c\left(\int_{\mathbb{T}_\ell^2} d\mu\right)^2 \le \int_{\mathbb{T}_\ell^2}\int_{\mathbb{T}_\ell^2} G(x-y)\,d\mu(x)\,d\mu(y). \tag{3.21}$$

Therefore, by (3.5) we have that μ is bounded in the sense of measures, and so from (3.20) follows that $v \in L^2(\mathbb{T}_\ell^2)$ as well.

We may next show that (3.7) holds distributionally by testing v in (3.6) with $-\Delta\varphi + \kappa^2\varphi \in C^\infty(\mathbb{T}_\ell^2)$ and integrating by parts. Then, to conclude the proof of the lemma, we test (3.7) with $\varphi \in C^\infty(\mathbb{T}_\ell^2)$ and apply the Cauchy-Schwarz inequality to obtain

$$\left|\int_{\mathbb{T}_\ell^2} \varphi\,d\mu\right| = \left|\int_{\mathbb{T}_\ell^2} \left(\nabla\varphi\cdot\nabla v + \kappa^2\varphi v\right)dx\right| \le C\|v\|_{H^1(\mathbb{T}_\ell^2)}\|\varphi\|_{H^1(\mathbb{T}_\ell^2)}, \tag{3.22}$$

for some $C > 0$. This yields (i), and, hence, (3.7) also holds weakly in $H^1(\mathbb{T}_\ell^2)$. Finally, to obtain (iii), we interpret μ in (3.7) as an element of $H^{-1}(\mathbb{T}_\ell^2)$ and test (3.7) with either 1 or v itself. $\square$

Remark 3.3. *It is not difficult to extend the proof of Lemma 3.2 to the case of measures with finite Coulombic energy defined on a sufficiently regular domain* Ω *with either Dirichlet or Neumann boundary conditions for the potential. In this case the role of* H *would be played by the kernel of the Neumann-to-Dirichlet map for the operator* $-\Delta + \kappa^2$ *extended to* $\Omega \times \mathbb{R}^+$.

Observe that for the nontrivial minimizers we know from [38] that $\bar{E}^\varepsilon = O(1)$, $A_i = O(1)$ and $P_i = O(1)$ (and even more precisely $A_i \simeq 3^{2/3}\pi$ and $P_i \simeq 2\cdot 3^{1/3}\pi$), the number of droplets is $N = O(|\ln\varepsilon|)$, and μ closely approximates the sum of Dirac masses at the droplet centers with weights of order $|\ln\varepsilon|^{-1}$. If, on the other hand, the considered configurations only obey an energy bound under the optimal scaling, then the same estimates turn out to hold for the droplets on average. The precise result is stated in the following lemma.

Lemma 3.4. *Let* $(u^\varepsilon) \in \mathcal{A}$*, let* $\limsup_{\varepsilon\to 0} \bar{E}^\varepsilon[u^\varepsilon] < +\infty$*, and let* $\{A_i^\varepsilon\}$*,* $\{P_i^\varepsilon\}$ *and* μ^ε *be given by* (2.10) *and* (2.11) *with* $u = u^\varepsilon$*. Then*

$$\limsup_{\varepsilon\to 0} \frac{1}{|\ln\varepsilon|} \sum_i P_i^\varepsilon < +\infty, \qquad \limsup_{\varepsilon\to 0} \frac{1}{|\ln\varepsilon|} \sum_i A_i^\varepsilon < +\infty, \tag{3.23}$$

and

$$\limsup_{\varepsilon\to 0} \int_{\mathbb{T}_\ell^2} d\mu^\varepsilon < +\infty. \tag{3.24}$$

Proof. By (2.12) and the positivity of P_i^ε, we obtain the result, once we prove (3.24). To prove the latter, we simply note that if $\int_{\mathbb{T}_\ell^2} d\mu^\varepsilon \geq 2\bar{\delta}/(c\kappa^2)$, where c is the same as in (3.21), then by (2.12) we have from the definition of $\bar{E}^\varepsilon$ in (2.14):

$$\bar{E}^\varepsilon[u] \geq -\frac{2\bar{\delta}}{\kappa^2} \int_{\mathbb{T}_\ell^2} d\mu^\varepsilon + 2c \left(\int_{\mathbb{T}_\ell^2} d\mu^\varepsilon \right)^2 \geq c \left(\int_{\mathbb{T}_\ell^2} d\mu^\varepsilon \right)^2, \tag{3.25}$$

which yields (3.24). □

4 Proof of Theorem 1

Throughout all the proofs below, the values of A_i^ε and P_i^ε are always the rescaled areas and perimeters, defined in (2.10), of the connected components Ω_i^+ of $\Omega^+ = \{u = +1\}$ for a given $u = u^\varepsilon$, as in Lemma 3.4. The presentation is clarified by working with the rescaled energy $\bar{E}^\varepsilon$ defined by (2.14) rather than E^ε directly. We begin by proving Part i) of Theorem 1, the lower bound.

4.1 Proof of lower bound, Theorem 1 i)

Step 1: Estimate of $\bar{E}^\varepsilon$ in terms of A_i^ε and P_i^ε.

First, for a fixed $\gamma \in (0,1)$ we define a *truncated* rescaled droplet area:

$$\tilde{A}_i^\varepsilon := \begin{cases} A_i^\varepsilon, & \text{if } \ A_i^\varepsilon < 3^{2/3}\pi\gamma^{-1} \\ (3^{2/3}\pi\gamma^{-1})^{1/2}|A_i^\varepsilon|^{1/2} & \text{if } \ A_i^\varepsilon \geq 3^{2/3}\pi\gamma^{-1}, \end{cases} \tag{4.1}$$

and the isoperimetric deficit

$$I_{\mathrm{def}}^\varepsilon := \frac{1}{|\ln\varepsilon|}\sum_i \left(P_i^\varepsilon - \sqrt{4\pi A_i^\varepsilon}\right) \geq 0, \tag{4.2}$$

which will be used throughout the proof. The purpose of defining the truncated droplet area in (4.1) will become clear later.

We start by writing $\mu^\varepsilon = \sum_i \mu_i^\varepsilon$, with

$$d\mu_i^\varepsilon(x) := \varepsilon^{-2/3}|\ln\varepsilon|^{-1/3}\chi_{\Omega_i^+}(x)dx, \tag{4.3}$$

where Ω_i^+ are the connected components of $\Omega^+ = \{u^\varepsilon = +1\}$, and the index ε was omitted from Ω_i^+ to avoid cumbersome notation. For small enough ε this is justified by Lemma 3.1, in view of the fact that for some $C > 0$ we have

$$|\partial\Omega^+| \leq \varepsilon^{-1}E^\varepsilon[u^\varepsilon] \leq C\varepsilon^{1/3}|\ln\varepsilon|^{2/3}, \tag{4.4}$$

so $|\partial\Omega_i^+| \ll \ell$ whenever $\varepsilon \ll 1$. In particular, (3.2) holds for Ω_i^+ when ε is sufficiently small.

For a fixed $\rho > 0$ we introduce the "far field truncation" $G_\rho \in C^\infty(\mathbb{T}_\ell^2)$ of the Green's function G:

$$G_\rho(x-y) := G(x-y)\phi_\rho(|x-y|) \qquad \forall (x,y) \in \mathbb{T}_\ell^2 \times \mathbb{T}_\ell^2, \tag{4.5}$$

where $\phi_\rho \in C^\infty(\mathbb{R})$ is a monotonically increasing cutoff function such that $\phi_\rho(t) = 0$ for all $t < \frac{1}{2}\rho$ and $\phi_\rho(t) = 1$ for all $t > \rho$. Then, for sufficiently small ε we have $|\partial\Omega_i^+| \leq \rho$ in view of (4.4), and from (2.14) and (3.2) we obtain

$$\begin{aligned} \bar{E}^\varepsilon[u^\varepsilon] \geq & I_{\mathrm{def}}^\varepsilon + \frac{1}{|\ln\varepsilon|}\left(\sum_i \sqrt{4\pi A_i^\varepsilon} - \frac{2\bar{\delta}}{\kappa^2}A_i^\varepsilon\right) \\ & + 2\sum_i \iint G(x-y)d\mu_i^\varepsilon(x)d\mu_i^\varepsilon(y) \\ & + 2\iint G_\rho(x-y)d\mu^\varepsilon(x)d\mu^\varepsilon(y), \end{aligned} \tag{4.6}$$

where we used (3.2) and the positivity of G (cf. e.g. [38]), and here and everywhere below we omit $\mathbb{T}^2_\ell \times \mathbb{T}^2_\ell$ as the domain of integration for double integrals to simplify the notation.

We recall that the Green's function for $-\Delta + \kappa^2$ on $\mathbb{T}^2_\ell$ can be written as $G(x-y) = -\frac{1}{2\pi}\ln|x-y| + O(|x-y|)$ [38]. With the help of this fact, together with (4.4) and (3.2), for ε sufficiently small we have the following estimate for the self-interaction energy:

$$
\begin{aligned}
\bar{E}^\varepsilon_{\text{self}} &:= 2\sum_i \iint G(x-y)\,d\mu^\varepsilon_i(x)\,d\mu^\varepsilon_i(y) \\
&\geq -\frac{1}{\pi}\sum_i \iint \left(\ln|x-y| + C\right) d\mu^\varepsilon_i(x)\,d\mu^\varepsilon_i(y). \\
&= -\frac{1}{\pi|\ln\varepsilon|^2}\sum_i \int_{\overline{\Omega}^+_i}\int_{\overline{\Omega}^+_i} \left(\ln(\varepsilon^{1/3}|\ln\varepsilon|^{2/3}|\overline{x}-\overline{y}|) + C\right) d\bar{x}\,d\bar{y},
\end{aligned} \tag{4.7}
$$

for some $C > 0$ independent of ε, where in equation (4.7) we have rescaled coordinates $\bar{x} = \varepsilon^{-1/3}|\ln\varepsilon|^{1/3}x$, $\bar{y} = \varepsilon^{-1/3}|\ln\varepsilon|^{1/3}$ and introduced the rescaled versions $\overline{\Omega}^+_i$ of Ω^+_i. Expanding the logarithm in (4.7) and using (3.23) and (3.2), we obtain that $\bar{E}^\varepsilon_{\text{self}}$ can be bounded from below as follows:

$$
\begin{aligned}
\bar{E}^\varepsilon_{\text{self}} &\geq \frac{1}{|\ln\varepsilon|}\sum_i |A^\varepsilon_i|^2\left(\frac{1}{3\pi} - C\left(\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|}\right) - \frac{1}{\pi|A^\varepsilon_i|^2|\ln\varepsilon|}\int_{\overline{\Omega}^+_i}\int_{\overline{\Omega}^+_i}\ln|\overline{x}-\overline{y}|\,d\bar{x}\,d\overline{y}\right) \\
&\geq \frac{1}{|\ln\varepsilon|}\sum_i |A^\varepsilon_i|^2\left(\frac{1}{3\pi} - C\left(\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|}\right) - \frac{1}{\pi|\ln\varepsilon|}\ln P^\varepsilon_i\right) \\
&\geq \frac{1}{|\ln\varepsilon|}\sum_i |A^\varepsilon_i|^2\left(\frac{1}{3\pi} - C\left(\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|}\right)\right),
\end{aligned} \tag{4.8}
$$

for some $C > 0$ independent of ε (which changes from line to line).

Now observe that the term in parentheses appearing in the right-hand side of (4.8) is positive for ε sufficiently small. Using this and the fact that $A^\varepsilon_i \geq \tilde{A}^\varepsilon_i$, from (4.8) we obtain

$$
\bar{E}^\varepsilon_{\text{self}} \geq \frac{1}{|\ln\varepsilon|}\sum_i |\tilde{A}^\varepsilon_i|^2\left(\frac{1}{3\pi} - C\left(\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|}\right)\right), \tag{4.9}
$$

where $C > 0$ is a constant independent of ε. It is also clear from the definition of $\tilde{A}^\varepsilon_i$ that there exists a constant $c > 0$ such that

$$
|\tilde{A}^\varepsilon_i|^2 \leq cA^\varepsilon_i. \tag{4.10}
$$

Combining this inequality with (4.9) and choosing any $\eta > 0$, for ε small enough we have

$\eta > Cc\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|^2}$ and, therefore, from (4.6) we obtain

$$\begin{aligned}\bar{E}^\varepsilon[u^\varepsilon] \geq & I^\varepsilon_{\text{def}} + \frac{1}{|\ln\varepsilon|}\sum_i \left(\sqrt{4\pi A_i^\varepsilon} - \left(\frac{2\bar{\delta}}{\kappa^2} + \eta\right) A_i^\varepsilon + \frac{1}{3\pi}|\tilde{A}_i^\varepsilon|^2\right) \\ & + 2\iint G_\rho(x-y)d\mu^\varepsilon(x)d\mu^\varepsilon(y). \end{aligned} \tag{4.11}$$

Step 2: Optimization over A_i^ε.

Focusing on the second term in the right-hand side of (4.11), we define

$$f(x) := \frac{2\sqrt{\pi}}{\sqrt{x}} + \frac{1}{3\pi}x, \tag{4.12}$$

and observe that f is strictly convex and attains its minimum of $3^{2/3}$ at $x = 3^{2/3}\pi$, with

$$f''(x) = \frac{3\sqrt{\pi}}{2x^{5/2}}. \tag{4.13}$$

We claim that we can bound the second term in the right-hand side of (4.11) from below by the sum $I + II + III$ of the following three terms:

$$I = \frac{1}{|\ln\varepsilon|}\left(3^{2/3} - \frac{2\bar{\delta}}{\kappa^2} - \eta\right)\sum_i A_i^\varepsilon + \frac{1}{|\ln\varepsilon|}\sum_{A_i^\varepsilon > 3^{2/3}\pi\gamma^{-1}} 3^{2/3}(3^{-1}\gamma^{-1} - 1)A_i^\varepsilon, \tag{4.14}$$

$$II = \frac{1}{|\ln\varepsilon|}\frac{\gamma^{5/2}}{4\pi^2\cdot 3^{2/3}}\sum_{A_i^\varepsilon < 3^{2/3}\pi\gamma} A_i^\varepsilon(A_i^\varepsilon - 3^{2/3}\pi)^2, \tag{4.15}$$

$$III = \frac{1}{|\ln\varepsilon|}\frac{\gamma^{7/2}}{4\pi}\sum_{3^{2/3}\pi\gamma \leq A_i^\varepsilon \leq 3^{2/3}\pi\gamma^{-1}} (A_i^\varepsilon - 3^{2/3}\pi)^2. \tag{4.16}$$

Before proving this, observe that defining

$$M^\varepsilon := \bar{E}^\varepsilon[u^\varepsilon] - \frac{1}{|\ln\varepsilon|}\left(3^{2/3} - \frac{2\bar{\delta}}{\kappa^2} - \eta\right)\sum_i A_i^\varepsilon - 2\iint G_\rho(x-y)d\mu^\varepsilon(x)d\mu^\varepsilon(y), \tag{4.17}$$

we have from (4.11) and (4.14)–(4.16) that if I_γ^ε is as in Theorem 2, then

$$M^\varepsilon \geq \frac{c_1}{|\ln\varepsilon|}\sum_{i\notin I_\gamma^\varepsilon} A_i^\varepsilon + \frac{c_2}{|\ln\varepsilon|}\sum_{i\in I_\gamma^\varepsilon}(A_i^\varepsilon - 3^{2/3}\pi)^2 + I^\varepsilon_{\text{def}} \geq 0 \qquad \forall\gamma\in(0,\tfrac{1}{3}), \tag{4.18}$$

for some constants $c_1, c_2 > 0$ depending only on γ.

We now argue in favor of the lower bound based on (4.14)–(4.16). First observe that by (4.1) we have for all $A_i^\varepsilon \geq 3^{2/3}\pi\gamma^{-1}$:

$$\sqrt{4\pi A_i^\varepsilon} + \frac{1}{3\pi}|\tilde{A}_i^\varepsilon|^2 - \left(\frac{2\bar{\delta}}{\kappa^2} + \eta\right) A_i^\varepsilon \geq \left(3^{2/3} - \frac{2\bar{\delta}}{\kappa^2} - \eta\right) A_i^\varepsilon + 3^{2/3}(3^{-1}\gamma^{-1} - 1)A_i^\varepsilon. \quad (4.19)$$

When $A_i^\varepsilon < 3^{2/3}\pi\gamma^{-1}$, which corresponds to both (4.15) and (4.16), we use the convexity of f and (4.13):

$$\begin{aligned}
\sqrt{4\pi A_i^\varepsilon} + \frac{1}{3\pi}|\tilde{A}_i^\varepsilon|^2 - \left(\frac{2\bar{\delta}}{\kappa^2} + \eta\right) A_i^\varepsilon &= A_i^\varepsilon \left(\frac{2\sqrt{\pi}}{\sqrt{A_i^\varepsilon}} + \frac{1}{3\pi}A_i^\varepsilon - \frac{2\bar{\delta}}{\kappa^2} - \eta\right) \\
&= A_i^\varepsilon \left(f(A_i^\varepsilon) - \frac{2\bar{\delta}}{\kappa^2} - \eta\right) \quad (4.20) \\
&\geq \left(3^{2/3} - \frac{2\bar{\delta}}{\kappa^2} - \eta\right) A_i^\varepsilon + \frac{1}{2}A_i^\varepsilon f''\left(3^{2/3}\pi\gamma^{-1}\right)(A_i^\varepsilon - 3^{2/3}\pi)^2,
\end{aligned}$$

where the last line follows from the second order Taylor formula for $f(x)$ about $x = 3^{2/3}\pi$ and the fact that $f''(x)$ is decreasing. Combining (4.17), (4.19) and (4.20) yields $M^\varepsilon \geq I + II + III$.

Now using (4.17) and (4.18) with γ sufficiently small, we deduce that

$$\bar{E}^\varepsilon[u^\varepsilon] \geq \frac{1}{|\ln\varepsilon|}\left(3^{2/3} - \frac{2\bar{\delta}}{\kappa^2} - \eta\right)\sum_i A_i^\varepsilon + 2\iint G_\rho(x-y)d\mu^\varepsilon(x)d\mu^\varepsilon(y). \quad (4.21)$$

Step 3: Passage to the limit.

We may now conclude from (2.13)–(2.15), (2.17), (3.7), (3.9) and (3.23) that

$$\limsup_{\varepsilon\to 0}\int_{\mathbb{T}_l^2}(|\nabla v^\varepsilon|^2 + \kappa^2|v^\varepsilon|^2)dx < +\infty, \quad (4.22)$$

while (μ^ε) are bounded in the sense of measures from (3.24). Consequently, up to a subsequence

$$v^\varepsilon \rightharpoonup v \text{ in } H^1(\mathbb{T}_\ell^2), \quad (4.23)$$

$$\mu^\varepsilon \overset{*}{\rightharpoonup} \mu \text{ in } C(\mathbb{T}_\ell^2), \quad (4.24)$$

where

$$-\Delta v + \kappa^2 v = \mu \quad (4.25)$$

holds in the distributional sense. Now passing to the limit in (4.21) and recalling (2.12), we obtain

$$\liminf_{\varepsilon\to 0} \bar{E}^\varepsilon[u^\varepsilon] \geq \left(3^{2/3} - \frac{2\bar{\delta}}{\kappa^2} - \eta\right) \int d\mu + 2 \iint G_\rho(x-y) d\mu(x) d\mu(y), \tag{4.26}$$

using continuity of G_ρ. On the other hand, we have $G_\rho(x-y) \to G(x-y)$ monotonically from below for each $x \neq y$ as $\rho \to 0$. Moreover, since μ satisfies (3.5), the set $\{(x,y) \in \mathbb{T}^2_\ell \times \mathbb{T}^2_\ell : x = y\}$ is $\mu \otimes \mu$-negligible. An application of monotone convergence theorem then yields

$$\liminf_{\varepsilon\to 0} \bar{E}^\varepsilon[u^\varepsilon] \geq \left(3^{2/3} - \frac{2\bar{\delta}}{\kappa^2}\right) \int d\mu + 2 \iint G(x-y) d\mu(x) d\mu(y), \tag{4.27}$$

upon sending $\rho \to 0$ and then $\eta \to 0$. □

We now argue in favor of the corresponding upper bound in Theorem 1. The construction resembles quite closely that of the vortex construction in [47] for the two dimensional Ginzburg-Landau functional and indeed we borrow several ideas from that proof and occasionally refer the reader to that paper for details.

4.2 Proof of the Upper Bound, Theorem 1 ii)

As in the proof of the lower bound, we set $d\mu_i^\varepsilon(x)$ as in (4.3), so that $\mu^\varepsilon = \sum \mu_i^\varepsilon$. If $\int_{\mathbb{T}^2_\ell} d\mu = 0$, there is nothing to prove. Otherwise, using a mollification with a strictly positive mollifier we can always approximate the measure μ by a measure with a smooth strictly positive density and retrieve a recovery sequence by a standard diagonal argument. Hence without loss of generality in this section we assume that

$$d\mu(x) = g(x)dx, \qquad c \leq g \leq C, \tag{4.28}$$

for some $C > c > 0$.

Step 1: Construction of the configuration.

We claim that for ε sufficiently small it is possible to place a total of $N(\varepsilon)$ disjoint spherical droplets, where

$$N(\varepsilon) = \frac{1}{3^{2/3}} \frac{|\ln \varepsilon|}{\pi} \mu(\mathbb{T}^2_\ell) + o(|\ln \varepsilon|), \tag{4.29}$$

with centers $\{a_i\}$ in $\mathbb{T}^2_\ell$ and radius

$$r = 3^{1/3} \varepsilon^{1/3} |\ln \varepsilon|^{-1/3}, \tag{4.30}$$

and satisfying for all $i \neq j$

$$d(\varepsilon) := \min |a_i - a_j| \geq \frac{C}{\sqrt{N(\varepsilon)}}, \tag{4.31}$$

for some constant $C > 0$ depending only on μ. Indeed, given μ satisfying (4.28), for ε sufficiently small we can partition $\mathbb{T}_\ell^2$ into disjoint squares $\{K_i\}$ of side length $\eta_\varepsilon > 0$ (hereafter simply denoted η) satisfying

$$|\ln \varepsilon|^{-1/2} \ll \eta \ll 1. \tag{4.32}$$

In each K_i we place

$$N_{K_i}(\varepsilon) = \left\lfloor \frac{1}{3^{2/3}} \frac{|\ln \varepsilon|}{\pi} \mu(K_i) \right\rfloor \tag{4.33}$$

points a_i (here $m = \lfloor x \rfloor$ denotes the smallest integer $m \leq x$) satisfying (4.31) and in addition

$$\text{dist } (a_i, \partial K_i) \geq \frac{C}{\sqrt{N(\varepsilon)}}, \qquad N(\varepsilon) := \sum_i N_{K_i}. \tag{4.34}$$

As argued in [47], our ability to do this follows from the estimate:

$$c\eta^2 \leq \mu(K_i) \leq C\eta^2, \tag{4.35}$$

which follows from (4.28) together with (4.32). We finally define our configuration u^ε by setting the connected components Ω_i^+ of $\Omega^+ = \{u^\varepsilon = +1\}$ to be balls of radius r from (4.30) centered at a_i, i.e. $\Omega_i^+ := B_r(a_i)$. We set $u^\varepsilon = -1$ in the complement of these balls.

With these choices we have

$$\begin{aligned} \bar{E}^\varepsilon[u^\varepsilon] &= \frac{2\pi \cdot 3^{1/3} N(\varepsilon)}{|\ln \varepsilon|} - \frac{2\pi \cdot 3^{2/3} N(\varepsilon) \bar{\delta}}{|\ln \varepsilon| \kappa^2} + 2 \iint G(x-y) d\mu^\varepsilon(x) d\mu^\varepsilon(y) \\ &= \frac{2}{3^{1/3}} \mu(\mathbb{T}_\ell^2) - \frac{2\bar{\delta}}{\kappa^2} \mu(\mathbb{T}_\ell^2) + 2 \iint G(x-y) d\mu^\varepsilon(x) d\mu^\varepsilon(y) + o(1). \end{aligned} \tag{4.36}$$

The main point of the rest of the proof is to show that the integral term in (4.36) converges to $\iint G(x-y) d\mu(x) d\mu(y) + 3^{-1/3} \int d\mu$, with the non-trivial last term coming from the self-interaction of the droplets. To prove that these are the only contributions to the limit energy, we need to use the fact that the droplets do not concentrate too much as $\varepsilon \to 0$.

Step 2: Convergence of the configurations.

Defining μ^ε as before, it is clear from the construction that

$$\mu^\varepsilon \rightharpoonup \mu \text{ in } (C(\mathbb{T}_\ell^2))^*. \tag{4.37}$$

Fix $\rho > 0$ sufficiently small (depending only on κ and ℓ) and consider $G_\rho(x-y)$ defined as in (4.5). By the continuity of G_ρ in $\mathbb{T}^2_\ell$ we have

$$\lim_{\varepsilon\to 0} \iint G_\rho(x-y)d\mu^\varepsilon(x)d\mu^\varepsilon(y) = \iint G_\rho(x-y)d\mu(x)d\mu(y). \tag{4.38}$$

Now, let I_ρ be the collection of indices (i,j) such that $0 < |a_i - a_j| < \rho$. Then for ε small enough we can write

$$\begin{aligned}
&\iint \big(G(x-y) - G_\rho(x-y)\big)d\mu^\varepsilon(x)d\mu^\varepsilon(y) \\
&\quad \le \sum_{i=1}^{N(\varepsilon)} \iint G(x-y)d\mu_i^\varepsilon(x)d\mu_i^\varepsilon(y) + \sum_{(i,j)\in I_\rho} \iint G(x-y)d\mu_i^\varepsilon(x)d\mu_j^\varepsilon(y) \qquad (4.39)\\
&\quad \le \frac{1}{6\pi|\ln\varepsilon|}\sum_{i=1}^{N(\varepsilon)}|A_i^\varepsilon|^2 + \frac{C\ln|\ln\varepsilon|}{|\ln\varepsilon|} + \frac{C'}{|\ln\varepsilon|^2}\sum_{(i,j)\in I_\rho} A_i^\varepsilon A_j^\varepsilon \left|\ln \operatorname{dist}\,(\Omega_i^+,\Omega_j^+)\right|,
\end{aligned}$$

for some $C, C' > 0$ independent of ε or ρ, where $A_i^\varepsilon = 3^{2/3}\pi$ and we expanded the Green's function as in (4.7) in the proof of the lower bound. Now, for $k = 1, 2, \ldots, K_\rho(\varepsilon)$, with $K_\rho(\varepsilon) := \lfloor \rho/d(\varepsilon)\rfloor$, let $I_\rho^k \subset I_\rho$ be disjoint sets consisting of all indices (i,j) such that $kd(\varepsilon) \le |a_i - a_j| < (k+1)d(\varepsilon)$. Since by the result on optimal packing density of disks in the plane [18] we have $|I_\rho^k| \le ckN(\varepsilon)$ for some universal $c > 0$ (here again $|I_\rho^k|$ denotes the cardinality of I_ρ^k), in view of (4.29) it holds that

$$\begin{aligned}
&\frac{1}{|\ln\varepsilon|^2}\sum_{(i,j)\in I_\rho} A_i^\varepsilon A_j^\varepsilon \left|\ln \operatorname{dist}\,(\Omega_i^+,\Omega_j^+)\right| \le \frac{CN(\varepsilon)}{|\ln\varepsilon|^2}\sum_{k=1}^{K_\rho(\varepsilon)} k|\ln(kd(\varepsilon))| \\
&\le \frac{2CN(\varepsilon)}{|\ln\varepsilon|^2 d^2(\varepsilon)}\int_{d(\varepsilon)}^{\rho} t|\ln t|dt \le C'\left(\frac{|\ln d(\varepsilon)|}{|\ln\varepsilon|} + \rho^2|\ln\rho|\right) \le 2C'\rho^2|\ln\rho|, \qquad (4.40)
\end{aligned}$$

for some $C, C' > 0$ independent of ε or ρ, when ε and ρ are sufficiently small. Therefore, from (4.30) and (4.39) we obtain

$$\limsup_{\varepsilon\to 0} \iint \big(G(x-y) - G_\rho(x-y)\big)d\mu^\varepsilon(x)d\mu^\varepsilon(y) \le 2^{-1}\cdot 3^{-1/3} + o(\rho). \tag{4.41}$$

Finally combining (4.41) with (4.36) and (4.38), upon sending $\varepsilon \to 0$, then $\rho \to 0$ and applying the monotone convergence theorem we have

$$\lim_{\varepsilon\to 0} \bar{E}^\varepsilon[u^\varepsilon] \le \left(3^{2/3} - \frac{2\bar{\delta}}{\kappa^2}\right)\int d\mu + 2\iint G(x-y)d\mu(x)d\mu(y), \tag{4.42}$$

as required. The fact that $v^\varepsilon \rightharpoonup v$ follows from (4.37) and the uniform bounds just demonstrated on the terms involving the Green's function in (4.36), from which it follows that (2.18) is satisfied distributionally. $\square$

5 Proof of Theorem 2

In the proof of Sec. 4, we have in fact established Theorem 2, which is clear by (4.18). Indeed, we have for a sequence of almost minimizers (u^ε):

$$\lim_{\varepsilon \to 0} E^\varepsilon[u^\varepsilon] - E_0[\mu] = 0. \tag{5.1}$$

Observing that M^ε defined in (4.17) does not contribute to $E_0[\mu]$, we have established that $M^\varepsilon \to 0$ as $\varepsilon \to 0$ for any $\gamma < \frac{1}{3}$ and, as a consequence, we obtain (2.22)–(2.24) for, say, $\gamma = \frac{1}{6}$. Then it is easy to see from the definition of I^ε_γ that the statement of the Theorem, in fact, holds for any $\gamma \in (0,1)$. $\square$

6 Proof of Theorem 3

We now turn to the proof of Theorem 3 extending the result of Theorem 1 for the sharp interface energy E^ε to the diffuse interface energy $\mathcal{E}^\varepsilon$. The proof proceeds by a refinement of the ideas of [38, Sec. 4] to establish matching upper and lower bounds for $\mathcal{E}^\varepsilon$ in terms of E^ε for sequences with bounded energy.

Step 1: Approximate lower bound.

In the following, it is convenient to rewrite the energy (2.4) in an equivalent form

$$\mathcal{E}^\varepsilon[u^\varepsilon] = \int_{\mathbb{T}^2_\ell} \left(\frac{\varepsilon^2}{2}|\nabla u^\varepsilon|^2 + W(u^\varepsilon) + \frac{1}{2}|\nabla v^\varepsilon|^2 \right) dx, \qquad -\Delta v^\varepsilon = u^\varepsilon - \bar{u}^\varepsilon, \qquad \int_{\mathbb{T}^2_\ell} v^\varepsilon dx = 0. \tag{6.1}$$

Fix any $\delta \in (0,1)$ and consider a sequence $(u^\varepsilon) \in \mathcal{A}^\varepsilon$ such that $\limsup_{\varepsilon\to 0} \|u^\varepsilon\|_{L^\infty(\mathbb{T}^2_\ell)} \leq 1$ and $\mathcal{E}^\varepsilon[u^\varepsilon] \leq C\varepsilon^{4/3}|\ln\varepsilon|^{2/3}$ for some $C > 0$ independent of ε. Then we claim that

$$\limsup_{\varepsilon \to 0} \|u^\varepsilon\|_{L^\infty(\mathbb{T}^2_\ell)} = 1, \qquad \lim_{\varepsilon\to 0} \|v^\varepsilon\|_{L^\infty(\mathbb{T}^2_\ell)} = 0. \tag{6.2}$$

Indeed, for the first statement we have from the definition of $\mathcal{E}^\varepsilon$ in (2.4) that

$$|\Omega^\delta_0| \leq C\varepsilon^{4/3}|\ln\varepsilon|^{2/3}\delta^{-2}, \qquad \Omega^\delta_0 := \{-1+\delta \leq u^\varepsilon \leq 1-\delta\}, \tag{6.3}$$

for some $C > 0$ independent of ε. Hence, in particular, $\limsup_{\varepsilon\to 0} \|u^\varepsilon\|_{L^\infty(\mathbb{T}^2_\ell)} \geq 1$, proving the first statement of (6.2). To prove the second statement in (6.2), we note that by standard elliptic theory (see, e.g., [21, Theorem 9.9]) we have $\|v^\varepsilon\|_{W^{2,p}(\mathbb{T}^2_\ell)} \leq C'$ for any $p > 2$ and some $C' > 0$ independent of ε and, hence, by Sobolev embedding $\|\nabla v^\varepsilon\|_{L^\infty(\mathbb{T}^2_\ell)} \leq$

C'' for some $C'' > 0$ independent of ε as well. Therefore, applying Poincaré's inequality, we obtain

$$C\varepsilon^{4/3}|\ln\varepsilon|^{2/3} \geq \mathcal{E}^\varepsilon[u^\varepsilon] \geq C' \int_{\mathbb{T}_\ell^2} |v^\varepsilon|^2 dx \geq C''\|v^\varepsilon\|^4_{L^\infty(\mathbb{T}_\ell^2)}, \tag{6.4}$$

for some $C', C'' > 0$ independent of ε, yielding the claim.

In view of (6.2), for small enough ε we have $\|u^\varepsilon\|_{L^\infty(\mathbb{T}_\ell^2)} \leq 1 + \delta^3$ and $\|v^\varepsilon\|_{L^\infty(\mathbb{T}_\ell^2)} \leq \delta^3$, and by the assumption on energy we may further assume that $\mathcal{E}^\varepsilon[u^\varepsilon] \leq \delta^{12}$. Therefore, by [38, Proposition 4.2] there exists a function $\tilde{u}_0^\varepsilon \in \mathcal{A}$ such that

$$\mathcal{E}^\varepsilon[u^\varepsilon] \geq (1 - \delta^{1/2})E^\varepsilon[\tilde{u}_0^\varepsilon]. \tag{6.5}$$

In particular, $(\tilde{u}_0^\varepsilon)$ satisfy the assumptions of Theorem 1, and, therefore, upon extraction of subsequences we have $\tilde{\mu}_0^\varepsilon \rightharpoonup \mu \in \mathcal{M}^+(\mathbb{T}_\ell^2) \cap H^{-1}(\mathbb{T}_\ell^2)$ in $(C(\mathbb{T}_\ell^2))^*$, where

$$d\tilde{\mu}_0^\varepsilon(x) := \tfrac{1}{2}\varepsilon^{-2/3}|\ln\varepsilon|^{-1/3}(1 + \tilde{u}_0^\varepsilon(x))dx. \tag{6.6}$$

Furthermore, recalling that by construction the jump set of $\tilde{u}_0^\varepsilon$ is either contained in Ω_0^δ or empty, see the proof of [38, Lemma 4.1], from (6.3) we have

$$\|\tilde{u}_0^\varepsilon - u_0^\varepsilon\|_{L^1(\mathbb{T}_\ell^2)} \leq C\varepsilon^{4/3}|\ln\varepsilon|^{2/3}\delta^{-2}, \tag{6.7}$$

where u_0^ε is given by (2.29), for some $C > 0$ independent of ε. Comparing (6.7) with (6.6), we then see that $\mu_0^\varepsilon \rightharpoonup \mu$ in $(C(\mathbb{T}_\ell^2))^*$ as well. The result of part (i) of Theorem 3 then follows by the arbitrariness of $\delta > 0$ via a diagonal process. □

Step 2: Approximate upper bound.

First note that if $\mu = 0$, we can choose $u^\varepsilon = \bar{u}^\varepsilon$. Indeed, we have $\varepsilon^{-4/3}|\ln\varepsilon|^{-2/3}\mathcal{E}^\varepsilon[\bar{u}^\varepsilon] = \ell^2 W(\bar{u}^\varepsilon) = \frac{\ell^2\bar{\delta}^2}{2\kappa^2} + o(1)$ and $\bar{u}^\varepsilon \to -1$. On the other hand, if $\int_{\mathbb{T}_\ell^2} d\mu > 0$, we can construct the approximate upper bounds for a suitable lifting of the recovery sequences in the proof of Theorem 1(ii) to $\mathcal{A}^\varepsilon$. Let $(\tilde{u}_0^\varepsilon) \in \mathcal{A}$ be a recovery sequence constructed in Sec. 4.2. This sequence consists of circular droplets of the optimal radius $r = 3^{1/3}\varepsilon^{1/3}|\ln\varepsilon|^{-1/3} \gg \varepsilon^{1/2}$ and mutual distance $d \geq C|\ln\varepsilon|^{-1/2} \gg \varepsilon^{1/2}$, for some $C > 0$ independent of ε. In addition, since

$$E^\varepsilon[\tilde{u}_0^\varepsilon] = \frac{\varepsilon}{2}\int_{\mathbb{T}_\ell^2} |\nabla\tilde{u}_0^\varepsilon|\, dx + 2\int_{\mathbb{T}_\ell^2} \Big(|\nabla\tilde{v}^\varepsilon|^2 + \kappa^2|\tilde{v}^\varepsilon|^2\Big)dx \leq C\varepsilon^{4/3}|\ln\varepsilon|^{2/3}, \tag{6.8}$$

where $\tilde{v}^\varepsilon(x) = \int_{\mathbb{T}_\ell^2} G(x-y)(\tilde{u}_0^\varepsilon(y) - \bar{u}^\varepsilon)dy$, for some $C > 0$ independent of ε, by the argument of (6.4) one can see that $\lim_{\varepsilon\to 0}\|\tilde{v}^\varepsilon\|_{L^\infty(\mathbb{T}_\ell^2)} = 0$. Therefore, for any $\delta \in (0,1)$

and $\varepsilon > 0$ sufficiently small we have $\|\tilde{v}^\varepsilon\|_{L^\infty(\mathbb{T}^2_\ell)} \leq \delta$ and $E^\varepsilon[\tilde{u}^\varepsilon_0] \leq \delta^{5/2}$. We can then apply [38, Proposition 4.3] to obtain a function $u^\varepsilon \in \mathcal{A}^\varepsilon$ such that

$$\mathcal{E}^\varepsilon[u^\varepsilon] \leq (1 + \delta^{1/2}) E^\varepsilon[\tilde{u}^\varepsilon_0]. \tag{6.9}$$

Furthermore, by the construction of u^ε (see [38, Eqs. (4.31)–(4.33)]) and arbitrariness of $\delta > 0$, we also have $\limsup_{\varepsilon \to 0} \|u^\varepsilon\|_{L^\infty(\mathbb{T}^2_\ell)} = 1$, and

$$\|\tilde{u}^\varepsilon_0 - u^\varepsilon_0\|_{L^1(\mathbb{T}^2_\ell)} \leq C \varepsilon^{4/3} |\ln \varepsilon|^{2/3}, \tag{6.10}$$

for some $C > 0$ independent of ε, where u^ε_0 is given by (2.29), and we used (6.8). Hence $\mu^\varepsilon_0 \rightharpoonup \mu = \lim_{\varepsilon \to 0} \tilde{\mu}^\varepsilon_0$ in $(C(\mathbb{T}^2_\ell))^*$. The result of part (ii) of Theorem 3 again follows by arbitrariness of $\delta > 0$ via a diagonal process. $\square$

Remark 6.1. *It is possible to chose $\delta = \varepsilon^\alpha$ for $\alpha > 0$ sufficiently small in the arguments of the proof of Theorem 3. Therefore, given a sequence of minimizers $(u^\varepsilon) \in \mathcal{A}^\varepsilon$ of $\mathcal{E}^\varepsilon$ and the corresponding sequence $(u^\varepsilon_0) \in \mathcal{A}$ of minimizers of E^ε, one has*

$$\varepsilon^{-4/3} |\ln \varepsilon|^{-2/3} \mathcal{E}^\varepsilon[u^\varepsilon] = \varepsilon^{-4/3} |\ln \varepsilon|^{-2/3} E^\varepsilon[u^\varepsilon_0] + O(\varepsilon^\alpha), \tag{6.11}$$

for some $\alpha \ll 1$, as $\varepsilon \to 0$.

7 Proof of Theorem 4

Let (u^ε) be a sequence from Theorem 3(ii). Arguing as in Step 1 of the proof of Theorem 3, for every $\delta > 0$ sufficiently small there exists a sequence $(\tilde{u}^\varepsilon_0) \in \mathcal{A}$ such that (6.5) holds, the jump set of $\tilde{u}^\varepsilon_0$ is contained in $\{-1 + \delta \leq u^\varepsilon \leq 1 - \delta\}$, and if $\tilde{\mu}^\varepsilon_0$ is defined via (6.6), then $\tilde{\mu}^\varepsilon_0 \rightharpoonup \mu$ in $(C(\mathbb{T}^2_\ell))^*$. On the other hand, applying the result of Theorem 1(i), we obtain

$$\begin{aligned} E^0[\mu] &\geq \limsup_{\varepsilon \to 0} \varepsilon^{-4/3} |\ln \varepsilon|^{-2/3} \mathcal{E}^\varepsilon[u^\varepsilon] \geq (1 - \delta^{1/2}) \limsup_{\varepsilon \to 0} \varepsilon^{-4/3} |\ln \varepsilon|^{-2/3} E^\varepsilon[\tilde{u}^\varepsilon_0] \\ &\geq (1 - \delta^{1/2}) \liminf_{\varepsilon \to 0} \varepsilon^{-4/3} |\ln \varepsilon|^{-2/3} E^\varepsilon[\tilde{u}^\varepsilon_0] \geq (1 - \delta^{1/2}) E^0[\mu]. \end{aligned} \tag{7.1}$$

Therefore, in view of arbitrariness of $\delta > 0$ we conclude that $(\tilde{u}^\varepsilon_0)$ is a sequence of almost minimizers of E^ε with prescribed density μ by a diagonal process. As a consequence, Theorem 2 applies to $(\tilde{u}^\varepsilon_0)$. Moreover, by (6.7) and the fact that

$$\lim_{\varepsilon \to 0} \varepsilon^{-2/3} |\ln \varepsilon|^{-1/3} |\{u^\varepsilon > 0\}| = \int_{\mathbb{T}^2_\ell} d\mu > 0, \tag{7.2}$$

we obtain (2.34). $\square$

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References

[1] G. Alberti, R. Choksi, and F. Otto. Uniform energy distribution for an isoperimetric problem with long-range interactions. *J. Amer. Math. Soc.*, 22:569–605, 2009.

[2] L. Ambrosio, V. Caselles, S. Masnou, and J.-M. Morel. Connected components of sets of finite perimeter and applications to image processing. *J. Eur. Math. Soc.*, 3:39–92, 2001.

[3] F. S. Bates and G. H. Fredrickson. Block copolymers – designer soft materials. *Physics Today*, 52:32–38, 1999.

[4] A. Braides. Γ-*convergence for beginners*, volume 22 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2002.

[5] A. Braides and L. Truskinovsky. Asymptotic expansions by Γ-convergence. *Continuum Mech. Thermodyn.*, 20:21–62, 2008.

[6] L. Q. Chen and A. G. Khachaturyan. Dynamics of simultaneous ordering and phase separation and effect of long-range Coulomb interactions. *Phys. Rev. Lett.*, 70:1477–1480, 1993.

[7] R. Choksi. Scaling laws in microphase separation of diblock copolymers. *J. Nonlinear Sci.*, 11:223–236, 2001.

[8] R. Choksi, S. Conti, R. V. Kohn, and F. Otto. Ground state energy scaling laws during the onset and destruction of the intermediate state in a Type-I superconductor. *Comm. Pure Appl. Math.*, 61:595–626, 2008.

[9] R. Choksi and R. V. Kohn. Bounds on the micromagnetic energy of a uniaxial ferromagnet. *Comm. Pure Appl. Math.*, 51:259–289, 1998.

[10] R. Choksi, R. V. Kohn, and F. Otto. Domain branching in uniaxial ferromagnets: a scaling law for the minimum energy. *Commun. Math. Phys.*, 201:61–79, 1999.

[11] R. Choksi, R. V. Kohn, and F. Otto. Energy minimization and flux domain structure in the intermediate state of a Type-I superconductor. *J. Nonlinear Sci.*, 14:119–171, 2004.

[12] R. Choksi, M. Maras, and J. F. Williams. 2D phase diagram for minimizers of a Cahn–Hilliard functional with long-range interactions. *SIAM J. Appl. Dyn. Syst.*, 10:1344–1362, 2011.

[13] R. Choksi and M. A. Peletier. Small volume fraction limit of the diblock copolymer problem: I. Sharp interface functional. *SIAM J. Math. Anal.*, 42:1334–1370, 2010.

[14] R. Choksi and M. A. Peletier. Small volume fraction limit of the diblock copolymer problem: II. Diffuse interface functional. *SIAM J. Math. Anal.*, 43:739–763, 2011.

[15] P. G. de Gennes. Effect of cross-links on a mixture of polymers. *J. de Physique – Lett.*, 40:69–72, 1979.

[16] A. DeSimone, R. V. Kohn, S. Müller, and F. Otto. Magnetic microstructures—a paradigm of multiscale problems. In *ICIAM 99 (Edinburgh)*, pages 175–190. Oxford Univ. Press, 2000.

[17] V. J. Emery and S. A. Kivelson. Frustrated electronic phase-separation and high-temperature superconductors. *Physica C*, 209:597–621, 1993.

[18] L. Fejes Tóth. Über einen geometrischen Satz. *Math. Z.*, 46:83–85, 1940.

[19] G. Friesecke, R. D. James, and S. Müller. A hierarchy of plate models derived from nonlinear elasticity by Gamma-convergence. *Arch. Ration. Mech. Anal.*, 180:183–236, 2006.

[20] N. Fusco, F. Maggi, and A. Pratelli. The sharp quantitative isoperimetric inequality. *Ann. of Math.*, 168:941–980, 2008.

[21] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer-Verlag, Berlin, 1983.

[22] S. Glotzer, E. A. Di Marzio, and M. Muthukumar. Reaction-controlled morphology of phase-separating mixtures. *Phys. Rev. Lett.*, 74:2034–2037, 1995.

[23] D. Goldman, C. B. Muratov, and S. Serfaty. The Γ-limit of the two-dimensional Ohta-Kawasaki energy. II. Droplet arrangement at the sharp interface level via the renormalized energy. *(submitted to Arch. Rational Mech. Anal.)*, 2012. Preprint. arXiv:1210.5098.

[24] C. C. Grimes and G. Adams. Evidence for a liquid-to-crystal phase transition in a classical, two-dimensional sheet of electrons. *Phys. Rev. Lett.*, 42:795–798, 1979.

[25] A. Hubert and R. Schäfer. *Magnetic Domains*. Springer, Berlin, 1998.

[26] R. P. Huebener. *Magnetic flux structures in superconductors*. Springer-Verlag, Berlin, 1979.

[27] H. Knüpfer and C. B. Muratov. Domain structure of bulk ferromagnetic crystals in applied fields near saturation. *J. Nonlinear Sci.*, 21:921–962, 2011.

[28] H. Knüpfer and C. B. Muratov. On an isoperimetric problem with a competing non-local term. II. The general case. *Commun. Pure Appl. Math.*, 2013 (to appear).

[29] R. V. Kohn. Energy-driven pattern formation. In *International Congress of Mathematicians. Vol. I*, pages 359–383. Eur. Math. Soc., Zürich, 2007.

[30] C. Le Bris and P.-L. Lions. From atoms to crystals: a mathematical journey. *Bull. Amer. Math. Soc. (N.S.)*, 42:291–363, 2005.

[31] E. H. Lieb. Thomas-Fermi and related theories of atoms and molecules. *Rev. Mod. Phys.*, 53:603–641, 1981.

[32] S. Lundqvist and N. H. March, editors. *Theory of inhomogeneous electron gas*. Plenum Press, New York, 1983.

[33] L. Modica. The gradient theory of phase transitions and the minimal interface criterion. *Arch. Rational Mech. Anal.*, 98:123–142, 1987.

[34] L. Modica and S. Mortola. Un esempio di Γ-convergenza. *Boll. Un. Mat. Ital. B*, 14:285–299, 1977.

[35] S. Müller. Singular perturbations as a selection criterion for periodic minimizing sequences. *Calc. Var. PDE*, 1:169–204, 1993.

[36] C. B. Muratov. *Theory of domain patterns in systems with long-range interactions of Coulombic type*. Ph. D. Thesis, Boston University, 1998.

[37] C. B. Muratov. Theory of domain patterns in systems with long-range interactions of Coulomb type. *Phys. Rev. E*, 66:066108 pp. 1–25, 2002.

[38] C. B. Muratov. Droplet phases in non-local Ginzburg-Landau models with Coulomb repulsion in two dimensions. *Comm. Math. Phys.*, 299:45–87, 2010.

[39] M. Muthukumar, C. K. Ober, and E. L. Thomas. Competing interactions and levels of ordering in self-organizing polymeric materials. *Science*, 277:1225–1232, 1997.

[40] E. L. Nagaev. Phase separation in high-temperature superconductors and related magnetic systems. *Phys. Uspekhi*, 38:497–521, 1995.

[41] I. A. Nyrkova, A. R. Khokhlov, and M. Doi. Microdomain structures in polyelectrolyte systems: calculation of the phase diagrams by direct minimization of the free energy. *Macromolecules*, 27:4220–4230, 1994.

[42] T. Ohta and K. Kawasaki. Equilibrium morphologies of block copolymer melts. *Macromolecules*, 19:2621–2632, 1986.

[43] C. Radin. The ground state for soft disks. *J. Statist. Phys.*, 26:365–373, 1981.

[44] X. Ren and L. Truskinovsky. Finite scale microstructures in nonlocal elasticity. *J. Elasticity*, 59:319–355, 2000.

[45] X. Ren and J. Wei. Many droplet pattern in the cylindrical phase of diblock copolymer morphology. *Rev. Math. Phys.*, 19:879–921, 2007.

[46] X. F. Ren and J. C. Wei. On the multiplicity of solutions of two nonlocal variational problems. *SIAM J. Math. Anal.*, 31:909–924, 2000.

[47] E. Sandier and S. Serfaty. A rigorous derivation of a free-boundary problem arising in superconductivity. *Ann. Sci. École Norm. Sup. (4)*, 33:561–592, 2000.

[48] E. Sandier and S. Serfaty. *Vortices in the magnetic Ginzburg-Landau model.* Progress in Nonlinear Differential Equations and their Applications, 70. Birkhäuser Boston Inc., Boston, MA, 2007.

[49] E. Sandier and S. Serfaty. From the Ginbzurg-Landau model to vortex lattice problems. *Comm. Math. Phys.*, 313:635–743, 2012.

[50] M. Seul and D. Andelman. Domain shapes and patterns: the phenomenology of modulated phases. *Science*, 267:476–483, 1995.

[51] E. Spadaro. Uniform energy and density distribution: diblock copolymers' functional. *Interfaces Free Bound.*, 11:447–474, 2009.

[52] F. H. Stillinger. Variational model for micelle structure. *J. Chem. Phys.*, 78:4654–4661, 1983.

[53] B. A. Strukov and A. P. Levanyuk. *Ferroelectric Phenomena in Crystals: Physical Foundations.* Springer, New York, 1998.

[54] F. Theil. A proof of crystallization in two dimensions. *Comm. Math. Phys.*, 262:209–236, 2006.

[55] E. Y. Vedmedenko. *Competing Interactions and Pattern Formation in Nanoworld.* Wiley, Weinheim, Germany, 2007.

[56] H.-J. Wagner. Crystallinity in two dimensions: a note on a paper of C. Radin: "The ground state for soft disks" [J. Statist. Phys. **26** (1981), 365–373)]. *J. Stat. Phys.*, 33:523–526, 1983.

[57] E. Wigner. On the interaction of electrons in metals. *Phys. Rev.*, 46:1002–1011, 1934.

[58] N. K. Yip. Structure of stable solutions of a one-dimensional variational problem. *ESAIM Control Optim. Calc. Var.*, 12:721–751, 2006.

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