REPEATED GAMES FOR NON-LINEAR PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS AND INTEGRAL CURVATURE FLOWS

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Abstract. The main purpose of this paper is to approximate several non-local evolution equations by zero-sum repeated games in the spirit of the previous works of Kohn and the second author (2006 and 2009): general fully non-linear parabolic integro-differential equations on the one hand, and the integral curvature flow of an on the other hand. In order to do so, we start by constructing such a game for eikonal equations whose speed has a non-constant sign. This provides a (discrete) deterministic control interpretation of these evolution equations.

In all our games, two players choose positions successively, and their final payoff is determined by their positions and additional parameters of choice. Because of the non-locality of the problems approximated, by contrast with local problems, their choices have to "collect" information far from their current position. For parabolic integro-differential equations, players choose smooth functions on the whole space. For integral curvature flows, players choose hypersurfaces in the whole space and positions on these hypersurfaces.

1. General introduction

Kohn and the second author gave in [20] a deterministic control interpretation for motion by mean curvature and some other geometric laws. In particular, given an initial set $\Omega_0 \subset \mathbb{R}^N$, they prove that the repeated game invented by Joel Spencer (originally called "pusher-chooser" game, now sometimes known as the "Paul-Carol" game) [27] converges towards the mean curvature motion of $\partial\Omega_0$. In a second paper [21], they construct analogous approximations of general fully non-linear parabolic and elliptic equations.

This paper is concerned with extending this approach to several non-local evolutions, for which we construct zero-sum repeated games with two players.

Our main motivation for constructing such games is to show that viscosity solutions of an even wider class of equations have a deterministic control representation; while previously this was known to be true only for first order Hamilton-Jacobi equations, and then since [20, 21] for general local second order PDE's. Seen differently, it shows that a wide class of non-local evolutions have a minimax formulation.

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Parabolic Integro-Differential Equations. The first natural class to consider is that of general parabolic integro-differential equations (in short parabolic PIDE). The study of non-linear PIDE has attracted a lot of attention; as far as parabolic or elliptic fully non-linear PIDE equations are concerned, the viscosity theory has been well developed since the seminal paper by Soner [25]. Without giving an exhaustive list of references, it is natural to quote works by Sayah [24], Jakobsen and Karlsen [19], Barles and the first author [3] and the series of papers by Caffarelli and Silvestre about regularity, e.g. [7]. The interested reader is referred to the introduction of [1] where numerous additional references are given.

The solution to finding a deterministic control approach in that case turns out to be a natural generalization of that of [21] for the case of (local) fully non-linear parabolic equations. In the game of [21] for second order local problems, the choices of one of the players serve as proxies for the first and second order derivatives of the value function at the current point. In contrast, in the case of nonlocal problems, the players's choices have to "collect" information far from their current position: they will choose smooth functions on the whole space.

In the case where the equation happens to be local, only the first and second derivatives of that function at the current location matter, so in effect it really amounts to the game of [21].

Integral curvature flows. Secondly, we consider a class of nonlocal geometric evolutions: the integral curvature flows. Such evolutions were originally introduced to describe dislocation dynamics in [2]. This type of motion also appears in [8] where threshold dynamics associated with kernels decaying slowly at infinity are considered, and in [18, 6]. It was recently reformulated by the first author [17] in order to deal with singular interacting potentials. For more details on these flows, we refer to the introduction of [17].

The specificity of the integral curvature flow is that it is non-local in the sense that its normal speed at a boundary point x not only depends on the front close to x (such as the outer normal unit vector or the curvature tensor) but also on the whole curve. Indeed, the integral curvature is a singular integral operator.

In order to construct the game for integral curvature flows, we start with the simpler guiding case of the eikonal equation associated with a changing sign velocity, for which we give a game approximation. We are guided by the ideas of Evans and Souganidis [14]; they proved in particular that the solution of the eikonal equation can be represented by the value function of a differential game. Our first task is thus to give a discrete version of such a representation. The way we treat the change of sign of the velocity is analogous to a splitting method. There are two steps at each turn of the game: the first step retains the positive part of the velocity and is controlled by one of the players, the second step retains its negative part and is controlled by the other player.

In the case of the integral curvature equation, we thus split the integral curvature into two pieces (its positive and its negative parts) that are treated separately. Because the equation is non-local, proving that this splitting method permits to recover the full integral curvature equation is one of the many technical difficulties to overcome.

Technical framework. The framework of viscosity solutions [12, 11] and the level-set approach [23, 9, 15] are used in order to define properly the various geometric motions. We recall that the level-set approach consists in representing the initial interface as the 0-level set of a (Lipschitz) continuous function u_0, looking for the evolving interface under the same form, proving that the function $u(t, x)$ solves a partial differential equation and finally proving that the 0-level set of the function $u(t, \cdot)$ only depends on the 0-level set of u_0. The proofs of convergence follow the method of Barles-Souganidis [5] i.e. use the stability, monotonicity and consistency of the schemes provided by our games.

Open problems. The game we present for integral curvature flow, even though this is a geometric evolution, is much more complicated that the Paul-Carol game studied in [20]. It would be very nice to find a game whose rules are simpler and which would be a natural generalization of the Paul-Carol game. However, we do not know at this stage whether this is possible.

Organization of the article. The paper is organized as follows: In Section 2, we present the various equations that we study, state the definitions, present the games and give the main convergence results: first for parabolic PIDE, second for eikonal equations, and third for integral curvature flows. In Section 3, we return to these theorems in order and give their proofs.

Notation. The unit ball of $\mathbb{R}^N$ is denoted by B. A ball of radius r centered at x is denoted by $B_r(x)$. The function $\mathbf{1}_A(z)$ is defined as follows: $\mathbf{1}_A(z) = 1$ if $z \in A$ and 0 if not. The unit sphere of $\mathbb{R}^N$ is denoted by $\mathbb{S}^{N-1}$. The set of symmetric real $N \times N$ matrices is denoted by S_N.

Given two real numbers a, b, $a \wedge b$ denotes $\min(a, b)$ and $a \vee b$ denotes $\max(a, b)$. Moreover, a_+ denotes $\max(0, a)$ and $a_- = \max(0, -a)$.

The time derivative, space gradient and Hessian matrix of a function ϕ are respectively denoted by $\partial_t \phi$, $D\phi$ and $D^2\phi$.

$C^2_b(\mathbb{R}^N)$ denotes the space of C^2 bounded functions such that their first and second derivatives are also bounded.

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2. Main Results

This section is devoted to the description of the games we introduce to approximate the various geometric motions or solutions of parabolic PIDE.

Following [20, 21], in each game there are two opposing players Paul and Carol (or sometimes Helen and Mark). Paul starts at point x at time $t > 0$ with zero score. At each step n, the position x_n and time t_n are updated by using a small parameter $\varepsilon > 0$: $(t_n, x_n) = (t_n(\varepsilon), x_n(\varepsilon))$. The game continues until the running time t_N is larger than a given final time T. At the end of the game, Paul's final score is $u_T(x_N)$ where u_T is a given continuous function u_T defined on $\mathbb{R}^N$, and x_N is the final position. Paul's objective is to maximize his final score and Carol's is to obstruct him.

We define the value function u^ε of the game starting at x at time t as

$$u^\varepsilon(t, x) = \max\left(\text{final score for Paul starting from } (t, x)\right) . \tag{2.1}$$

The main results of this paper assert that the value functions associated with the games described in the next subsections converge to solutions of the corresponding evolution equations. As it is natural for control problems, the framework to use is that of viscosity solutions.

We present the games in increasing order of complexity, so we start by presenting the results for parabolic PIDE.

2.1. **General Parabolic Integro-Differential Equations.** The parabolic non-linear integro-differential equations at stake in this paper are of the following form

$$-\partial_t u + F(t,x,Du,D^2u,I[x,u]) = 0 \quad \text{in } (0,T]\times\mathbb{R}^N \tag{2.2}$$

where $T>0$ is a final time, F is a continuous non-linearity satisfying a proper ellipticity condition (see below) and $I[x,U]$ is a singular integral term defined for $U:\mathbb{R}^N\to\mathbb{R}$ as follows

$$I[x,U] = \int [U(x+z)-U(x)-DU(x)\cdot z\mathbf{1}_B(z)]\nu(dz) \tag{2.3}$$

where we recall B is the unit ball, $\mathbf{1}_B(z)=1$ if $|z|<1$ and 0 if not, and ν is a non-negative singular measure satisfying

$$\int_B |z|^2\nu(dz) < +\infty, \quad \int_{\mathbb{R}^N\setminus B} \nu(dz) < +\infty\,. \tag{2.4}$$

We also assume for simplicity that $\nu(dz)=\nu(-dz)$ but this is not a restriction. Such measures are referred to as (symmetric) Lévy measures and associated integral operators $I[x,U]$ as Lévy operators. Such equations appear in the context of mathematical finance for models driven by jump processes; see for instance [10]. Because of the games we construct, a *terminal* condition is associated with such a parabolic PIDE. Given a final time $T>0$, the solution u of (2.2) is submitted to the additional condition

$$u(T,x) = u_T(x) \tag{2.5}$$

where $u_T:\mathbb{R}^N\to\mathbb{R}$ is the terminal datum. The equation is called parabolic when the following ellipticity condition is fulfilled

$$A\le B, l\le m \Rightarrow F(t,x,p,A,l)\ge F(t,x,p,B,m)\,, \tag{2.6}$$

where $A\le B$ is meant with respect to the order on symmetric matrices. Under this condition, the equation with terminal condition (2.5) is well-posed in $(0,T]\times\mathbb{R}^N$.

We start with stating precisely the rules of our game, then we recall the definition of viscosity solutions in that context, and finally state the main convergence theorem.

2.1.1. *The game for parabolic PIDE.* We are given positive parameters $\varepsilon, R>0$. A truncated integral operator $I_R[x,\Phi]$ is defined by replacing in (2.3) $\nu(dz)$ with $\mathbf{1}_{B_R}(z)\nu(dz)$. We also consider a positive real number $\alpha\in(0,(\max(1,k_1,k_2))^{-1})$ where the constants k_1,k_2 appear in Assumption (A1). In this setting, for the sake of consistency with [21] where a financial interpretation was given, the players should be Helen (standing for hedger) and Mark (standing for market), with Helen trying to maximize her final score under the opposition of Mark.

Game 1 (Parabolic PIDE)**.** *At time $t \in (0,T)$, the game starts at x and Helen has a zero score. Her objective is to get the highest final score.*

(1) *Helen chooses a function $\Phi \in C_b^2(\mathbb{R}^N)$ such that $\|\Phi\|_\infty \leq \varepsilon^{-\alpha}$, $|D\Phi(x)| \leq \varepsilon^{-\alpha}$ and $|D^2\Phi(x)| \leq \varepsilon^{-\alpha}$.*
(2) *Mark chooses the new position $y \in B_R(x)$.*
(3) *Helen's score is increased by*

$$\Phi(x) - \Phi(y) - \varepsilon F(t, x, D\Phi(x), D^2\Phi(x), I_R[x, \Phi])\,.$$

Time is reset to $t+\varepsilon$. Then we repeat the previous steps until time is larger than T. At that time, Helen collects the bonus $u_T(x)$, where x is the current position of the game.

As in [21], the value function $u^\varepsilon(t,x)$ associated to the game is defined as the maximal final score for Helen (under the best opposition of Mark) when the game starts from position x at time t. It is characterized by the one-step dynamic programming principle:

$$u^\varepsilon(t,x) = \sup_{\substack{\Phi \in C^2(\mathbb{R}^N) \\ \|\Phi\|_\infty, |D\Phi(x)|, |D^2\Phi(x)| \leq \varepsilon^{-\alpha}}} \inf_{y \in B_R(x)} \Big\{ u^\varepsilon(t+\varepsilon, y) + \Phi(x) - \Phi(y) - \varepsilon F(t, x, D\Phi(x), D^2\Phi(x), I_R[x,\Phi]) \Big\}\,. \tag{2.7}$$

2.1.2. *Viscosity solutions for PIDE.* In this section, we recall the definition and framework of viscosity solutions for (2.2). Since the pioneering paper by Soner [25], many references are devoted to the question of the well-posedness of fully non-linear integro-differential equations in the viscosity solution framework. There are many equivalent definitions (see for instance [24, 3]), many sets of assumptions to ensure uniqueness (see for instance [19]) or to study regularity (see for instance [7]).

Since we will work with bounded viscosity solutions, we give a definition in this framework.

Definition 1 (Viscosity solutions for PIDE)**.** *Consider $u : (0,T) \times \mathbb{R}^N \to \mathbb{R}$, a bounded function.*

(1) *It is a* viscosity sub-solution *of* (2.2) *if it is upper semi-continuous and if for every bounded test-function $\phi \in C^2$ such that $u - \phi$ admits a global maximum 0 at $(t,x) \in (0,T) \times \mathbb{R}^N$, we have*

$$-\partial_t \phi(t,x) + F(t, x, D\phi(x), D^2\phi(x), I[x,\phi]) \leq 0\,. \tag{2.8}$$

(2) *It is a* viscosity super-solution *of* (2.16) *if it is lower semi-continuous and if for every bounded test-function $\phi \in C^2$ such that $u - \phi$ admits a global minimum 0 at $(t,x) \in (0,T) \times \mathbb{R}^N$, we have*

$$-\partial_t \phi(t,x) + F(t, x, D\phi(x), D^2\phi(x), I[x,\phi]) \geq 0\,. \tag{2.9}$$

(3) *A continuous function u is a* viscosity solution *of* (2.2) *if it is both a sub and super-solution.*

Remark 2.1. If the mesure ν is supported in B_R, then the global maximum/minimum 0 of $u-\phi$ at (t,x) can be replaced with a strict maximum/minimum 0 in $(0,T)\times B_{R'}(x)$ for any $R' \geq R$. Indeed, changing ϕ outside $B_R(x)$ does not change the value of $I[x,\phi]$ in this case.

On the one hand, in order for the value of the repeated game we are going to construct to be finite, we need to make some growth assumption on the nonlinearity F. On the other hand, in order to get the convergence of the value of the repeated game, the comparison principle for (2.2) has to hold. For these reasons we assume that F satisfies the ellipticity condition given above together with the following set of assumptions (see [3]):

Assumptions (A).

- (A0) F is continuous on $\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N\times S_N\times\mathbb{R}$.
- (A1) There exist constants $k_1>0$, $k_2>0$ and $C>0$ such that for all $(t,x,p,A)\in\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N\times S_N$, we have
$$|F(t,x,p,A,0)| \leq C(1+|p|^{k_1}+|X|^{k_2})\,.$$
- (A2-1) For all $R>0$, there exist moduli of continuity ω,ω_R such that, for all $|x|,|y|\leq R$, $|v|\leq R$, $l\in\mathbb{R}$ and for all $X,Y\in S_N$ satisfying
$$\begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \leq \frac{1}{\varepsilon}\begin{bmatrix} I & -I \\ -I & I \end{bmatrix} + r(\beta)\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{2.10}$$
for some $\varepsilon>0$ and $r(\beta)\to 0$ as $\beta\to 0$ (in the sense of matrices in S_{2N}), then, if $s(\beta)\to 0$ as $\beta\to 0$, we have
$$\begin{aligned} F(t,y,v,\varepsilon^{-1}(x-y),Y,l) - F(t,x,v,\varepsilon^{-1}(x-y)+s(\beta),X,l) \\ \leq \omega(\beta)+\omega_R(|x-y|+\varepsilon^{-1}|x-y|^2) \end{aligned} \tag{2.11}$$
or
- (A2-2) For all $R>0$, F is uniformly continuous on $[-R,R]\times\mathbb{R}^n\times B_R\times D_R\times\mathbb{R}$ where $D_R := \{X\in S_N;\ |X|\leq R\}$ and there exist a modulus of continuity ω_R such that, for all $x,y\in\mathbb{R}^N$, $|v|\leq R$, $l\in\mathbb{R}$ and for all $X,Y\in S_N$ satisfying (2.10) and $\varepsilon>0$, we have
$$\begin{aligned} F(t,y,v,\varepsilon^{-1}(x-y),Y,l) - F(t,x,v,\varepsilon^{-1}(x-y),X,l) \\ \leq \omega_R(|x-y|+\varepsilon^{-1}|x-y|^2)\,. \end{aligned} \tag{2.12}$$
- (A3) $F(t,x,u,p,X,l)$ is Lipschitz continuous in l, uniformly with respect to all the other variables.

Assumptions (A0)-(A1) are all we need to show that the relaxed semi-limits of our value functions are viscosity sub- (resp. super-)solutions to (2.2). Assumptions (A2)-(A3) are meant to ensure that a comparison principle holds for (2.2), *i.e.* that viscosity sub-solutions are smaller than viscosity super-solutions, which guarantees the final convergence.

2.1.3. *Theorem and comments.* It is possible to construct a repeated game that approximates a parabolic PIDE where F also depends on u itself, but its formulation is a bit more complicated. This is important from the point of view of applications but since, with the previous game at hand, ideas from [21] can be applied readily, we prefer to present it in this simpler framework.

As usual in such control problems, the convergence of u^ε to u stated below relies heavily on (2.7). We will present right after the statement of the theorem an easy formal argument that allows to predict this convergence.

Theorem 1. *Assume that F is elliptic and satisfies* (A0) *and* (A1)*. Assume also that $u_T \in W^{2,\infty}(\mathbb{R}^N)$. Then the upper (resp. lower) relaxed semi-limit $\overline{u}$ (resp. $\underline{u}$) of $(u^\varepsilon)_{\varepsilon>0}$ is a sub-solution (resp. super-solution) of* (2.2) *and*

$$\overline{u}(T,x) \leq u_T(x) \leq \underline{u}(T,x)\,.$$

In particular, if F also satisfies (A2), (A3)*, then u^ε converges locally uniformly in $\mathbb{R}\times\mathbb{R}^N$ towards the viscosity solution u of* (2.2), (2.5) *as $\varepsilon \to 0$ and $R \to +\infty$ successively.*

Remark 2.2. As we mentioned, the second statement follows from the fact that (A2)–(A3) together with (A0) imply that the comparison principle for (2.2) holds true in the class of bounded functions.

Remark 2.3. We are in fact going to prove that under the same assumptions, u^ε converges locally uniformly in $\mathbb{R}\times\mathbb{R}^N$ as $\varepsilon \to 0$ towards the viscosity solution of (2.2), (2.5) where I is replaced with the truncated integral operator. Theorem 1 is then a direct consequence of this fact by using stability results such as the ones proved in [3].

Remark 2.4. We assume that u_T lies in $W^{2,\infty}(\mathbb{R}^N)$ for simplicity but one can consider terminal data that are much less regular, for instance bounded and uniformly continuous. However, this implies further technicalities that we prefer to avoid here.

Formal argument for Theorem 1. Arguing formally, through an expansion in time of $u^\varepsilon(t,x)$, the result reduces to showing that the following equality holds true

$$\begin{aligned}(2.13)\quad u^\varepsilon(t,x) &= u^\varepsilon(t+\varepsilon,x)\\ &\quad - \varepsilon F(t,x,Du^\varepsilon(t+\varepsilon,x),D^2u^\varepsilon(t+\varepsilon,x),I[x,u^\varepsilon(t+\varepsilon,\cdot)]) + o(\varepsilon).\end{aligned}$$

Indeed, after rearranging terms, dividing by ε and passing to the limit, we get

$$-\partial_t u(t,x) + F(t,x,Du(t,x),D^2u(t,x),I[x,u(t,\cdot)]) = 0\,.$$

It is easy to see that if Helen chooses $\Phi = u^\varepsilon(t+\varepsilon,\cdot)$, Mark cannot change the score by acting on y. Indeed, the dynamic programming principle implies that $u^\varepsilon(t,x)$ is larger than the right-hand side of (2.13).

It turns out that it is optimal for Helen to choose $\Phi = u^\varepsilon(t+\varepsilon,\cdot)$. In other words, the converse inequality holds true (and thus (2.13) holds true too). To see this, the dynamic programming principle tells us that it is enough to prove that, for

$\Phi \in C^2(\mathbb{R}^N)$ fixed (with proper bounds), we have

$$\inf_{y \in B(x,R)} \{u^\varepsilon(t+\varepsilon, y) + \Phi(x) - \Phi(y) - \varepsilon F(t, x, D\Phi(x), D^2\Phi(x), I[x, \Phi])\}$$
$$\leq u^\varepsilon(t+\varepsilon, x) - \varepsilon F(t, x, Du^\varepsilon(t+\varepsilon, x), D^2 u^\varepsilon(t+\varepsilon, x), I[x, u^\varepsilon(t+\varepsilon, \cdot)]) + o(\varepsilon).$$

The following crucial lemma permits to conclude. We recall that we assume that the singular measure is supported in $B(0, R)$ for some $R > 0$.

Lemma 2.1 (Crucial lemma for PIDE)**.** *Let F be continuous and $\Phi, \psi \in C^2(\mathbb{R}^N)$ be two bounded functions. Let K be a compact subset of $\mathbb{R}^N$ and let $x \in K$. For all $\varepsilon > 0$, there exists $y = y_\varepsilon \in B_R(x)$ such that*

$$\psi(y) + \Phi(x) - \Phi(y) - \varepsilon F(t, x, D\Phi(x), D^2\Phi(x), I_R[x, \Phi]) \leq \psi(x) - \varepsilon F(t, x, D\psi(x), D^2\psi(x), I_R[x, \psi]) + o(\varepsilon) \tag{2.14}$$

where the $o(\varepsilon)$ depends on F, ψ, Φ and K but not on t, x, y.

The rigourous proof of this lemma is postponed until Subsection 3.1. However, we can motivate this result by giving a (formal) sketch of its proof. Assume that the conclusion of the lemma is false. Then there exists $\eta > 0$ and we have for all $y \in K$

$$\psi(y) - \psi(x) > \Phi(y) - \Phi(x) + \varepsilon(F(\dots) - F(\dots)) + \eta\varepsilon.$$

In particular, $\psi(y) - \psi(x) > \Phi(y) - \Phi(x) + O(\varepsilon)$. This implies (at least formally)

$$\begin{aligned} D\psi(x) &= D\Phi(x) + o(1) \\ D^2\psi(x) &\leq D^2\Phi(x) + o(1) \\ I[x, \psi] &\leq I[x, \Phi] + o(1). \end{aligned}$$

Then the ellipticity of F implies that $F(\dots) - F(\dots) \geq o(1)$ and we get the following contradiction: $0 \geq \eta o(1) + \eta\varepsilon$. □

One can observe that this very simple game is a natural generalization of the game constructed in [21] for fully non-linear parabolic equations. Indeed, if F does not depend on $I[\Phi]$, then all is needed is proxies for $D\Phi(x), D^2\Phi(x)$. So instead of choosing a whole function Φ, Helen only needs to choose a vector p (proxy for $D\Phi(x)$) and a symmetric matrix Γ (proxy for $D^2\Phi(x)$), and replace $\Phi(y) - \Phi(x)$ in the score updating by its quadratic approximation

$$p \cdot (y - x) + \frac{1}{2}\langle \Gamma(y - x), (y - x)\rangle.$$

One then recovers the game of [21] (except there y is constrained to $B_{\varepsilon^{1-\alpha}}(x)$). Of course it is natural that for a non-local equation, local information at x does not suffice and information in the whole space needs to be collected at each step.

2.2. **Level-set approach to geometric motions.** Before stating our results for the geometric flows (eikonal equations and integral curvature flow), we recall the level set framework for such geometric evolutions.

The level-set approach [23, 9, 15] consists in defining properly motions of interfaces associated with geometric laws. More precisely, given an initial interface Γ_0, *i.e.* the boundary of a bounded open set Ω_0, their time evolutions $\{\Gamma_t\}_{t>0}$ and $\{\Omega_t\}_{t>0}$ are defined by prescribing the velocity V of Ω_t at $x \in \Gamma_t$ along its normal direction $n(x)$ as a function of time t, position x, normal direction $n(x)$, curvature tensor $Dn(x)$, or even the whole set Ω_t at time t. The geometrical law thus writes

$$V = G(t, x, n(x), Dn(x), \Omega_t) \; . \tag{2.15}$$

The level-set approach consists in describing Γ_0 and $\{\Gamma_t\}_{t>0}$ as zero-level sets of continous functions u_0 (such as the signed distance function to Γ_0) and $u(t, \cdot)$ respectively

$$\Gamma_0 = \{x \in \mathbb{R}^N : u_0(x) = 0\} \qquad \text{and} \qquad \Omega_0 = \{x \in \mathbb{R}^N : u_0(x) > 0\}$$
$$\Gamma_t = \{x \in \mathbb{R}^N : u(t,x) = 0\} \qquad \text{and} \qquad \Omega_t = \{x \in \mathbb{R}^N : u(t,x) > 0\} \; .$$

The geometric law (2.15) translates into a fully non-linear parabolic equation for u:

$$\partial_t u = G(t, x, \widehat{Du}, (I - \widehat{Du} \otimes \widehat{Du})D^2u, \Omega_t)|Du| := -F(t, x, Du, D^2u, \Omega_t) \tag{2.16}$$

(where $\hat{p} = |p|^{-1}p$ for $p \in \mathbb{R}^N$, $p \neq 0$) supplemented with the initial condition $u(0, x) = u_0(x)$. If proper assumptions are made on the nonlinearity F, the level-set approach is *consistent* in the sense that, for two different initial conditions u_0 and v_0 with the same 0-level set, the associated (viscosity) solutions u and v have the same zero-level sets at all times as well. The interested reader is referred to [23, 9, 15] for fundamental results, [4] for extensions and [26] for a survey paper.

In the present paper, we deal with terminal conditions instead of initial conditions. This is the reason why, for a given terminal time $T > 0$, we consider the equation $-\partial_t u + F = 0$ supplemented with the terminal condition (2.5). We will consider two special cases of (2.16)

- the eikonal equation

$$-\partial_t u - v(x)|Du| = 0 \tag{2.17}$$

- and the integral curvature equation

$$-\partial_t u - \kappa[x, u]|Du| = 0 \tag{2.18}$$

  where $\kappa[x, u]$ is the integral curvature of u at x (see below for a definition).

2.3. **Eikonal equation.** The first geometric law (2.15) we are interested in is the simple case where $V = v(x)$ and

$$v : \mathbb{R}^N \to \mathbb{R} \text{ is a Lipschitz continuous function}$$

and we do not assume that it has a constant sign. In this case, the geometric equation (2.16) reduces to the standard eikonal equation (2.17).

The solution of an eikonal equation can be represented as the value function of a deterministic control problem when v has a constant sign [22]. If v changes sign, it can be represented as the value function of a deterministic differential game problem, *i.e.*, loosely speaking, a control problem with two opposing players [14].

2.3.1. *The game for the eikonal equation.* The idea of the game consists in using the time counter to control the velocity of the motion (note that in previous games, the timesteps were always fixed and constant). The desired motion of the level set is with prescribed normal velocity equal to $v(x)$. Instead of moving of length $|v(x)|\varepsilon$ in a time step ε, the players will move by ε in time $\varepsilon/|v(x)|$. The direction of the motion, following the sign of v, will be controlled by the rule that determines which player chooses the next move. More precisely, we have two players Paul and Carol, and we want, starting from x at time t:

- if $v(x) > 0$, Paul chooses the next point in $B(x,\varepsilon)$, and time gets reset to $t+\varepsilon/|v(x)|$;
- if $v(x) < 0$, Carol chooses the next point in $B(x,\varepsilon)$, and time gets reset to $t+\varepsilon/|v(x)|$;
- at final time T the final score is $u_T(x)$.

One can check that formally, this leads to the desired evolution for the value function equal to the maximal final score for Paul. This game needs to be slightly modified to handle the cases where v gets close to 0. We use cut-off functions to truncate possibly too large or too small time-increments: for $\varepsilon > 0$ and $r > 0$, we define

$$C_\varepsilon(r) = (r \vee \varepsilon^{\frac{3}{2}}) \wedge \varepsilon^{\frac{1}{2}} = \begin{cases} \varepsilon^{\frac{3}{2}} & \text{if } 0 < r < \varepsilon^{\frac{3}{2}}, \\ r & \text{if } \varepsilon^{\frac{3}{2}} < r < \varepsilon^{\frac{1}{2}}, \\ \varepsilon^{\frac{1}{2}} & \text{if } r > \varepsilon^{\frac{1}{2}}. \end{cases} \tag{2.19}$$

This function is non-decreasing and for every r we have $\varepsilon^{\frac{3}{2}} \le C_\varepsilon(r) \le \varepsilon^{\frac{1}{2}}$. We may now state the rigorous game that we will study.

Game 2 (Eikonal equation)**.** *At time $t \in (0,T)$, Paul starts at x with zero score. His objective is to get the highest final score.*

(1) *Either $B_\varepsilon(x) \cap \{v > 0\} \neq \varnothing$, then Paul chooses a point $x_P \in B_\varepsilon(x) \cap \{v > 0\}$ and time gets reset to $t_P = t + C_\varepsilon[\varepsilon(v_+(x_P))^{-1}]$.*
Or $B_\varepsilon(x) \cap \{v > 0\} = \varnothing$, then Paul stays at $x_P = x$ and time gets reset to $t_P = t + \varepsilon^2$.
(2) *Either $B_\varepsilon(x_P) \cap \{v < 0\} \neq \varnothing$, then Carol chooses a point $x_C \in B_\varepsilon(x_P) \cap \{v < 0\}$ and time gets reset to $t_C = t_P + C_\varepsilon[\varepsilon(v_-(x_C))^{-1}]$.*
Or $B_\varepsilon(x_P) \cap \{v < 0\} = \varnothing$, then Paul stays at $x_C = x_P$ and time gets reset to $t_C = t_P + \varepsilon^2$.
(3) *The players repeat the two previous steps until $t_C \geq T$. Paul's final score is $u_T(x_C)$ where x_C s the final position of the game.*

As previously, the value function $u^\varepsilon(t,x)$ for the game, when starting from x at time t, is defined as Paul's maximal possible final score under the best opposition of Carol, and is characterized by the one-step dynamic programming principle: let for short E^+ and E^- denote the sets

$$E^\pm(x) = \begin{cases} B_\varepsilon(x) \cap \{\pm v > 0\} & \text{if } B_\varepsilon(x) \cap \{\pm v > 0\} \neq \varnothing \\ \{x\} & \text{if not.} \end{cases} \tag{2.20}$$

Recall also that $(\cdot)_+$ denotes the positive part and $(\cdot)_-$ the negative part of a quantity. With this notation, the dynamic programming principle is

$$u^\varepsilon(t,x) = \sup_{x_P \in E^+(x)} \left\{ \inf_{x_C \in E^-(x_P)} \{u^\varepsilon(t_C, x_C)\} \right\}, \tag{2.21}$$

where

$$\begin{cases} t_P = t + \begin{cases} C_\varepsilon[\varepsilon(v_+(x_P))^{-1}] & \text{if } B_\varepsilon(x) \cap \{v > 0\} \neq \varnothing \\ \varepsilon^2 & \text{if not} \end{cases} \\ t_C = t_P + \begin{cases} C_\varepsilon[\varepsilon(v_-(x_C))^{-1}] & \text{if } B_\varepsilon(x_P) \cap \{v < 0\} \neq \varnothing \\ \varepsilon^2 & \text{if not} \end{cases} \end{cases} \tag{2.22}$$

and

$$u^\varepsilon(t,x) = u_T(x) \text{ if } t \geq T\ .$$

2.3.2. *Viscosity solutions to the eikonal equation.* We recall the definition of a viscosity solution to the eikonal equation (2.17).

Definition 2 (Viscosity solution for (2.17))**.** *Given a function $u : (0,T) \times \mathbb{R}^N \to \mathbb{R}$, we say that*

(1) *It is a* viscosity sub-solution *of (2.17) if it is upper semi-continuous and if for every test-function $\phi \in C^2$ such that $u - \phi$ admits a local maximum at $(t,x) \in (0,T) \times \mathbb{R}^N$, we have*

$$-\partial_t \phi(t,x) - v(x)|\nabla\phi|(t,x) \leq 0. \tag{2.23}$$

(2) *It is a* viscosity super-solution *of (2.17) if it is lower semi-continuous and if for every test-function $\phi \in C^2$ such that $u - \phi$ admits a local minimum at $(t,x) \in (0,T) \times \mathbb{R}^N$, we have*

$$-\partial_t \phi(t,x) - v(x)|\nabla\phi|(t,x) \geq 0. \tag{2.24}$$

(3) *It is a viscosity solution of (2.17) if it is both a sub and super-solution.*

2.3.3. *Convergence result.* We next claim that the following convergence result holds true; again the limiting equation can be predicted by a formal argument from (2.21) (see below).

Theorem 2. *Assume that v is Lipschitz continuous and u_T is bounded and Lipschitz continuous. Then the function u^ε converges locally uniformly as $\varepsilon \to 0$ towards the unique viscosity solution of* (2.17), (2.5)*.*

Remark 2.5. Let us mention that the parameters $\alpha = \frac{1}{2}$ and $\beta = \frac{3}{2}$ in the definition of C_ε (2.19) really only need to satisfy $1 < \alpha < \beta < 2$.

We next give the formal argument which permits to predict the convergence result for the eikonal equation.

Formal argument for Theorem 2. We first rewrite the dynamic programming principle as follows

$$0 = \sup_{x_P \in E^+(x)} \left\{ u^\varepsilon(t_P, x_P) - u^\varepsilon(t,x) + \inf_{x_C \in E^-(x_P)} \{u^\varepsilon(t_C, x_C) - u^\varepsilon(t_P, x_P)\} \right\}.$$

We only treat the case $v(x) > 0$ because the argument is completely analogous in the case $v(x) < 0$. Hence, for ε small enough, $B_\varepsilon(x) \subset \{v > 0\}$, $B_\varepsilon(x_P) \cap \{v < 0\} = \varnothing$ and $(t_C, x_C) = (t_P + \varepsilon^2, x_P)$. The previous equality then yields (approximating $C_\varepsilon(r)$ by r)

$$\begin{aligned}
0 &= \sup_{x_P \in B_\varepsilon(x)} \left\{ u^\varepsilon(t_P, x_P) - u^\varepsilon(t,x) + O(\varepsilon^2) \right\} \\
&= \sup_{x_P \in B_\varepsilon(x)} \left\{ \partial_t u^\varepsilon(t,x)(t_P - t) + Du^\varepsilon(t,x)(x_P - x) \right\} + o(\varepsilon) \\
&= \frac{\varepsilon}{v(x_P)} (\partial_t u^\varepsilon(t,x) + v(x_P)|Du^\varepsilon(t,x)|) + o(\varepsilon) \\
&= \frac{\varepsilon}{v(x_P)} (\partial_t u^\varepsilon(t,x) + v(x)|Du^\varepsilon(t,x)|) + o(\varepsilon)\,.
\end{aligned}$$

Hence, dividing by $\varepsilon / v(x_P)$ and letting $\varepsilon \to 0$, we obtain formally

$$\partial_t u(t,x) + v(x)|Du|(t,x) = 0\,.$$

□

2.4. **Integral curvature flow.**

2.4.1. *Formal discussion.* The integral curvature equation (2.18) is a particular case of the eikonal equation (2.17), where $v(x)$ is replaced by the nonlocal curvature of the level-set of the solution, which is not given a priori. We will give a precise definition of this nonlocal curvature below, but without going into these details yet, we start by giving an idea of the game, then a precise definition of it.

Mimicking the game for the eikonal equation with prescribed velocity, we will still have two players Paul and Carol; when the curvature κ is positive at x, Paul will determine the next move in $B(x,\varepsilon)$, and when it is negative, Carol will. The time counter will be increased by $\varepsilon/|\kappa|$. The difficulty is that the definition of κ can only be implicit: in the end κ should be a proxy for the integral curvature $\kappa[x,\Gamma]$ of the level set Γ of the value function at x. The idea is to let the curve Γ be chosen by the players, in such a way that it becomes a proxy for the level set of the value function. Assuming Paul is the one choosing the curve, this will be achieved be letting Carol jump to any point on (one side of) Γ. This forces Paul to choose Γ within the sublevel set $\{u^\varepsilon(y) \le u^\varepsilon(x)\}$, otherwise Carol could take advantage of it. Then, Paul will have to choose Γ exactly equal to the level set $\{u^\varepsilon(y) = u^\varepsilon(x)\}$, because it is that choice which is most favorable with respect to the advancement of the time counter.

With these ingredients, we are led to the formal definition of the game.
Formal definition of the game. The game proceeds in two main steps; in each step, Paul and Carol play successively. It starts from x at time t.

- Paul can decide to play (see Figure 2.4.1); in this case, he chooses a point x_P in $B(x,\varepsilon)$, then a curve Γ^+ passing through x_P, whose integral curvature at x_P is well defined and positive.

- Paul can also decide not to play. In the game, this will be implemented by allowing Paul to choose curves whose integral curvature is either not well defined or non-positive.
- Carol chooses the next position "outside" the curve Γ^+. Time gets reset to $t+\varepsilon/|\kappa[x_P,\Gamma^+]|$.
- Next, the roles of the players are reversed, and Carol can decide to play or not. If she does, she chooses a point at distance ε from the current location, and a curve Γ^- passing through that point of negative integral curvature. Paul chooses the next position "inside" Γ^- and time gets reset to $t+\varepsilon/|\kappa[\Gamma^-]|$.
- All the previous steps are repeated until the final time T is reached. Then Paul collects the final score $u_T(x_{cur})$ where x_{cur} denotes the current position.

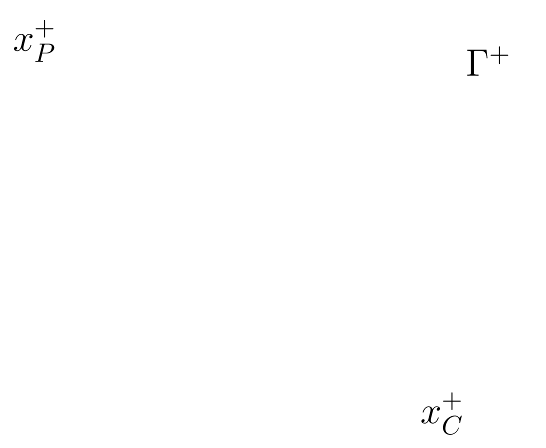

FIGURE 1. **Paul plays:** he chooses a curve Γ^+ passing through $x_P^+ \in B(x,\varepsilon)$, whose integral curvature is well defined and positive. Then Carol chooses the new current point x_C^+ "outside" the curve Γ^+.

As explained above, players are expected to make the following choices:

- If the curvature of the set $\Gamma = \{u^\varepsilon(t,\cdot) = u^\varepsilon(t,x)\}$ is positive, then in the first step, we expect Paul to move first at distance ε in the direction of $\nabla u^\varepsilon(x)$, then to choose Γ^+ equal to the level set of u^ε at the new point, and Carol to choose any point on Γ^+. In the next step, we expect Carol not to play.
- If the curvature is negative, then we expect Paul not to play, and in the next step, Carol to move first at distance ε in the direction of $-\nabla u^\varepsilon(x)$, then to choose Γ^- equal to the level set of u^ε at that new point, and Paul to pick a point on Γ^-.

These facts will be stated and proved rigourously for the rigourous game in technical lemmas that we will refer to as *consistency lemmas* (see Lemmas 3.7, 3.8, 3.9, 3.10 in Section 3.3).

2.4.2. *Definition of integral curvature and viscosity solutions.* Even if most authors do not use this word, the notion of integral curvature is considered in papers such as [13, 16, 8, 17, 6]. Here is the definition we will take.

Consider a function $K : \mathbb{R}^N \to (0, +\infty)$ such that

$$\begin{cases} K \text{ is even, supported in } B_R(0) \\ K \in W^{1,1}(\mathbb{R}^N \setminus B_\delta(0)) \text{ for all } \delta > 0 \\ \int_{B_\delta(0)} K = o\left(\frac{1}{\delta}\right) \\ \int_{\mathcal{Q}(r,e)} K < +\infty \text{ for all } r > 0, e \in \mathbb{S}^{N-1} \\ \int_{\mathcal{Q}(r,e)} K = o\left(\frac{1}{r}\right) \end{cases} \tag{2.25}$$

where $\mathcal{Q}(r,e)$ is a paraboloid defined as follows

$$\mathcal{Q}(r,e) = \{z \in \mathbb{R}^N : r|z \cdot e| \leq |z - (z \cdot e)e|^2\}.$$

Interesting examples of such K's include

$$K(z) = \frac{C(z)}{1 + |z|^{N+\alpha}} \quad \text{or} \quad K(z) = \frac{C(z)}{|z|^{N+\alpha}}$$

for some cut-off function $C : \mathbb{R}^N \to \mathbb{R}$ which is even, smooth and supported in $B_R(0)$.

Remark 2.6. It is not necessary to assume that K has a compact support in order to define the non-local geometric flow. However, we need this assumption in order to construct the game and prove that it approximates the geometric flow. We can then later follow what we did when dealing with parabolic PIDE: approximate any integral curvature flow by first approximating K by kernels K^R compactly supported in $B_R(0)$ and by taking next the limit of the corresponding value functions as $\varepsilon \to 0$ and $R \to \infty$ respectively.

Consider $U \in C^2$ such that $DU(x) \neq 0$. We define

$$\kappa^*[x,U] = K * \mathbf{1}_{\{U \geq U(x)\}} - K * \mathbf{1}_{\{U < U(x)\}}$$
$$\kappa_*[x,U] = K * \mathbf{1}_{\{U > U(x)\}} - K * \mathbf{1}_{\{U \leq U(x)\}}.$$

These functions coincide if for instance $DU \neq 0$ on $\{U = U(x)\}$. They define the integral curvature of the "hypersurface" $\{U(z) = U(x)\}$ at the point x. The reader can notice that this "hypersurface" is oriented *via* the sign of the function U. The classical curvature can be recovered if $K(z) = \frac{1-\alpha}{|z|^{N+\alpha}}$ and $\alpha \to 1$, $\alpha < 1$; see [17].

Functions κ^* and κ_* enjoy the following properties (see [17]):

(1) SEMI-CONTINUITY: functions $\kappa^*[\cdot,U]$ and $\kappa_*[\cdot,U]$ are respectively upper and lower semi-continuous

$$\kappa^*[x,U] \geq \limsup_{y \to x} \kappa^*[y,U]; \tag{2.26}$$

$$\kappa_*[x,U] \leq \liminf_{y \to x} \kappa^*[y,U]; \tag{2.27}$$

(2) Monotonicity property:

$$\{U \geq U(x)\} \subset \{V \geq V(x)\} \;\Rightarrow\; \kappa^*[x,U] \leq \kappa^*[x,V], \tag{2.28}$$
$$\{U > U(x)\} \subset \{V > V(x)\} \;\Rightarrow\; \kappa_*[x,U] \leq \kappa_*[x,V].$$

We next make precise the notion of viscosity solutions for (2.18), (2.5) that will be used in the present paper.

Definition 3 (Viscosity solutions for (2.18))**.** *Given a function $u : (0,T) \times \mathbb{R}^N \to \mathbb{R}$, we say that*

(1) *It is a* sub-solution *of* (2.18) *if it is upper semi-continuous and if for every test-function $\phi \in C^2$ such that $u - \phi$ admits a strict maximum in $(0,T) \times B_{R+1}(x)$ at (t,x), we have*

$$-\partial_t \phi(t,x) - \kappa^*[x, \phi(t,\cdot)]|D\phi|(t,x) \leq 0 \tag{2.29}$$

if $D\phi(t,x) \neq 0$ and $-\partial_t\phi(t,x) \leq 0$ if $D\phi(t,x) = 0$;

(2) *It is a* super-solution *of* (2.18) *if it is lower semi-continuous and if for every test-function $\phi \in C^2$ such that $u - \phi$ admits a strict minimum in $(0,T) \times B_{R+1}(x)$ at (t,x), we have*

$$-\partial_t \phi(t,x) - \kappa_*[x, \phi(t,\cdot)]|D\phi|(t,x) \geq 0 \tag{2.30}$$

if $D\phi(t,x) \neq 0$ and $-\partial_t\phi(t,x) \geq 0$ if $D\phi(t,x) = 0$;

(3) *It is a* solution *of* (2.18) *if it is both a sub and super-solution.*

It is proved in [17] that a comparison principle holds true for such super- and sub-solutions.

2.4.3. *Rigourous definition of the game.* We now give a precise and rigourous description of the repeated game. Here and throughout the paper, a hypersurface refers to the 0-level set of a smooth function ϕ. We recall the definition of the cut-off function we considered in the repeated game for the eikonal case.

$$C_\varepsilon(r) = (r \vee \varepsilon^{\frac{3}{2}}) \wedge \varepsilon^{\frac{1}{2}} = \begin{cases} \varepsilon^{\frac{3}{2}} & \text{if } 0 < r < \varepsilon^{\frac{3}{2}}, \\ r & \text{if } \varepsilon^{\frac{3}{2}} < r < \varepsilon^{\frac{1}{2}}, \\ \varepsilon^{\frac{1}{2}} & \text{if } r > \varepsilon^{\frac{1}{2}}. \end{cases} \tag{2.31}$$

We also recall that R is the size of the support of κ as in (2.25).

Game 3 (Integral curvature equation)**.** *At time $t \in (0,T)$, Paul starts at x with zero score. His objective is to get the highest final score.*

(1) *Paul chooses a point $x_P^+ \in B_\varepsilon(x)$ and a hypersurface Γ^+ passing through x_P^+ defined by*
$$\Gamma^+ = \{z \in \mathbb{R}^N : \phi^+(z) = \phi^+(x_P^+)\}$$
with $\phi^+ \in C^2(\mathbb{R}^N)$, oriented through the requirement $\phi^+(x) \leq \phi^+(x_P^+)$.
   - *If $D\phi^+(x_P^+) \neq 0$ and $\kappa^*[x_P^+, \Gamma^+] > 0$, Carol chooses the new position point x_C^+ in the half-space delimited by Γ^+ i.e. in $\{z \in B_R(x_P^+) : \phi^+(z) \geq \phi^+(x_P^+)\}$. Time gets reset to $t^+ = t + C_\varepsilon(\varepsilon\kappa^*[x_P^+, \Gamma^+]^{-1})$.*
   - *If $D\phi^+(x_P^+) = 0$ or $\kappa^*[x_P^+, \Gamma^+] \leq 0$, then the game stays at x: $x_C^+ = x$. Time gets reset to $t^+ = t + \varepsilon^2$.*

(2) *From the new position* x_C^+ *and time* t^+ *determined above, Carol chooses a point* $x_C^- \in B_\varepsilon(x_C^+)$ *and a hypersurface* Γ^- *passing through* x_C^- *defined by*

$$\Gamma^- = \{z \in \mathbb{R}^N : \phi^-(z) = \phi^-(x_C^-)\}$$

with $\phi^- \in C^2(\mathbb{R}^N)$*, and oriented through the requirement* ϕ^- *is such that* $\phi^-(x_C^+) \geq \phi^-(x_C^-)$*.*

- *If* $D\phi^-(x_C^-) \neq 0$ *and* $\kappa_*[x_C^-, \Gamma^-] < 0$*, Paul chooses the new position point* x_P^- *in the half-space delimited by* Γ^- *i.e. in* $\{z \in B_R(x_C^-) : \phi^-(z) \leq \phi^-(x_C^-)\}$*. Time gets reset to* $t^- = t^+ + C_\varepsilon(\varepsilon|\kappa_*[x_C^-, \Gamma]|^{-1})$*.*
- *If* $D\phi^-(x_C^-) = 0$ *or* $\kappa_*[x_C^-, \Gamma^-] \geq 0$*, then the game stays at* x_C^+ *(*$x_C^- = x_C^+$*) and time gets reset to* $t^- = t^+ + \varepsilon^2$*.*

(3) *Then previous steps are repeated as long as* $t^- < T$*. Paul's final score is* $u_T(x_P^-)$*.*

Remark in particular that in Step 1, the value of the function ϕ^+ is successively increased while in Step 2, the value of the function ϕ^- is successively decreased. Precisely,

$$\phi^+(x) \leq \phi^+(x_P^+) \leq \phi^+(x_C^+)\,,$$
$$\phi^-(x_C^+) \geq \phi^-(x_C^-) \geq \phi^-(x_P^-)\,.$$

The value function for the game $u^\varepsilon(t,x)$, when starting from x at time t is defined as Paul's maximal possible final score under the best opposition of Carol, and is characterize by the one-step dynamic programming principle. In order to state it, we first introduce admissible sets of points and half-spaces for both players. Precisely, we consider

$$\mathcal{C}^\pm(x) = \{(y,\varphi) \in B_\varepsilon(x) \times C^2(\mathbb{R}^N) : \pm\varphi(y) \geq \pm\varphi(x)\}, \tag{2.32}$$

$$\mathcal{P}^+(x,y,\varphi) = \begin{cases} \{z \in B_R(y) : \varphi(z) \geq \varphi(y)\} & \text{if } D\varphi(y) \neq 0 \text{ and } \kappa^*[y,\varphi] > 0 \\ \{x\} & \text{if not,} \end{cases} \tag{2.33}$$

$$\mathcal{P}^-(x,y,\varphi) = \begin{cases} \{z \in B_R(y) : \varphi(z) \leq \varphi(y)\} & \text{if } D\varphi(y) \neq 0 \text{ and } \kappa_*[y,\varphi] < 0 \\ \{x\} & \text{if not.} \end{cases} \tag{2.34}$$

Hence, the dynamic programming principle associated to the game is

$$u^\varepsilon(t,x) = \sup_{(x_P^+,\phi^+)\in\mathcal{C}^+(x)} \left\{ \inf_{x_C^+\in\mathcal{P}^+(x,x_P^+,\phi^+)} \left\{ \inf_{(x_C^-,\phi^-)\in\mathcal{C}^-(x_C^+)} \left\{ \sup_{x_P^-\in\mathcal{P}^-(x_C^+,x_C^-,\phi^-)} \left\{u^\varepsilon\left(t^-, x_P^-\right)\right\} \right\} \right\} \right\} \tag{2.35}$$

where

$$\begin{cases} t^+ = t & + \begin{cases} C_\varepsilon(\varepsilon\kappa^*[x_P^+,\Gamma^+]^{-1}) & \text{if } D\phi^+(x_P^+) \neq 0 \text{ and } \kappa^*[x_P^+,\Gamma^+] > 0, \\ \varepsilon^2 & \text{if not,} \end{cases} \\ t^- = t^+ & + \begin{cases} C_\varepsilon(\varepsilon|\kappa_*[x_C^-,\Gamma^-]|^{-1}) & \text{if } D\phi^-(x_C^-) \neq 0 \text{ and } \kappa_*[x_C^-,\Gamma^-] < 0, \\ \varepsilon^2 & \text{if not.} \end{cases} \end{cases} \tag{2.36}$$

2.4.4. *Convergence result.* The last main result is

Theorem 3. *Assume that $u_T \in W^{2,\infty}(\mathbb{R}^N)$. Then the sequence u^ε converges locally uniformly as $\varepsilon \to 0$ towards the unique viscosity solution of* (2.16), (2.5).

Remark 2.7. To avoid further technicalities, we assume that the terminal datum is very regular; it can be shown that the result still holds true for much less regular functions u_T such as bounded uniformly continuous ones. This extension is left to the reader.

2.5. **Open problems.** As we mentioned above, the games we have constructed have much more complicated rules than the Paul-Carol game for mean curvature flow. It is then a natural open problem to find simpler games and in particular a game for the integral curvature flow associated with the singular measure $\nu(dz) = (1-\alpha)dz/|z|^{N+\alpha}$ which converges (in some sense) as $\alpha \to 1$ to the original Paul and Carol game. The reason to look for such as game is that it is known [17] that the integral curvature flow converges towards the mean curvature flow as $\alpha \to 1$. The same question can be raised for the fractional Laplacian operators in the situation of PIDE's: find a game a natural game associated to Δ^α operators, which coincides with a natural game for $\alpha = 1$.

3. Proofs of convergence results

3.1. **Proof of Theorem 1.** As explained in Remark 2.3, it is enough to prove the convergence as $\varepsilon \to 0$ by assuming that ν is supported in B_R for some fixed $R > 0$.

For fixed $\varepsilon > 0$ and $x \in \mathbb{R}^N$, the value function $u^\varepsilon(t,x)$ is finite for t close to T thanks to the following lemma. Proposition 3.2 below is needed to prove that $u^\varepsilon(t,x)$ is finite for all $t \in (0,T)$.

Lemma 3.1 (The functions u^ε are well defined)**.** *For all $\Phi \in C^2(\mathbb{R}^N)$ such that*

$$\|\Phi\|_\infty \leq \varepsilon^{-\alpha}, \; |D\Phi(x)| \leq \varepsilon^{-\alpha}, \; |D^2\Phi(x)| \leq \varepsilon^{-\alpha}\,, \tag{3.1}$$

we have

$$-\varepsilon F(t,x,D\Phi(x),D^2\Phi(x),I_R[x,\Phi]) \leq C\varepsilon^\gamma$$

with $\gamma = 1 - \alpha \max(1,k_1,k_2) \in (0,1)$ and C depends on F, ν and R.

Proof of Lemma 3.1. We consider a bounded C^2 function Φ such that (3.1) holds. ¿From the definition of $I_R[x,\cdot]$ (see (2.3)), it is clear that there exists a constant C only depending on R and ν such that

$$|I_R[x,\Phi]| \leq C\varepsilon^{-\alpha}\,.$$

We thus get from (A1) and (A3)

$$-\varepsilon F(t,x,D\Phi(x),D^2\Phi(x),I_R[x,\Phi]) \;\; \leq \;\; C\varepsilon(1+\varepsilon^{-\alpha k_1}+\varepsilon^{-\alpha k_2}+\varepsilon^{-\alpha})$$

and the lemma follows at once. □

Let us define as usual the semi-relaxed limits $\underline{u} = \liminf^*_{\varepsilon\to 0} u^\varepsilon$ and $\overline{u} = \limsup^*_{\varepsilon\to 0} u^\varepsilon$. Theorem 1 will follow from the following two propositions.

Proposition 3.1. *The functions $\underline{u}$ and $\overline{u}$ are finite and are respectively a super-solution and a sub-solution of* (2.2)*.*

Proposition 3.2. *Given $R > 0$, there exists a constant $C > 0$ such that for all $\varepsilon > 0$, all $(t, x) \in (0, T) \times B_R$, we have*

$$|u^\varepsilon(T - t, x) - u_T(x)| \leq Ct\,.$$

In particular, $\underline{u}$ and $\overline{u}$ are finite and they satisfy at time $t = T$ and for all $x \in \mathbb{R}^N$

$$\underline{u}(T, x) = \overline{u}(T, x) = u_T(x)\,.$$

These two propositions together with the comparison principle imply that $\underline{u} = \overline{u}$, *i.e.* u^ε converges locally uniformly towards a continuous function denoted u. This implies that u is a (continuous) viscosity solution of (2.2) satisfying (2.5). This finishes the proof of the theorem.

We now prove the two propositions.

Proof of Proposition 3.1. We use the general method proposed by Barles and Souganidis [5] in order to prove that $\underline{u}$ is a super-solution of (2.2). This is the reason why, given a function $U : \mathbb{R}^N \to \mathbb{R}$, we introduce

$$\mathcal{S}^\varepsilon[U](t, x) = \sup_{\substack{\Phi \in C^2(\mathbb{R}^N) \\ \|\Phi\|_\infty, |D\Phi(x)|, |D^2\Phi(x)| \leq \varepsilon^{-\alpha}}} \inf_{y \in B_R(x)} \Big\{ U(y) + \Phi(x) - \Phi(y) - \varepsilon F(t, x, D\Phi(x), D^2\Phi(x), I_R[x, \Phi]) \Big\}\,.$$

The two important properties of $\mathcal{S}^\varepsilon$ are:

$$\text{it commutes with constants: } \mathcal{S}^\varepsilon[U + C] = \mathcal{R}^\varepsilon[U] + C \text{ for any } C \in \mathbb{R}; \tag{3.2}$$

$$\text{it is monotone: if } U \leq V \text{ then } \mathcal{S}^\varepsilon[U] \leq \mathcal{S}^\varepsilon[V]. \tag{3.3}$$

The dynamic programming principle (2.7) is rewritten as follows

$$u^\varepsilon(t, x) = \mathcal{S}^\varepsilon[u^\varepsilon(t + \varepsilon, \cdot)](t, x). \tag{3.4}$$

We now explain how to prove that $\underline{u}$ is a super-solution of (2.2). The case of $\overline{u}$ is proven analogously thanks to a "consistency lemma" (see below Lemma 3.2). Following Definition 1 and Remark 2.1, consider a C^2 bounded test function ϕ and a point (t_0, x_0) with $t_0 > 0$ such that $\underline{u} - \phi$ admits a strict minimum 0 at (t_0, x_0) on $\mathcal{V}_0 = (0, T) \times B_{R+1}(x_0)$. By definition of $\underline{u}$, there exists $(\tau_\varepsilon, y_\varepsilon)$ such that $(\tau_\varepsilon, y_\varepsilon) \to (t_0, x_0)$ and $u^\varepsilon(\tau_\varepsilon, y_\varepsilon) \to \underline{u}(t_0, x_0)$ as $\varepsilon \to 0$, up to a subsequence. Let then $(t_\varepsilon, x_\varepsilon)$ be a point of minimum of $u^\varepsilon - \phi$ on $\mathcal{V}_0$. We have

$$u^\varepsilon(t_\varepsilon, x_\varepsilon) - \phi(t_\varepsilon, x_\varepsilon) \leq u^\varepsilon(\tau_\varepsilon, y_\varepsilon) - \phi(\tau_\varepsilon, y_\varepsilon) \to \underline{u}(t_0, x_0) - \phi(t_0, x_0) = 0$$

hence by definition of $\underline{u}$ and (t_0, x_0) as a strict local minimum, we conclude that we must have $(t_\varepsilon, x_\varepsilon) \to (t_0, x_0)$ as $\varepsilon \to 0$. In addition, for all $(t, x) \in \mathcal{V}_0$, we have

$$u^\varepsilon(t, x) \geq \phi(t, x) + (u^\varepsilon(t_\varepsilon, x_\varepsilon) - \phi(t_\varepsilon, x_\varepsilon)) =: \phi(t, x) + \xi_\varepsilon.$$

In particular, if ε is small enough, this inequality holds true on $(0,T)\times B_R(x_\varepsilon)$. From the definition of $\mathcal{S}^\varepsilon$, the dynamic programming principle (3.4), and the properties (3.2)–(3.3), the previous inequality implies

$$\begin{aligned} u^\varepsilon(t_\varepsilon,x_\varepsilon) &= \mathcal{S}^\varepsilon[u^\varepsilon(t_\varepsilon+\varepsilon,\cdot)](t_\varepsilon,x_\varepsilon) \\ &\geq \mathcal{S}^\varepsilon[\phi(t_\varepsilon+\varepsilon,\cdot)+\xi_\varepsilon](t_\varepsilon,x_\varepsilon) = \mathcal{S}^\varepsilon[\phi(t_\varepsilon+\varepsilon,\cdot)](t_\varepsilon,x_\varepsilon)+\xi_\varepsilon. \end{aligned}$$

Since $u^\varepsilon(t_\varepsilon,x_\varepsilon)=\phi(t_\varepsilon,x_\varepsilon)+\xi_\varepsilon$, we get

$$\phi(t_\varepsilon,x_\varepsilon)\geq \mathcal{S}^\varepsilon[\phi(t_\varepsilon+\varepsilon,\cdot)](t_\varepsilon,x_\varepsilon).$$

We claim Proposition 3.1 is proved if the following lemma holds true.

Lemma 3.2 (Consistency lemma for PIDE)**.** *Consider a C^2 bounded test function ψ. Given a compact subset K of $(0,T)\times\mathbb{R}^N$, there then exists a function $o(\varepsilon)$ such that $o(\varepsilon)/\varepsilon\to 0$ as $\varepsilon\to 0$ and for all $(t,x)\in K$, we have*

$$\mathcal{S}^\varepsilon[\psi](t,x)=\psi(x)-\varepsilon F(t,x,D\psi(x),D^2\psi(x),I_R[x,\psi])+o(\varepsilon).$$

Indeed, applying this lemma to $\psi=\phi(t_\varepsilon+\varepsilon,\cdot)$, we are led to

$$\begin{aligned} \phi(t_\varepsilon,x_\varepsilon) &\geq \phi(t_\varepsilon+\varepsilon,x_\varepsilon) \\ &\quad -\varepsilon F(t_\varepsilon,x_\varepsilon,D\phi(t_\varepsilon+\varepsilon,x_\varepsilon),D^2\phi(t_\varepsilon+\varepsilon,x_\varepsilon),I_R[x_\varepsilon,\phi(t_\varepsilon+\varepsilon,\cdot)])+o(\varepsilon)\,. \end{aligned}$$

Dividing by ε and letting $\varepsilon\to 0$ we obtain (using the C^2 character of ϕ and continuity of F),

$$-\partial_t\phi(t_0,x_0)+F(t_0,x_0,D\phi(t_0,x_0),D^2\phi(t_0,x_0),I_R[x_0,\phi(t_0,\cdot)])\geq 0.$$

This allows us to conclude that $\underline{u}$ is a viscosity super-solution of (2.2). The proof that $\overline{u}$ is a sub-solution is analogous. □

We now turn to the proof of the consistency lemma, *i.e.* Lemma 3.2.

Proof of Lemma 3.2. This lemma easily follows from Lemma 2.1. Indeed, Lemma 2.1 implies that for $\psi\in C^2(\mathbb{R}^N)$, there exists $o(\varepsilon)$ depending on ψ such that

$$\mathcal{S}^\varepsilon[\psi](t,x)\leq\psi(x)-\varepsilon F(t,x,D\psi(x),D^2\psi(x),I_R[x,\psi])+o(\varepsilon).$$

On the other hand, by choosing $\Phi=\psi$ in the definition of $\mathcal{S}^\varepsilon$, we immediately get

$$\mathcal{S}^\varepsilon[\psi](t,x)\geq\psi(x)-\varepsilon F(t,x,D\psi(x),D^2\psi(x),I_R[x,\psi])\,.$$

Combining the two previous inequalities yield the desired result. □

We next turn to the proof of the crucial Lemma, *i.e.* Lemma 2.1.

Proof of Lemma 2.1. By contradiction assume this is wrong, hence there exists $\eta>0$ and $\varepsilon_n\to 0$ such that for all $y\in B_R(x)$,

$$\begin{aligned} (3.5)\quad & \Phi(x)-\psi(x)-\Phi(y)+\psi(y) \\ &> -\varepsilon_n\Big(F(t,x,D\Phi(x),D^2\Phi(x),I_R[x,\Phi]) \\ &\qquad - F(t,x,D\psi(x),D^2\psi(x),I_R[x,\psi])\Big)+\eta\varepsilon_n\,. \end{aligned}$$

In order to simplify notation, ε_n is simply denoted by ε.

Let us first take $y = x + \varepsilon^{\frac{1}{2}} w$ with $\|w\| = 1$. Inserting into (3.5) and using the C^2 character of Φ and ψ gives

$$-\varepsilon^{\frac{1}{2}} D(\Phi - \psi)(x) \cdot w \geq -C\varepsilon$$

where C depends only on Φ, ψ, F, x and not on ε. Dividing by $\varepsilon^{\frac{1}{2}}$, and using the fact that this is true for every $w \in \mathbb{R}^N$ with $\|w\| = 1$, we find that

$$|D(\Phi - \psi)(x)| \leq C\varepsilon^{\frac{1}{2}} . \tag{3.6}$$

Using now $y = x + \varepsilon^{\frac{1}{3}} w$ and doing a second order Taylor expansion of $\Phi - \psi$, we find

$$-\varepsilon^{\frac{1}{3}} D(\Phi - \psi)(x) \cdot w - \frac{\varepsilon^{\frac{2}{3}}}{2} \langle D^2(\Phi - \psi)(x) w, w \rangle \geq O(\varepsilon)$$

and using (3.6) we obtain

$$\langle D^2(\Phi - \psi)(x) w, w \rangle \leq O(\varepsilon^{\frac{1}{6}}) .$$

Since this is true for any w of norm 1, we deduce that

$$|D^2(\Phi - \psi)(x)| \leq O(\varepsilon^{\frac{1}{6}}) \tag{3.7}$$

where $|A| = \lambda_{max}(A)$ denotes here the largest eigenvalue of a symmetric matrix. Finally setting $y = x + z$ in (3.5) with $z \in B_R(0)$ gives

$$\Phi(x + z) - \Phi(x) \leq \psi(x + z) - \psi(x) + C\varepsilon$$

where the constant C again depends on x, Φ, ψ, F but not on z. Integrating this inequality with respect to z's such that $\gamma \leq |z| \leq R$ (with $\gamma > 0$ to be chosen later), we obtain

$$\int_{|z| \geq \gamma} (\Phi(x + z) - \Phi(x)) \nu(dz) \leq \int_{|z| \geq \gamma} (\psi(x + z) - \psi(x)) \nu(dz) + C\varepsilon \nu(|z| \geq \gamma) .$$

By (2.4), we know that $\gamma^2 \nu(|z| \geq \gamma) \leq C_\nu$ where C_ν is a constant that only depends on ν, we conclude that

$$\int_{|z| \geq \gamma} (\Phi(x + z) - \Phi(x)) \nu(dz) \leq \int_{|z| \geq \gamma} (\psi(x + z) - \psi(x)) \nu(dz) + C\varepsilon\gamma^{-2} \tag{3.8}$$

where C depends on x, Φ, ψ, F and ν (we do not change the name of the constant). On the other hand, by using the C^2 regularity of Φ and ψ, we obtain

$$\begin{aligned} \int_{|z| \leq \gamma} & (\Phi(x + z) - \Phi(x) - D\Phi(x) \cdot z) \nu(dz) \\ & \leq \int_{|z| \leq \gamma} (\psi(x + z) - \psi(x) - D\psi(x) \cdot z) \nu(dz) \\ & \quad + \left(\frac{1}{2} |D^2\Phi(x) - D^2\psi(x)| + C\gamma \right) \int_{|z| \leq \gamma} |z|^2 \nu(dz) \end{aligned}$$

where C only depends on Φ and ψ. Now choosing $\gamma = \varepsilon^{\frac{1}{6}}$ and using (3.7), we get

$$\int_{|z|\le\gamma}(\Phi(x+z)-\Phi(x)-D\Phi(x)\cdot z)\nu(dz) \le \int_{|z|\le\gamma}(\psi(x+z)-\psi(x)-D\psi(x)\cdot z)\nu(dz)+C\varepsilon^{\frac{1}{6}}$$

where C depends only on Φ, ψ, x, F, ν. Finally, from (3.7), we have

$$\left|\int_{\varepsilon^{1/6}\le|z|\le 1}(-D\Phi(x)\cdot z + D\psi(x)\cdot z)\nu(dz)\right| \le C\varepsilon^{1/6}.$$

Combining the above estimates with (3.8), we conclude that

$$I_R[x,\Phi] \le I_R[x,\psi] + C\varepsilon^{\frac{1}{6}} \tag{3.9}$$

where C depends on Φ, ψ, F, x and ν. Combining (3.6), (3.7) and (3.9), the continuity of F and its monotonicity condition (2.6) yield that

$$F(t,x,D\Phi(x),D^2\Phi(x),I_R[x,\Phi]) - F(t,x,D\psi(x),D^2\psi(x),I_R[x,\psi]) \ge o(1)\,. \tag{3.10}$$

Inserting this into (3.5) and choosing $y = x$, we find

$$0 > \varepsilon o(1) + \eta\varepsilon$$

a contradiction for ε small enough. Hence the lemma is proved. □

We conclude the proof of the convergence theorem (Theorem 1) by proving that the terminal condition is satisfied (Proposition 3.2).

Proof of Proposition 3.2. It is enough to prove that for some constant $C > 0$ and for all $k \in \mathbb{N}$, we have

$$\forall (t,x) \in (0,T)\times\mathbb{R}^N, \quad |u^\varepsilon(T-k\varepsilon,x) - u_T(x)| \le Ck\varepsilon\,. \tag{3.11}$$

We argue by induction. The relation (3.11) is clear for $k = 0$. We assume it is true for k and we prove it for $k+1$.

We first consider a family u_T^η of bounded C^2 functions such that $\|u_T^\eta - u_T\|_{W^{2,\infty}} \le \eta$. From the one-step dynamic programming principle (2.7) and the choice $\Phi = u_T^\eta$, we easily deduce that

$$\begin{aligned} u^\varepsilon(T-(k+1)\varepsilon,x) &\ge \inf_{y\in B_R(x)}\{u^\varepsilon(T-k\varepsilon,y) + u_T^\eta(x) - u_T^\eta(y) \\ &\quad -\varepsilon F(T-(k+1)\varepsilon, x, Du_T^\eta(x), D^2u_T^\eta(x), I_R[x,u_T^\eta])\} \\ &\ge \inf_{y\in B_R(x)}\{u^\varepsilon(T-k\varepsilon,y) - u_T(y)\} + u_T(x) - C_1\varepsilon - 2\eta \\ &\ge -Ck\varepsilon + u_T(x) - C_1\varepsilon - 2\eta \end{aligned}$$

where we used (3.11) and we chose

$$C_1 = \max\{F(t,x,p,A,l) : t\in(0,T), x\in B_R, |p|+|A| \le 2\|u_T\|_{W^{2,\infty}}, |l| \le C_\nu\|u_T\|_{W^{2,\infty}}\} + 1\,.$$

Changing C if necessary in (3.11) we can assume that $C \geq C_1$. Since η is arbitrary, we easily get an estimate from below:

$$u^\varepsilon(T(k+1)\varepsilon, x) - u_T(x) \geq -C(k+1)\varepsilon.$$

Using once again the one-step dynamic programming principle (2.7) and (3.11), we next get

$$\begin{aligned} u^\varepsilon(T-(k+1)\varepsilon, x) & \\ \leq \sup_{\substack{\Phi \in C^2(\mathbb{R}^N) \\ \|\Phi\|_\infty, |D\Phi(x)|, |D^2\Phi(x)| \leq \varepsilon^{-\alpha}}} \inf_{y \in B_R(x)} \Big(& u_T^\eta(y) + \Phi(x) - \Phi(y) \\ & - \varepsilon F(T-(k+1)\varepsilon, x, D\Phi(x), D^2\Phi(x), I_R[x,\Phi]) \Big) + Ck\varepsilon + \eta\,. \end{aligned} \tag{3.12}$$

In order to get the upper bound in (3.11), we use the consistency Lemma 3.2. Applying it to Φ, $\psi = u_T^\eta$, $t = T-(k+1)\varepsilon$, we get from (3.12)

$$\begin{aligned} u^\varepsilon(T-(k+1)\varepsilon, x) &\leq u_T(x) + Ck\varepsilon + 2\eta + o(\varepsilon) \\ &\quad -\varepsilon(F(T-(k+1)\varepsilon, x, Du_T^\eta, D^2u_T^\eta(x), I_R[x, u_T^\eta]) \\ &\leq u_T(x) + C_1\varepsilon + Ck\varepsilon + 2\eta \\ &\leq u_T(x) + C(k+1)\varepsilon + 2\eta \end{aligned}$$

and since η is arbitrary, the proof of the proposition is now complete. □

3.2. **Proof of Theorem 2.** We first remark that for all $\varepsilon > 0$ and all $(t,x) \in (0,T] \times \mathbb{R}^N$, $\inf u_T \leq u^\varepsilon(t,x) \leq \sup u_T$. We thus can consider the upper and lower relaxed limits $\overline{u}$ and $\underline{u}$ (they are finite) and we will prove below the following result.

Proposition 3.3. *The functions $\underline{u}$ and $\overline{u}$ are respectively a super-solution and a sub-solution of* (2.17)*.*

In order to conclude that u^ε converges towards the unique solution of (2.17), (2.5), it is then enough to prove that $\overline{u}(T,x) \leq u_T(x) \leq \underline{u}(T,x)$. This is an easy consequence of the following proposition whose proof is postponed too.

Lemma 3.3. *Given $\delta, R > 0$ there exists $C > 0$ such that for all $t \in (T-\delta, T]$ and all $x \in B_R(0)$*

$$|u^\varepsilon(t,x) - u_T(x)| \leq C(T - t + \varepsilon^{\frac{1}{2}})\, \|u_T\|_{Lip}. \tag{3.13}$$

The comparison principle for (2.17) then permits to conclude.

It remains to prove Proposition 3.3 and Lemma 3.3. In order to do so, we proceed as in the proof of Proposition 3.1 by introducing an operator $\mathcal{S}^\varepsilon[\phi]$. More precisely, if $\phi : (0,T] \times \mathbb{R}^N \to \mathbb{R}$ is a bounded function, we let

$$\mathcal{S}^\varepsilon[\phi](t,x) = \sup_{x_P \in E^+(x)} \left\{ \inf_{x_C \in E^-(x_P)} \phi(t_C, x_C) \right\} \tag{3.14}$$

where t_C is defined by (2.22) and $E^\pm$ is defined by (2.20). It is convenient to write

$$\begin{aligned} t_P &= t + T_P(x, x_P) \\ t_C &= t_P + T_C(x_P, x_C) \end{aligned}$$

where T_P and T_C are defined as follows

$$T_P(x, x_P) = \begin{cases} C_\varepsilon[\varepsilon(v_+(x_P))^{-1}] & \text{if } B_\varepsilon(x) \cap \{v > 0\} \neq \emptyset \\ \varepsilon^2 & \text{if not ,} \end{cases} \tag{3.15}$$

$$T_C(x_P, x_C) = \begin{cases} C_\varepsilon[\varepsilon(v_-(x_C))^{-1}] & \text{if } B_\varepsilon(x_P) \cap \{v < 0\} \neq \emptyset \\ \varepsilon^2 & \text{if not .} \end{cases}$$

We also introduce

$$\begin{aligned} \mathcal{R}^\varepsilon[\phi](t,x) &= \sup_{x_P \in E^+(x)} \phi(t_P, x_P) \\ \mathcal{R}_\varepsilon[\phi](t,x) &= \inf_{x_C \in E^-(x_P)} \phi(t_C, x_C). \end{aligned}$$

The two important properties of $\mathcal{R}^\varepsilon$ are:

$$\text{it commutes with constants:}$$
$$\mathcal{R}^\varepsilon[\phi + c] = \mathcal{R}^\varepsilon[\phi] + c \text{ for any constant } c \in \mathbb{R}; \tag{3.16}$$
$$\text{it is monotone:}$$
$$\text{if } \phi \leq \psi \text{ then } \mathcal{R}^\varepsilon[\phi] \leq \mathcal{R}^\varepsilon[\psi]. \tag{3.17}$$

We now rewrite $\mathcal{S}_\varepsilon$ as

$$\mathcal{S}_\varepsilon[\phi] = \mathcal{R}_\varepsilon[\mathcal{R}^\varepsilon[\phi]]\,.$$

We remark that $\mathcal{R}_\varepsilon$ and $\mathcal{S}_\varepsilon$ also commute with constants and are monotone. One can also observe that $\mathcal{R}_\varepsilon$ and $\mathcal{R}^\varepsilon$ have opposite values if v is changed into $-v$ and ϕ into $-\phi$.

Recall that the dynamic programming principle in this context is (2.21) *i.e.* with the new terminology

$$u^\varepsilon(t,x) = \mathcal{S}_\varepsilon[u^\varepsilon](t,x) = \mathcal{R}_\varepsilon[\mathcal{R}^\varepsilon[u^\varepsilon]](t,x). \tag{3.18}$$

The core of the proof of of Proposition 3.3 lies in the following "consistency lemma".

Lemma 3.4 (Consistency lemma for the eikonal equation)**.** *Consider a C^1 bounded smooth function $\phi : (0,T] \times \mathbb{R}$. Given $r > 0$ small enough and $(t_0, x_0) \in (0, T-r) \times \mathbb{R}^N$, there exists a function $o(1)$ depending only on (ε, r), ϕ, (t_0, x_0) and the speed function v such that $o(1) \to 0$ as $(\varepsilon, r) \to 0$, and the following holds: for all $(t,x) \in B_r(t_0, x_0)$, there exists $x_P, y_P, x_C, y_C \in B_\varepsilon(x)$, such that*

$$\mathcal{R}^\varepsilon[\phi](t,x) - \phi(t,x) \leq T_P(x, x_P)\Big(\partial_t \phi(t,x) + v_+(x)|D\phi(t,x)| + o(1)\Big), \tag{3.19}$$

$$\mathcal{R}^\varepsilon[\phi](t,x) - \phi(t,x) \geq T_P(x, y)\Big(\partial_t \phi(t,x) + v_+(x)|D\phi(t,x)| + o(1)\Big), \tag{3.20}$$

and

$$\mathcal{R}_\varepsilon[\phi](t,x) - \phi(t,x) \leq T_C(x,y_C)\bigg(\partial_t\phi(t,x) + v_-(x)|D\phi(t,x)| + o(1)\bigg), \tag{3.21}$$

$$\mathcal{R}_\varepsilon[\phi](t,x) - \phi(t,x) \geq T_C(x,x_C)\bigg(\partial_t\phi(t,x) + v_-(x)|D\phi(t,x)| + o(1)\bigg). \tag{3.22}$$

We can deduce from this lemma the following one

Lemma 3.5 (Consistency lemma for the eikonal equation - second version)**.** *We consider a function $\phi : (0,T] \times \mathbb{R}$ that is bounded and C^1 and $(t_0,x_0) \in (0,T) \times \mathbb{R}^N$. There exist a function $o(1)$ such that $o(1) \to 0$ as $(\varepsilon, r) \to 0$ and positive numbers m_ε, M_ε such that for all $(t,x) \in B_r(t_0,x_0)$ and all (ε, r) small enough*

$$\begin{aligned} m_\varepsilon\bigg(\partial_t\phi(t_0,x_0) + v(x_0)|D\phi(t_0,x_0)| + o(1)\bigg) &\leq \mathcal{S}^\varepsilon[\phi](t,x) - \phi(t,x) \\ &\leq M_\varepsilon\bigg(\partial_t\phi(t_0,x_0) + v(x_0)|D\phi(t_0,x_0)| + o(1)\bigg). \end{aligned} \tag{3.23}$$

The proofs of the two previous lemmas are postponed. We first explain how to derive Proposition 3.3 from Lemma 3.5.

Proof of Proposition 3.3. We are going show that $\underline{u}$ is a super-solution. Following Definition 2, let ϕ be a C^1 function such that $\underline{u} - \phi$ admits a minimum 0 at (t_0,x_0) on $\mathcal{V}_0 = B_\delta(t_0,x_0)$. Without loss of generality, we can assume that this minimum is strict, see [11]. Arguing as in the proof of Proposition 3.1, we deduce that $u^\varepsilon - \phi$ admits a minimum at $(t_\varepsilon, x_\varepsilon)$ on $\mathcal{V}_0$ with $(t_\varepsilon,x_\varepsilon) \to (t_0,x_0)$ as $\varepsilon \to 0$; and for all $(t,x) \in \mathcal{V}_0$

$$u^\varepsilon(t,x) \geq \phi(t,x) + (u^\varepsilon(t_\varepsilon,x_\varepsilon) - \phi(t_\varepsilon,x_\varepsilon)) := \phi(t,x) + \xi_\varepsilon$$

¿From the properties (3.16), (3.17) of $\mathcal{S}^\varepsilon$ and the dynamic programming principle (3.18), we have

$$u^\varepsilon(t_\varepsilon,x_\varepsilon) \geq \mathcal{S}^\varepsilon[\phi + \xi_\varepsilon](t_\varepsilon,x_\varepsilon) = \mathcal{S}^\varepsilon[\phi](t_\varepsilon,x_\varepsilon) + \xi_\varepsilon.$$

Since $u^\varepsilon(t_\varepsilon,x_\varepsilon) = \phi(t_\varepsilon,x_\varepsilon) + \xi_\varepsilon$ it follows that

$$\phi(t_\varepsilon,x_\varepsilon) \geq \mathcal{S}^\varepsilon[\phi](t_\varepsilon,x_\varepsilon).$$

Using Lemma 3.5 applied at $(t_\varepsilon,x_\varepsilon)$ we deduce the existence of m_ε such that

$$m_\varepsilon\left(\partial_t\phi(t_0,x_0) + v(x_0)|D\phi(t_0,x_0)| + o(1)\right) \leq 0.$$

Dividing by $m_\varepsilon > 0$ then letting $\varepsilon \to 0$ and $r \to 0$, we obtain

$$\partial_t\phi(t_0,x_0) + v(x_0)|D\phi(t_0,x_0)| \leq 0.$$

We thus conclude that $\underline{u}$ is a super-solution. The proof that $\overline{u}$ is a sub-solution is entirely parallel and Proposition 3.3 is proved. □

We now turn to the core of the argument, *i.e.*

Proof of Lemma 3.4. First we note that it suffices to prove (3.19), (3.20), because (3.21) and (3.22) follow by changing ϕ into $-\phi$ and v into $-v$.

Consider $(t,x) \in B_r(t_0,x_0)$ and x_P an ε^3-optimal position starting from x at time t i.e. such that $x_P \in E^+(x)$ and

$$\mathcal{R}^\varepsilon[\phi](t,x) = \phi(t_P, y) + O(\varepsilon^3)\,.$$

First, remark that $x_P \in B_\varepsilon(x) \subset B_{r+\varepsilon}(x_0)$. In particular, $|v(x_P)| \le |v(x_0)| + L_v(r+\varepsilon)$ where L_v is the Lipschitz constant of the function v. Hence, $|v(y)| \le \frac{1}{\varepsilon^{\frac{1}{2}}}$ for ε small enough (only depending on L_v, r and x_0). In particular in view of the definitions (2.19) and (3.15), $T_P(x,x_P) = \varepsilon^2$, $T_P(x,x_P) = \varepsilon v_+(x_P)^{-1}$ or $T_P(x,x_P) = \varepsilon^{\frac{1}{2}}$.

We now distinguish these three cases.

Case $T_P(x,x_P) = \varepsilon^2$. This can only happen if $x_P = x$, $v_+(x) = 0$ and $t_P = t + \varepsilon^2$, so we may write, by Taylor expansion of ϕ,

$$\begin{aligned}
\mathcal{R}^\varepsilon[\phi](t,x) - \phi(t,x) &= \phi(t_P,x_P) - \phi(t,x) + O(\varepsilon^3)\\
&= \phi(t+\varepsilon^2, x) - \phi(t,x) + O(\varepsilon^3)\\
&= \varepsilon^2(\partial_t\phi(t,x) + v_+(x)|D\phi(t,x)| + o_\varepsilon(1))
\end{aligned}$$

and we obtain the desired result (3.19)–(3.20).

Case $T_P(x,x_P) = \varepsilon^{\frac{1}{2}}$.. This case happens only if $0 < v(x_P) \le \varepsilon^{\frac{1}{2}}$. This implies in particular that $|v(x)| \le \varepsilon^{\frac{1}{2}} + L_v\varepsilon$. Then we simply write

$$\begin{aligned}
\mathcal{R}^\varepsilon[\phi](t,x) - \phi(t,x) &= \phi(t+\varepsilon^{\frac{1}{2}}, x_P) - \phi(t,x) + O(\varepsilon)\\
&= \varepsilon^{\frac{1}{2}}(\partial_t\phi(t,x) + o_\varepsilon(1))\\
&= \varepsilon^{\frac{1}{2}}(\partial_t\phi(t,x) + v_+(x)|D\phi(t,x)| + o_\varepsilon(1))
\end{aligned}$$

and we obtain the desired result in this case too.

Case $T_P(x,x_P) = \varepsilon v(x_P)^{-1}$. Then $v(x_P) \ge \varepsilon^{\frac{1}{2}}$. This implies in particular that $v(x) \ge \varepsilon^{\frac{1}{2}} - L_v\varepsilon$. We write in this case, Taylor expanding ϕ again

$$\begin{aligned}
\mathcal{R}^\varepsilon[\phi](t,x) - \phi(t,x) &= \phi(t_P,x_P) - \phi(t,x) + O(\varepsilon^3)\\
&= \frac{\varepsilon}{v(x_P)}(\partial_t\phi(t,x) + o_\varepsilon(1)) + (x_P - x)\cdot D\phi(t,x) + O(\varepsilon^2)\,.
\end{aligned}$$

Hence, we are done if $D\phi(t,x) = 0$. If not, we can write

$$\begin{aligned}
\mathcal{R}^\varepsilon[\phi](t,x) - \phi(t,x) &\le \frac{\varepsilon}{v(x_P)}(\partial_t\phi(t,x) + o_\varepsilon(1)) + \varepsilon|D\phi(t,x)| + O(\varepsilon^2)\\
&\le \frac{\varepsilon}{v(x_P)}(\partial_t\phi(t,x) + v_+(x)|D\phi(t,x)| + o_\varepsilon(1))\,.
\end{aligned}$$

To get the reversed inequality in this last case, we consider $y_P = x + \varepsilon\frac{D\phi(t,x)}{|D\phi(t,x)|}$. Remark that $v(y_P) \ge v(x) - L_v\varepsilon \ge \varepsilon^{\frac{1}{2}} - 2L_v\varepsilon > 0$ for ε small enough (only depending on L_v).

Then, since $y_P \in E^+(x)$, we have

$$\begin{aligned} \mathcal{R}^\varepsilon[\phi](t,x) - \phi(t,x) &\geq \phi(t + T_P(x,y_P), y_P) - \phi(t,x) \\ &\geq T_P(x,y_P)(\partial_t \phi(t,x) + o_\varepsilon(1)) + \varepsilon |D\phi(t,x)| + O(\varepsilon^2) \\ &\geq T_P(x,y_P)(\partial_t \phi(t,x) + \frac{\varepsilon}{T_P(x,y_P)} |D\phi(t,x)| + o_\varepsilon(1)) \\ &\geq T_P(x,y_P)(\partial_t \phi(t,x) + \max(\varepsilon^{\frac12}, v(y_P)) |D\phi(t,x)| + o_\varepsilon(1)). \end{aligned}$$

If $v(y_P) \geq \varepsilon^{\frac12}$, then we use the Lipschitz continuity of v_+ in order to get

$$\mathcal{R}^\varepsilon[\phi](t,x) - \phi(t,x) \geq T_P(x,y_P)(\partial_t \phi(t,x) + v_+(x)|D\phi(t,x)| + o_\varepsilon(1)).$$

If $v(y_P) < \varepsilon^{\frac12}$, then $\varepsilon^{\frac12} - L_v \varepsilon \leq v(x) \leq \varepsilon^{\frac12} + L_v \varepsilon$ and we conclude that $v(x) = o(1)$ and the result is obtained in this case too. ☐

Proof of Lemma 3.5. Recall that $\mathcal{S}_\varepsilon[\phi] = \mathcal{R}_\varepsilon[\mathcal{R}^\varepsilon[\phi]]$. We distinguish cases.
Case $v(x_0) > 0$. In this case, we can write $v(x_0) \geq 2\delta_0 > 0$. For ε small enough and $r \leq \frac12$, we have for all $x_P \in E^+(x) \subset B_\varepsilon(x)$, $\delta_0 \leq v(x_P) \leq v(x_0) + 1$ and $T_P(x,x_P) = \frac{\varepsilon}{v(x_P)}$. We thus obtain from Lemma 3.4 for all $(t,x) \in B_r(t_0,x_0)$, the existence of $x_P \in B_\varepsilon(x)$ such that

$$\begin{aligned} \mathcal{R}^\varepsilon[\phi](t,x) &\leq \phi(t,x) + \frac{\varepsilon}{v(x_P)}(\partial_t \phi(t,x) + v_+(x)|D\phi(t,x)| + o(1)) \\ &\leq \phi(t,x) + \frac{\varepsilon}{\delta_1}(\partial_t \phi(t_0,x_0) + v(x_0)|D\phi(t_0,x_0)| + o(1)) \end{aligned}$$

as $(\varepsilon, r) \to 0$, where

$$\delta_1 = \begin{cases} \delta_0 & \text{if } \partial_t \phi(t_0,x_0) + v(x_0)|D\phi(t_0,x_0)| > 0, \\ v(x_0) + 1 & \text{if } \partial_t \phi(t_0,x_0) + v(x_0)|D\phi(t_0,x_0)| \leq 0. \end{cases}$$

Since $\mathcal{R}_\varepsilon$ commutes with constants and is monotone, the previous inequality implies the following one

$$\begin{aligned} \mathcal{S}_\varepsilon[\phi](t,x) - \phi(t,x) &\leq \mathcal{R}_\varepsilon[\phi](t,x) - \phi(t,x) \\ &\quad + \frac{\varepsilon}{\delta_1}(\partial_t \phi(t_0,x_0) + v(x_0)|D\phi(t_0,x_0)| + o(1)). \end{aligned}$$

Now from (3.21) we deduce

$$\mathcal{R}_\varepsilon[\phi](t,x) \leq T_C(x,y_C)\,(\partial_t \phi(t_0,x_0) + o(1))$$

for some $y_C \in B_\varepsilon(x)$ and since $v(x_0) \geq 2\delta_0 > 0$ we have $B_\varepsilon(x) \cap \{v < 0\} = \varnothing$ for r small enough and thus $T_C(x,y_C) = \varepsilon^2$. It follows that

$$\begin{aligned} \mathcal{S}_\varepsilon[\phi](t,x) - \phi(t,x) &\leq \varepsilon^2 \partial_t \phi(t_0,x_0) + o(\varepsilon^2) \\ &\quad + \frac{\varepsilon}{\delta_1}(\partial_t \phi(t_0,x_0) + v(x_0)|D\phi(t_0,x_0)| + o(1)) \\ &\leq \frac{\varepsilon}{\delta_1}(\partial_t \phi(t_0,x_0) + v(x_0)|D\phi(t_0,x_0)| + o(1))\,. \end{aligned}$$

which establishes the upper bound part in (3.23). The case $v(x_0) < 0$ is analogous.

Case $v(x_0) = 0$. By (3.19), we may write

$$\begin{aligned}\mathcal{R}^\varepsilon[\phi](t,x) &\leq \phi(t,x) + T_P(x,x_P)(\partial_t\phi(t,x) + o(1)) \\ &\leq \phi(t,x) + M_{\varepsilon,1}(\partial_t\phi(t,x) + o(1))\end{aligned}$$

for some positive constant $M_{\varepsilon,1}$, since T_P is bounded above and below by positive constants depending on ε (the last relation is obtained by discussing according to the sign of $\partial_t\phi(t,x)$). ¿From (3.21) we obtain similarly that

$$\mathcal{R}_\varepsilon[\phi](t,x) \leq \phi(t,x) + M_{\varepsilon,2}(\partial_t\phi(t,x) + o(1))$$

for some positive $M_{\varepsilon,2}$. Combining the two relations we obtain

$$\mathcal{S}^\varepsilon[\phi](t,x) - \phi(t,x) \leq (M_{\varepsilon,1} + M_{\varepsilon,2})(\partial_t\phi(t,x) + o(1)).$$

The lower bound is entirely parallel, and the desired result follows in this case. The proof is now complete. □

Proof of Lemma 3.3. It is enough to study the sequence of iterated positions and times starting from $(t,x) \in (T-\delta,T] \times B_R(0)$. The dynamic programming principle (2.21) gives optimal positions x_P, x_C such that $u^\varepsilon(t,x) = u^\varepsilon(t_C,x_C)$ for the corresponding time t_C. Letting $t_0 = t$ and $x_0 = x$, and iterating this, we may define for $k \in \{1,\dots,K\}$ optimal positions x_k (corresponding to the x_C) and corresponding times t_k with K such that $t_K \geq T$ and $t_{K-1} < T$ such that

$$u^\varepsilon(t_0,x_0) = u^\varepsilon(t_k,x_k) = u^\varepsilon(t_K,x_K) = u_T(x_K).$$

It follows that

$$u^\varepsilon(t,x) - u_T(x) = u_T(x_K) - u_T(x)\ .$$

We conclude from the previous equality that, in order to prove (3.13), it is enough to prove that for any $k \in \{0, K-1\}$

$$|x_{k+1} - x_k| \leq C(t_{k+1} - t_k) \tag{3.24}$$

with C not depending on ε and (t,x) (but possibly on δ and R).

We notice that the supremum and the infimum defining u^ε may not be attained. In this case, we simply choose first an ε^2-optimal position, then an ε^3-optimal position and iterating this, we obtain an error which is smaller than ε as soon as $\varepsilon \leq \frac{1}{2}$.

In order to prove such a result, we first remark that it suffices to prove that

$$|x_P - x| \leq C(t_P - t) \quad \text{and} \quad |x_C - x_P| \leq C(t_C - t_P)\ .$$

Hence, when $x_P = x$ and $x_C = x_P$, this is automatically satisfied. If not, we always have $|x_P - x| \leq \varepsilon$ and $|x_C - x_P| \leq \varepsilon$. Hence, we only need to check that for such time steps

$$C_\varepsilon(\varepsilon|v(x_P)|^{-1}) \geq C^{-1}\varepsilon \quad \text{and} \quad C_\varepsilon(\varepsilon|v(x_C)|^{-1}) \geq C^{-1}\varepsilon$$

for $C > 0$ well chosen. This is equivalent to showing

$$|v(x_i)| \leq C \quad \text{for } i = P, C\ .$$

By Lipschitz continuity of v, there exists C_R such that for all $y \in B_{R+1}(0)$

$$|v(y)| \leq C_R\ .$$

Recall that $x \in B_R(0)$. If the finite sequence $(x_k)_{k=0,\dots,K}$ remains in $B_{R/2}(x) \subset B_{R+1}(0)$, we are done: we choose $C = C_R$. We now claim that the finite sequence does remain in $B_{R/2}$.

We argue by contradiction. If not, consider k_0, the smallest integer $k \leq K$ such that $x_k \in B_R(x) \setminus B_{R/2}(x) \subset B_{R+1}(0)$. Consider also the number k_1 ($\leq k_0$) of steps such that Paul and Carol move (*i.e.* $x_P \neq x$ and $x_C \neq x_P$). This implies that the corresponding time increments are at least ε / C_R. This also implies that $2k_1\varepsilon \geq R/2$. Indeed,

$$\frac{R}{2} \leq |x_{k_0} - x| \leq \sum_{i=0}^{k_0-1} |x_{i+1} - x_i| \leq k_1 \times (2\varepsilon) .$$

Recalling that $t \in (T-\delta, T]$, it is now enough to choose

$$\delta < \frac{R}{4C_R}$$

to conclude that $t_{k_0} - t \geq \frac{k_1 \varepsilon}{C_R} \geq \frac{R}{4C_R} > \delta$ and thus $t_{k_0} > T$, and get a contradiction. □

3.3. **Proof of Theorem 3.** First we denote $\underline{u} = \liminf_* u^\varepsilon$ and $\overline{u} = \limsup^* u^\varepsilon$. These relaxed semi-limits are finite since we always have $\inf u_T \leq u^\varepsilon \leq \sup u_T$ and u_T is assumed to be uniformly bounded. The theorem follows as above from the following two results

Proposition 3.4. *The functions* $\underline{u}$ *and* $\overline{u}$ *are respectively a super-solution and a sub-solution of* (2.18)*.*

Lemma 3.6. *Given* R, δ*, there exists* $C > 0$ *such that for all* $t \in (0, \delta)$ *and all* $x \in B_R(0)$

$$|u^\varepsilon(t,x) - u_T(x)| \leq C(T - t + \varepsilon^{\frac{1}{2}}) . \tag{3.25}$$

Lemma 3.6 implies that $\underline{u}(T,x) \geq u_T(x) \geq \overline{u}(T,x)$ and the comparison principle for (2.18) (see [17]) permits to conclude.

It remains to prove Proposition 3.4 and Lemma 3.6. We first introduce some notation, analogous to that of Section 3.2. Given $x \in \mathbb{R}^N$ and $\phi \in C^2$, T_P and T_C are defined by

$$T_P(x,\phi) = \begin{cases} C_\varepsilon(\varepsilon \kappa^*[x,\phi]^{-1}) & \text{if } D\phi(x) \neq 0 \text{ and } \kappa^*[x,\phi] > 0 \\ \varepsilon^2 & \text{if not} \end{cases} \tag{3.26}$$

$$T_C(x,\phi) = \begin{cases} C_\varepsilon(\varepsilon |\kappa_*[x,\phi]|^{-1}) & \text{if } D\phi(x) \neq 0 \text{ and } \kappa_*[x,\phi] < 0 \\ \varepsilon^2 & \text{if not} . \end{cases}$$

It is convenient to write

$$\begin{aligned} t^+ &= t + T_P(x_P^+, \phi^+) \\ t^- &= t^+ + T_C(x_C^-, \phi^-) \end{aligned}$$

where T_P and T_C are defined by (3.26).

We now introduce for any arbitrary function $\phi : (0,T) \times \mathbb{R}^N \to \mathbb{R}$ the following operator

$$\mathcal{S}^\varepsilon[\phi](t,x) = \sup_{(x_P^+,\phi^+)\in\mathcal{C}^+(x)} \left\{ \inf_{x_C^+\in\mathcal{P}^+(x,x_P^+,\phi^+)} \left\{ \inf_{(x_C^-,\phi^-)\in\mathcal{C}^-(x_C^+)} \left\{ \sup_{x_P^-\in\mathcal{P}^-(x_C^+,x_C^-,\phi^-)} \{\phi(t^-,x_P^-)\} \right\}\right\}\right\}$$

where t^- is defined in (2.36). The dynamic programming principle (2.35) can be rewritten as follows

$$u^\varepsilon(t,x) = \mathcal{S}^\varepsilon[u^\varepsilon](t,x). \tag{3.27}$$

For the reader's convenience, we recall here the definitions of $\mathcal{C}^\pm(x)$ and $\mathcal{P}^\pm(x,y,\varphi)$:

$$\begin{aligned}
\mathcal{C}^\pm(x) &= \{(y,\varphi) \in B_\varepsilon(x) \times C^2(\mathbb{R}^N) : \pm\varphi(y) \geq \pm\varphi(x)\}\,,\\
\mathcal{P}^+(x,y,\varphi) &= \begin{cases} \{z \in B_R(y) : \varphi(z) \geq \varphi(y)\} & \text{if } D\varphi(y) \neq 0 \text{ and } \kappa^*[y,\varphi] > 0\\ \{x\} & \text{if not,}\end{cases}\\
\mathcal{P}^-(x,y,\varphi) &= \begin{cases} \{z \in B_R(y) : \varphi(z) \leq \varphi(y)\} & \text{if } D\varphi(y) \neq 0 \text{ and } \kappa_*[y,\varphi] < 0\\ \{x\} & \text{if not.}\end{cases}
\end{aligned}$$

Let us also define the following operators

$$\begin{aligned}
\mathcal{R}^\varepsilon[\phi](t,x) &= \sup_{(y,\varphi)\in\mathcal{C}^+(x)} \inf_{z\in\mathcal{P}^+(x,y,\varphi)} \phi(t+T_P(y,\varphi),z)\,,\\
\mathcal{R}_\varepsilon[\phi](t,x) &= \inf_{(y,\varphi)\in\mathcal{C}^-(x)} \sup_{z\in\mathcal{P}^-(x,y,\varphi)} \phi(t+T_C(y,\varphi),z)\,.
\end{aligned}$$

The reader can notice that

$$\begin{aligned}
\mathcal{S}^\varepsilon[\phi](t,x) &= \mathcal{R}^\varepsilon[\mathcal{R}_\varepsilon[\phi]](t,x)\,,\\
\mathcal{R}_\varepsilon[\phi](t,x) &= -\mathcal{R}^\varepsilon[-\phi](t,x)
\end{aligned} \tag{3.28}$$

In order to get the second equality, we need to remark that

$$\begin{aligned}
(z,\varphi) \in \mathcal{C}^+(y) &\Leftrightarrow (z,-\varphi) \in \mathcal{C}^-(y)\,,\\
z \in \mathcal{P}^+(y,\varphi) &\Leftrightarrow z \in \mathcal{P}^-(y,-\varphi)\,,\\
T_P(z,\varphi) &= T_C(z,-\varphi)\,.
\end{aligned}$$

Moreover, the operator $\mathcal{R}^\varepsilon$ is monotone and commutes with constants:

$$\phi_1 \leq \phi_2 \;\Rightarrow\; \mathcal{R}^\varepsilon[\phi_1] \leq \mathcal{R}^\varepsilon[\phi_2] \tag{3.29}$$

$$\mathcal{R}^\varepsilon[\phi + c] = \mathcal{R}^\varepsilon[\phi] + c \tag{3.30}$$

for all $c \in \mathbb{R}$. The proof of Proposition 3.4 relies on four consistency lemmas. Before stating them, let us point out that we will write $\kappa[\cdot]_+$ for the positive part of $\kappa[\cdot]$ and $\kappa[\cdot]_-$ for the negative part (both being nonnegative).

Lemma 3.7 (Estimate from below for $\mathcal{R}^\varepsilon$)**.** *Consider a C^2 function $\phi : (0,T] \times \mathbb{R}^N \to \mathbb{R}$ and $(t_0,x_0) \in (0,T) \times \mathbb{R}^N$. There exists a function $o(1)$ depending on ϕ and (ε,r) such that $o(1) \to 0$ as $(\varepsilon,r) \to 0$ and such that for all $(t,x) \in B_r(t_0,x_0)$ there exists $(y,\varphi) \in \mathcal{C}^+(x)$ such that*

- *if* $D\phi(t_0,x_0)\neq 0$,

$$\mathcal{R}^\varepsilon[\phi](t,x)-\phi(t,x)\geq T_P(y,\varphi)\Big(\partial_t\phi(t_0,x_0)+\kappa_*[x_0,\phi(t_0,\cdot)]_+|D\phi(t_0,x_0)|+o(1)\Big), \tag{3.31}$$

$$|D\varphi(y)|\geq\frac{1}{2}|D\phi(t_0,x_0)| \tag{3.32}$$

and

$$\kappa_*[x_0,\phi(t_0,\cdot)]+o(1)\leq\kappa_*[y,\varphi]\leq\kappa^*[y,\varphi]\leq\kappa^*[x_0,\phi(t_0,\cdot)]+o(1); \tag{3.33}$$

- *if* $D\phi(t_0,x_0)=0$, (3.31) *still holds true with the convention*

$$\kappa_*[x_0,\phi(t_0,\cdot)]_+|D\phi(t_0,x_0)|=0$$

and $T_P(y,\varphi)=\varepsilon^2$.

Lemma 3.8 (Estimate from above for $\mathcal{R}^\varepsilon$)**.** *Consider a* C^2 *function* $\phi:(0,T]\times\mathbb{R}^N\to\mathbb{R}$ *and* $(t_0,x_0)\in(0,T)\times\mathbb{R}^N$. *There exists a function* $o(1)$ *such that* $o(1)\to 0$ *as* $(\varepsilon,r)\to 0$ *and such that for all* $(t,x)\in B_r(t_0,x_0)$

- *either*

$$\mathcal{R}^\varepsilon[\phi](t,x)-\phi(t,x)\leq\varepsilon^2(\partial_t\phi(t_0,x_0)+o(1))\,; \tag{3.34}$$

- *or* $D\phi(t_0,x_0)=0$ *and*

$$\mathcal{R}^\varepsilon[\phi](t,x)-\phi(t,x)\leq T_P(y,\varphi)(\partial_t\phi(t_0,x_0)+o(1)) \tag{3.35}$$

for some $(y,\varphi)\in\mathcal{C}^+(x)$;

- *or* $D\phi(t_0,x_0)\neq 0$ *and* $\kappa^*[x_0,\phi(t_0,\cdot)]\geq 0$ *and*

$$\mathcal{R}^\varepsilon[\phi](t,x)-\phi(t,x)\leq T_P(y,\varphi)(\partial_t\phi(t_0,x_0)+\kappa^*[x_0,\phi(t_0,\cdot)]|D\phi(t_0,x_0)|+o(1)) \tag{3.36}$$

for some $(y,\varphi)\in\mathcal{C}^+(x)$ *such that* $T_P(y,\varphi)=\min\left(\varepsilon/\kappa^*[y,\varphi],\varepsilon^{\frac{1}{2}}\right)$ *with*

$$0<\kappa^*[y,\varphi]\leq\kappa^*[x_0,\phi(t_0,\cdot)]+o(1). \tag{3.37}$$

By using the fact that $\mathcal{R}_\varepsilon[\phi]=-\mathcal{R}^\varepsilon[-\phi]$ and exchanging the roles of $+$ and $-$, we then can deduce from the two previous lemmas the two following ones.

Lemma 3.9 (Estimate from above for $\mathcal{R}_\varepsilon$)**.** *Consider a* C^2 *function* $\phi:(0,T]\times\mathbb{R}^N\to\mathbb{R}$ *and* $(t_0,x_0)\in(0,T)\times\mathbb{R}^N$. *There exists a function* $o(1)$ *depending on* ϕ *and* (ε,r) *such that* $o(1)\to 0$ *as* $(\varepsilon,r)\to 0$ *and such that for all* $(t,x)\in B_r(t_0,x_0)$ *there exists* $(y,\varphi)\in\mathcal{C}^-(x)$ *such that*

- *if* $D\phi(t_0,x_0)\neq 0$,

$$\mathcal{R}_\varepsilon[\phi](t,x)-\phi(t,x)\leq T_C(y,\varphi)\Big(\partial_t\phi(t_0,x_0)-\kappa^*[x_0,\phi(t_0,\cdot)]_-|D\phi(t_0,x_0)|+o(1)\Big) \tag{3.38}$$

$$|D\varphi(y)|\geq\frac{1}{2}|D\phi(t_0,x_0)| \tag{3.39}$$

and

$$\kappa_*[x_0, \phi(t_0, \cdot)] + o(1) \leq \kappa_*[y, \varphi] \leq \kappa^*[y, \varphi] \leq \kappa^*[x_0, \phi(t_0, \cdot)] + o(1); \tag{3.40}$$

- *if* $D\phi(t_0, x_0) = 0$, *then* (3.38) *still holds true with the convention*

$$\kappa^*[x_0, \phi(t_0, \cdot)]_- |D\phi(t_0, x_0)| = 0$$

and $T_C(y, \varphi) = \varepsilon^2$.

Lemma 3.10 (Estimate from below for $\mathcal{R}_\varepsilon$)**.** *Consider a* C^2 *function* $\phi : (0, T] \times \mathbb{R}^N \to \mathbb{R}$ *and* $(t_0, x_0) \in (0, T) \times \mathbb{R}^N$. *There exists a function* $o(1)$ *such that* $o(1) \to 0$ *as* $(\varepsilon, r) \to 0$ *and such that for all* $(t, x) \in B_r(t_0, x_0)$

- *either*

$$\mathcal{R}_\varepsilon[\phi](t, x) - \phi(t, x) \geq \varepsilon^2(\partial_t \phi(t_0, x_0) + o(1)); \tag{3.41}$$

- *or* $D\phi(t_0, x_0) = 0$ *and*

$$\mathcal{R}_\varepsilon[\phi](t, x) - \phi(t, x) \geq T_C(y, \varphi)(\partial_t \phi(t_0, x_0) + o(1)) \tag{3.42}$$

for some $(y, \varphi) \in \mathcal{C}^-(x)$;

- *or* $D\phi(t_0, x_0) \neq 0$ *and* $\kappa_*[x_0, \phi(t_0, \cdot)] \leq 0$ *and*

(3.43)
$$\mathcal{R}_\varepsilon[\phi](t, x) - \phi(t, x) \geq T_C(y, \varphi)(\partial_t \phi(t_0, x_0) + \kappa_*[x_0, \phi(t_0, \cdot)]|D\phi(t_0, x_0)| + o(1))$$

for some $(y, \varphi) \in \mathcal{C}^-(x)$ *such that* $T_C(y, \varphi) = \min\left(\varepsilon/|\kappa_*[y, \varphi]|, \varepsilon^{\frac{1}{2}}\right)$ *with*

$$0 > \kappa_*[y, \varphi] \geq \kappa_*[x_0, \phi(t_0, \cdot)] + o(1).$$

The proofs of Lemmas 3.7 and 3.8 are postponed. We now explain how to derive Proposition 3.4.

Proof of Proposition 3.4. We only prove that $\overline{u}$ is a sub-solution of (2.18) since a symmetric argument can be used to prove that $\underline{u}$ is a super-solution. In order to do so, we consider a $(t_0, x_0) \in (0, T) \times \mathbb{R}^N$ and a $\phi \in C^2$ such that $\underline{u} - \phi$ attains a strict maximum at (t_0, x_0) in $(0, T) \times B_{R+1}(x_0)$. We want to prove that $-\partial_t \phi(t_0, x_0) - \kappa^*[x_0, \phi(t_0, \cdot)]|D\phi(t_0, x_0)| \leq 0$ if $D\phi(t_0, x_0) \neq 0$ and $-\partial_t \phi(t_0, x_0) \leq 0$ if $D\phi(t_0, x_0) = 0$.

We know that there exists a sequence $(t_{\varepsilon_n}, x_{\varepsilon_n})$ such that $u^{\varepsilon_n} - \phi$ attains a maximum in $(0, T) \times B_R(x_{\varepsilon_n})$ at $(t_{\varepsilon_n}, x_{\varepsilon_n})$. For simplicity, we simply write (t, x) for $(t_{\varepsilon_n}, x_{\varepsilon_n})$ and ε for ε_n. With the same argument as in the proof of Propositions 3.1 and 3.3, the dynamic programming principle (3.27) and the monotonicity of $\mathcal{S}_\varepsilon$ imply that

$$\phi(t, x) \leq \mathcal{S}^\varepsilon[\phi](t, x). \tag{3.44}$$

We now estimate $\mathcal{S}^\varepsilon[\phi](t, x)$ from above. We distinguish cases.

Case $D\phi(t_0,x_0)\neq 0$ and $\kappa^*[x_0,\phi(t_0,\cdot)]>0$. Lemma 3.9 yields an $(y,\varphi)\in\mathcal{C}^-(x)$, with $\kappa_*[y,\varphi]>0$ (by (3.40)) for ε small enough, thus $T_C(y,\varphi)=\varepsilon^2$, so that

$$\mathcal{R}_\varepsilon[\phi](t,x)-\phi(t,x)\leq \varepsilon^2\left(\partial_t\phi(t_0,x_0)+o(1)\right)$$

and using properties (3.29)–(3.30), we find

$$\begin{aligned}\mathcal{S}^\varepsilon[\phi](t,x) &= \mathcal{R}^\varepsilon[\mathcal{R}_\varepsilon[\phi]](t,x)\\ &\leq \mathcal{R}^\varepsilon[\phi](t,x)+\varepsilon^2(\partial_t\phi(t_0,x_0)+o(1)).\end{aligned}$$

We now use Lemma 3.8. If (3.34) holds true, then (with (3.44))

$$0\leq \mathcal{S}^\varepsilon[\phi](t,x)-\phi(t,x)\leq 2\varepsilon^2(\partial_t\phi(t_0,x_0)+o(1))$$

and we conclude that $\partial_t\phi(t_0,x_0)\geq 0$. The result follows easily in this subcase. The subcase where (3.35) holds works similarly.

If now (3.36) holds true, we get

$$\begin{aligned}0\leq \mathcal{S}_\varepsilon[\phi](t,x)-\phi(t,x)\leq T_P(y,\varphi)(\partial_t\phi(t_0,x_0)+\kappa^*[x_0,\phi(t_0,\cdot)]|D\phi(t_0,x_0)|+o(1))\\ +\varepsilon^2\left(\partial_t\phi(t_0,x_0)+o(1)\right).\end{aligned}$$

Since $T_P(y,\varphi)=\min\left(\varepsilon/\kappa_*[y,\varphi],\varepsilon^{\frac12}\right)$, this can be written as

$$0\leq \mathcal{S}_\varepsilon[\phi](t,x)-\phi(t,x)\leq T_P(y,\varphi)(\partial_t\phi(t_0,x_0)+\kappa^*[x_0,\phi(t_0,\cdot)]|D\phi(t_0,x_0)|+o(1))$$

and dividing by $T_P(y,\varphi)$ and letting $\varepsilon\to 0$, the desired inequality follows.

Case $D\phi(t_0,x_0)\neq 0$ and $\kappa^*[x_0,\phi(t_0,\cdot)]<0$. We apply first Lemma 3.9 and find

$$\mathcal{R}_\varepsilon[\phi](t,x)-\phi(t,x)\leq T_C(y,\varphi)(\partial_t\phi(t_0,x_0)+\kappa^*[x_0,\phi(t_0,x_0)]|D\phi(t_0,x_0)|+o(1)).$$

We note that from (3.39) we have $D\varphi(y)\neq 0$. Now we cannot have $\kappa_*[y,\varphi]\geq 0$, otherwise a contradiction would follow from (3.40) and our assumption $\kappa^*[x_0,\phi(t_0,\cdot)]<0$. We deduce that the case $T_C(y,\varphi)=\varepsilon^2$ cannot happen and we must have $\kappa_*[y,\varphi]<0$ and $\varepsilon^2=o(T_C(y,\varphi))$. With this piece of information at hand, we can write, as previously

$$\begin{aligned}0\leq \mathcal{S}_\varepsilon[\phi](t,x)-\phi(t,x)\leq \mathcal{R}^\varepsilon[\phi](t,x)-\phi(t,x)\\ +T_C(y,\varphi)(\partial_t\phi(t_0,x_0)+\kappa^*[x_0,\phi(t_0,\cdot)]|D\phi(t_0,x_0)|+o(1)).\end{aligned}$$

On the other hand, Lemma 3.8 yields (we can only be in the first situation of the lemma)

$$\mathcal{R}^\varepsilon[\phi](t,x)\leq \phi(t,x)+O(\varepsilon^2)=\phi(t,x)+o(T_C(y,\varphi)),$$

and we can write, dividing by $T_C(y,\varphi)$,

$$0\leq \partial_t\phi(t_0,x_0)+\kappa^*[x_0,\phi(t_0,\cdot)]|D\phi(t_0,x_0)|+o(1)$$

and the desired inequality is thus obtained in this case too.

Case $D\phi(t_0,x_0) \neq 0$ and $\kappa^*[x_0,\phi(t_0,\cdot)] = 0$. Once again, we first apply Lemma 3.9 and we obtain

$$0 \leq \mathcal{S}_\varepsilon[\phi](t,x) - \phi(t,x) \leq \mathcal{R}^\varepsilon[\phi](t,x) - \phi(t,x) + T_C(y,\varphi)(\partial_t\phi(t_0,x_0) + o(1)).$$

Lemma 3.8 implies that

$$\mathcal{R}^\varepsilon[\phi](t,x) - \phi(t,x) \leq M_\varepsilon(\partial_t\phi(t_0,x_0) + o(1))$$

with $M_\varepsilon = \varepsilon^2$ or $M_\varepsilon = T_P(y,\varphi)$. Hence we obtain

$$0 \leq (T_C(y,\varphi) + M_\varepsilon)(\partial_t\phi(t_0,x_0) + o(1))$$

and we obtain the desired inequality in this case too.
Case $D\phi(t_0,x_0) = 0$. Then Lemmas 3.9 and 3.8 yield

$$0 \leq (\varepsilon^2 + T_P(y,\varphi))(\partial_t\phi(t_0,x_0) + o(1))$$

or $0 \leq 2\varepsilon^2(\partial_t\phi(t_0,x_0) + o(1))$, and the proof of the proposition is now complete. □

We now turn to the proofs of Lemmas 3.7 and 3.8. As the reader shall see, we follow along the lines of proofs of Lemmas 3.4 and 3.5 used in the eikonal case.

Proof of Lemma 3.7. We first assume that $D\phi(t_0,x_0) \neq 0$. So we can assume that, for ε small enough, $|D\phi(t,x)| \geq \theta_0 > 0$.

Consider $y = x + \varepsilon\frac{D\phi(t,x)}{|D\phi(t,x)|}$ and $\varphi(z) = \phi(t,z) - \alpha_\varepsilon(z)$ with $\alpha_\varepsilon : \mathbb{R}^N \to [0,+\infty)$ smooth and

$$\alpha_\varepsilon(z) = \begin{cases} 0 & \text{if } |z-x| \leq \varepsilon \\ \varepsilon^{\frac{1}{4}} & \text{if } |z-x| \geq 2\varepsilon. \end{cases}$$

We also can write for ε small enough

$$\varphi(y) = \phi(t,y) = \phi(t,x) + \varepsilon|D\phi(t,x)| + O(\varepsilon^2) \geq \phi(t,x) = \varphi(x)$$

(we used the fact that $|D\phi(t,x)| \geq \theta_0 > 0$). This means that $(y,\varphi) \in \mathcal{C}^+(x)$ (at least for ε small enough). Remark also that (3.32) holds. Hence

$$\mathcal{R}^\varepsilon[\phi](t,x) \geq \inf_{z\in\mathcal{P}^+(x,y,\varphi)} \phi(t + T_P(y,\varphi), z).$$

Since $(t,y) = (t_0,x_0) + o_\varepsilon(1)$ and $\varphi \leq \phi(t,\cdot)$, it follows that

$$\{z|\varphi(z) \geq \varphi(y) = \phi(t,y)\} \subset \{z|\phi(t,z) \geq \phi(t,y)\}$$

and, using the monotonicity of κ^* and its upper semi-continuity (see (2.28) and (2.26)), we conclude that

$$\kappa^*[y,\varphi] \leq \kappa^*[y,\phi(t,\cdot)] \leq \kappa^*[x_0,\phi(t_0,\cdot)] + o_\varepsilon(1) \tag{3.45}$$

for ε small enough; the other part of (3.33) follows similarly by lower semi-continuity of κ_* (2.27). (3.33) is thus proved. Moreover (3.45) implies $\kappa^*[y,\varphi] < \frac{1}{\sqrt{\varepsilon}}$ hence either $T_P(y,\varphi) = \varepsilon^2$ or $T_P(y,\varphi) = \frac{\varepsilon}{\kappa^*[y,\varphi]_+}$ or $T_P(y,\varphi) = \varepsilon^{\frac{1}{2}}$. We treat these cases separately.

Case $T_P(y,\varphi)=\varepsilon^2$. We know that in this case we have $\kappa^*[y,\varphi]\le 0$ and $\mathcal{P}^+(x,y,\varphi)=\{x\}$. Hence,

$$\begin{array}{rcl}\mathcal{R}^\varepsilon[\phi](t,x)-\phi(t,x) & \ge & \phi(t+\varepsilon^2,x)-\phi(t,x)+O(\varepsilon^3)\\ & \ge & \varepsilon^2(\partial_t\phi(t,x)+o_\varepsilon(1)).\end{array}$$

Since $\kappa^*[y,\varphi]\le 0$ we deduce from (3.33) $\kappa_*[x_0,\phi(t_0,\cdot)]\le 0$. hence (3.31) is proved in this case.

In the two remaining cases, we have $T_P(y,\varphi)\le\varepsilon^{\frac12}$ and $\mathcal{P}^+(x,y,\varphi)\ni y$. These two facts imply the following inequality

$$\mathcal{R}^\varepsilon[\phi](t,x)\ge \inf_{z\in\mathcal{P}^+(x,y,\varphi)\cap B_{2\varepsilon}(x)}\phi(t+T_P(y,\varphi),z). \tag{3.46}$$

i.e. the fact that the infimum can only be achieved in $B_{2\varepsilon}(x)$. To see this, we simply write for $z\in\mathcal{P}^+(x,y,\varphi)$ such that $z\notin B_{2\varepsilon}(x)$,

$$\begin{array}{rcl}\phi(t+T_P(y,\varphi),z) & = & \phi(t,z)+O(T_P(y,\varphi))\\ & \ge & \phi(t,y)+\varepsilon^{\frac14}+O(\varepsilon^{\frac12})\\ & \ge & \phi(t+T_P(y,\varphi),y)+\varepsilon^{\frac14}+O(\varepsilon^{\frac12})\\ & > & \displaystyle\inf_{z\in\mathcal{P}^+(x,y,\varphi)\cap B_{2\varepsilon}(x)}\phi(t+T_P(y,\varphi),z)\end{array}$$

and (3.46) follows.
Case $T_P(y,\varphi)=\varepsilon^{\frac12}$. By definition of C_ε, this happens if $\kappa^*[y,\varphi]\le\varepsilon^{\frac12}$ and from (3.33) it follows that $\kappa_*[x_0,\phi(t_0,\cdot)]\le 0$. For $z\in\mathcal{P}^+(x,y,\varphi)\cap B_{2\varepsilon}(x)$, we have

$$\phi(t+T_P(y,\varphi),z)-\phi(t,x)\ge\phi(t+\varepsilon^{\frac12},z)-\phi(t,z)+\phi(t,z)-\phi(t,x)+O(\varepsilon^3)$$

But $\varphi(z)\ge\varphi(y)$ since $z\in\mathcal{P}^+(x,y,\varphi)$, hence $\phi(t,z)-\alpha_\varepsilon(z)\ge\phi(t,y)$ so replacing in the above and using $\alpha_\varepsilon\ge 0$ we are led to

$$\begin{array}{rcl}\phi(t+T_P(y,\varphi),z)-\phi(t,x) & \ge & \phi(t+\varepsilon^{\frac12},z)-\phi(t,z)+\phi(t,y)-\phi(t,x)+O(\varepsilon^3)\\ & \ge & \varepsilon^{\frac12}\partial_t\phi(t,x)+\varepsilon|D\phi(t,x)|+o_\varepsilon(\varepsilon)\\ & \ge & \varepsilon^{\frac12}(\partial_t\phi(t,x)+o_\varepsilon(1))\\ & \ge & \varepsilon^{\frac12}(\partial_t\phi(t,x)+\kappa_*[x_0,\phi(t_0,\cdot)]_+|D\phi(t_0,x_0)|+o_\varepsilon(1))\end{array}$$

and we get (3.31) in this case too.
Case $T_P(y,\varphi)=\frac{\varepsilon}{\kappa^*[y,\varphi]_+}$. Observe that from (3.45), $\kappa^*(y,\varphi)$ is bounded above hence $T_P(y,\varphi)$ bounded below by $c\varepsilon$. As above, we may write, recalling the choice of

$y = x + \varepsilon \frac{D\phi(t,x)}{|D\phi(t,x)|}$

$$\begin{aligned}
\mathcal{R}^\varepsilon[\phi](t,x) - \phi(t,x) &\geq \inf_{z\in\mathcal{P}^+(x,y,\varphi)\cap B_{2\varepsilon}(x)} \phi(t+T_P(y,\varphi),z) - \phi(t,x) \\
&\geq \inf_{z\in\mathcal{P}^+(x,y,\varphi)\cap B_{2\varepsilon}(x)} \phi(t+T_P(y,\varphi),z) - \phi(t,z) + \phi(t,z) - \phi(t,x) \\
&\geq T_P(y,\varphi)(\partial_t\phi(t,x) + o_\varepsilon(1)) + \phi(t,y) - \phi(t,x) \\
&\geq T_P(y,\varphi)(\partial_t\phi(t,x) + o_\varepsilon(1)) + \varepsilon|D\phi(t,x)| + O(\varepsilon^2) \\
&\geq T_P(y,\varphi)\left(\partial_t\phi(t,x) + \frac{\varepsilon}{T_P(y,\varphi)}|D\phi(t,x)| + o_\varepsilon(1)\right) \\
&\geq T_P(y,\varphi)\left(\partial_t\phi(t,x) + \kappa^*[y,\varphi]_+|D\phi(t,x)| + o_\varepsilon(1)\right) \\
&\geq T_P(y,\varphi)\left(\partial_t\phi(t,x) + \kappa_*[x_0,\phi(t_0,\cdot)]_+|D\phi(t_0,x_0)| + o_\varepsilon(1)\right)
\end{aligned}$$

where the last inequality follows from (3.33) ; and we get (3.31) in all cases.

Assume now that $D\phi(t_0,x_0) = 0$. Then choose $y = x$ and $\varphi(z) = -\alpha_\varepsilon(z)$. This is admissible and $D\varphi(y) = 0$ and $T_P(y,\varphi) = \varepsilon^2$ and the conclusion follows easily. □

We now turn to the proof of Lemma 3.8.

Proof of Lemma 3.8. We recall that

$$\mathcal{R}^\varepsilon[\phi](t,x) = \sup_{(y,\varphi)\in\mathcal{C}^+(x)} \inf_{z\in\mathcal{P}^+(x,y,\varphi)} \phi(t+T_P(y,\varphi),z) \tag{3.47}$$

In view of the definition of $\mathcal{C}^+(x)$ and $\mathcal{P}^+(x,y,\varphi)$, we can write more precisely

$$\begin{aligned}
&\mathcal{R}^\varepsilon[\phi](t,x) \\
&\quad = \sup_{\varphi\in C^2(\mathbb{R}^N)} \max\left(\sup_{\substack{y\in B_\varepsilon(x):\varphi(y)\geq\varphi(x)\\ D\varphi(y)\neq 0,\ \kappa^*[y,\varphi]>0,}} \inf_{z\in\mathcal{P}^+(x,y,\varphi)} \phi(t+T_P(y,\varphi),z), \phi(t+\varepsilon^2,x)\right).
\end{aligned} \tag{3.48}$$

Let φ be a fixed C^2 test function.
Case 1. Assume first that the max above is $\phi(t+\varepsilon^2,x)$. Then we easily obtain (3.34) as desired.
Case 2. We then turn to the situation where

$$\sup_{\substack{y\in B_\varepsilon(x):\varphi(y)\geq\varphi(x)\\ D\varphi(y)\neq 0,\kappa^*[y,\varphi]>0}} \inf_{z\in\mathcal{P}^+(x,y,\varphi)} \phi(t+T_P(y,\varphi),z) > \phi(t+\varepsilon^2,x). \tag{3.49}$$

We need to prove that (3.35) or (3.36) holds true in this case. So let $y\in B_\varepsilon(x)$ be such that $\varphi(y)\geq\varphi(x)$, $D\varphi(y)\neq 0$ and $\kappa^*[y,\varphi]>0$, and such that

$$\inf_{z\in\mathcal{P}^+(x,y,\varphi)} \phi(t+T_P(y,\varphi),z) > \phi(t+\varepsilon^2,x) - \varepsilon^3. \tag{3.50}$$

For any $z \in \mathcal{P}^+(x,y,\varphi) = \{z \in B_R(y) : \varphi(z) \geq \varphi(y)\}$ we may write

$$\begin{aligned}\phi(t^+,z) - \phi(t,x) &= \phi(t^+,z) - \phi(t^+,y) + \phi(t^+,y) - \phi(t,x) \\ &= \phi(t^+,z) - \phi(t^+,y) \\ &\quad + T_P(y,\varphi)(\partial_t\phi(t_0,x_0) + o_\varepsilon(1)) + D\phi(t^+,y)\cdot(y-x) + O(\varepsilon^2),\end{aligned}$$

and using the fact that $|y-x| \leq \varepsilon$ and $O(\varepsilon^2) = o(t^+ - t)$ since $T_P(y,\varphi) \geq \varepsilon^{3/2}$ in this case; we obtain

$$\phi(t^+,z) - \phi(t,x) \leq \phi(t^+,z) - \phi(t^+,y) + \varepsilon|D\phi(t^+,y)| + T_P(y,\varphi)(\partial_t\phi(t_0,x_0) + o_\varepsilon(1)). \tag{3.51}$$

We now evaluate $\phi(t^+,z)-\phi(t^+,y)$. In view of (3.48), (3.49) and (3.51), the following lemma permits to conclude.

Lemma 3.11. *For any $(y,\varphi) \in \mathcal{C}^+(x)$ with $D\varphi(y) \neq 0$, $\kappa^*[y,\varphi] > 0$, such that* (3.50) *holds, we have*

$$\inf_{z\in\mathcal{P}^+(x,y,\varphi)} \phi(t^+,z) - \phi(t^+,y) \leq T_P(y,\varphi)(\kappa^*[x_0,\phi(t_0,\cdot)]_+|D\phi(t_0,x_0)| + o_\varepsilon(1)) - \varepsilon|D\phi(t^+,y)|. \tag{3.52}$$

Moreover, if $D\phi(t_0,x_0) \neq 0$, we have

$$0 < \kappa^*[y,\varphi] \leq \kappa^*[x_0,\phi(t_0,\cdot)] \text{ and } T_P(y,\varphi) = \min\left(\varepsilon/\kappa^*(y,\varphi), \varepsilon^{\frac{1}{2}}\right).$$

There now remains to give the proof of Lemma 3.11. We start by

Lemma 3.12. *If $D\phi(t_0,x_0) \neq 0$, then for any (y,φ) as in the above lemma, we have*

$$0 < \kappa^*[y,\varphi] \leq \kappa^*[x_0,\phi(t_0,\cdot)].$$

Proof of Lemma 3.12. We are in the case where (3.50) holds true, hence for all $z \in \{z \in B_R(y) : \varphi(z) \geq \varphi(y)\}$, we have $\phi(t+T_P(y,\varphi),z) \geq \phi(t+\varepsilon^2,x) - \varepsilon^3$. This implies

$$\{z \in B_R(0) : \varphi(y+z) \geq \varphi(y)\} \subset \{z \in B_R(0) : \phi(t_0,x_0+z) \geq \phi(t_0,x_0)\}. \tag{3.53}$$

Indeed, if $z \in B_R(0)$ is such that $\phi(t_0,x_0+z) < \phi(t_0,x_0)$, then for ε small enough and $(t,x)-(t_0,x_0)$ small enough, $\phi(t+T_P(y,\varphi),y+z) < \phi(t+\varepsilon^2,x) - \varepsilon^3$ and in view of the above this implies $\varphi(y+z) < \varphi(y)$. By properties of the non-local curvature, (3.53) implies (3.54) and the proof of the lemma is complete. □

We can now complete the proof of Lemma 3.11 by

Proof of (3.52). Remark first that Lemma 3.12 implies in particular that

$$\kappa^*[y,\varphi]|D\phi(t_0,x_0)| \leq \kappa^*[x_0,\phi(t_0,\cdot)]|D\phi(t_0,x_0)| \tag{3.54}$$

since this inequality is trivial when $D\phi(t_0,x_0) = 0$.

We now argue by contradiction. We thus assume that there exists $\eta > 0$, $\varepsilon_n \to 0$, $(t_n,x_n) \to (t_0,x_0)$ and $(y_n,\varphi_n) \in \mathcal{C}^+(x_n)$ such that $D\varphi_n(y_n) \neq 0$, $\kappa^*[y_n,\varphi] > 0$ and (3.50) holds, and for all $z \in B_R(y_n) \cap \{\varphi_n \geq \varphi_n(y_n)\}$, we have

$$\phi(t_n^+,z) - \phi(t_n^+,y_n) \geq T_P(y_n,\varphi_n)(\kappa^*[x_0,\phi(t_0,\cdot)]_+|D\phi(t_0,x_0)| + \eta) - \varepsilon_n|D\phi(t_n^+,y_n)|. \tag{3.55}$$

It then follows from (3.54) that for n large enough
(3.56)

$$\phi(t_n^+, z) - \phi(t_n^+, y_n) \geq T_P(y_n, \varphi_n)\left(\kappa^*[y_n, \varphi_n]|D\phi(t_n^+, y_n)| + \frac{\eta}{2}\right) - \varepsilon_n|D\phi(t_n^+, y_n)|.$$

Assume first that there exists a subsequence such that $T_P(y_n, \varphi_n) = \varepsilon_n^{\frac{1}{2}}$. In this case, for all $z \in B_R(y_n) \cap \{\varphi_n \geq \varphi_n(y_n)\}$, we find

$$\phi(t_n^+, z) - \phi(t_n^+, y_n) \geq \frac{\eta}{2}\varepsilon_n^{\frac{1}{2}} + O(\varepsilon_n) > 0$$

for n large enough. We obtain a contradiction by taking $z = y_n$.

Either we have $T_P(y_n, \varphi_n) = \varepsilon^{3/2}$ or $T_P(y_n, \varphi_n) = \varepsilon_n/\kappa^*[y_n, \varphi_n]$. In both situations we have $T_P(y_n, \varphi_n) \geq \varepsilon_n/\kappa^*[y_n, \varphi_n]$. Choosing $z = y_n$ in (3.56) yields

$$0 \geq T_P(y_n, \varphi_n)\frac{\eta}{2}$$

which is a contradiction. □

□

We next prove that the terminal condition is satisfied at the limit.

Proof of Lemma 3.6. The proof consists in proving the following estimate

$$|u^\varepsilon(t, x) - u_T(x)| \leq C(T - t) \tag{3.57}$$

for $t < T$ and $x \in \mathbb{R}^N$ with

$$C = \sup_{x\in\mathbb{R}^N} \max(|\kappa_*[x, u_T]||Du_T(x)| + 1, |\kappa^*[x, u_T]||Du_T(x)| + 1).$$

We remark that (3.57) is a consequence of the following lemma.

Lemma 3.13. *Consider $k \in \mathbb{N} \cap (0, \varepsilon^{-2}T)$. If,*

$$\forall (t, x) \in (T - k\varepsilon^2, T) \times \mathbb{R}^N, \quad |u^\varepsilon(t, x) - u_T(x)| \leq C(T - t), \tag{3.58}$$

then

$$\forall (t, x) \in (T - (k + 1)\varepsilon^2, T) \times \mathbb{R}^N, \quad |u^\varepsilon(t, x) - u_T(x)| \leq C(T - t). \tag{3.59}$$

□

It remains to prove Lemma 3.13.

Proof of Lemma 3.13. We only prove that for all $(t, x) \in (T - (k+1)\varepsilon^2, T) \times \mathbb{R}^N$, we have

$$u^\varepsilon(t, x) \geq u_T(x) - C(T - t)$$

and the reader can check that the proof of the reverse inequality is similar.

It is enough to consider $t \in (T - (k + 1)\varepsilon^2, T - k\varepsilon^2)$. We recall that the dynamic programming principle can be written as follows

$$u^\varepsilon(t, x) = \mathcal{S}^\varepsilon[u^\varepsilon](t, x) = \mathcal{R}^\varepsilon[\mathcal{R}_\varepsilon[u^\varepsilon]](t, x).$$

Thanks to Lemma 3.10 and (3.58), we know that there exists $(y, \varphi) \in \mathcal{C}^+(x)$ such that

$$\begin{aligned}\mathcal{R}_\varepsilon[u^\varepsilon](t,x) &\geq u_T(x) - C(T-(t+T_C(y,\varphi))) - T_C(y,\varphi)(\kappa_*[x,u_T]|Du_T(x)|+1) \\ &\geq u_T(x) - C(T-t).\end{aligned}$$

We now use that $\mathcal{R}^\varepsilon$ is monotone and commutes with constants (see (3.29) and (3.30)) in order to write

$$u^\varepsilon(t,x) \geq \mathcal{R}^\varepsilon[u_T](x) - C(T-t).$$

We remark next that $(x, u_T) \in \mathcal{C}^+(x)$ and we write

$$u^\varepsilon(t,x) \geq \inf_{z \in \mathcal{P}^+(x,x,u_T)} u_T(z) - C(T-t).$$

We distinguish cases.

If $Du_T(x) \neq 0$ and $\kappa^*[x,u_T] > 0$, then we have the desired inequality; indeed,

$$\begin{aligned}u^\varepsilon(t,x) &\geq \inf_{z : u_T(z) \geq u_T(x)} u_T(z) - C(T-t) \\ &\geq u_T(x) - C(T-t).\end{aligned}$$

If now $Du_T(x) = 0$ or $\kappa^*[x,u_T] \leq 0$, then we also have

$$u^\varepsilon(t,x) \geq u_T(x) - C(T-t).$$

The proof of Lemma 3.13 is now complete. □

References

[1] N. Alibaud and C. Imbert, *Fractional semi-linear parabolic equations with unbounded data*, Trans. Amer. Math. Soc. 361 (2009), pp. 2527–2566. MR2471928 (2010b:35254)

[2] O. Alvarez, P. Hoch, Y. Le Bouar, and R. Monneau, *Dislocation dynamics: short-time existence and uniqueness of the solution*, Arch. Ration. Mech. Anal., 181 (2006), pp. 449–504. MR2231781 (2007d:74019)

[3] G. Barles and C. Imbert, *Second-order elliptic integro-differential equations: Viscosity solutions' theory revisited*, Annales de l'Institut Henri Poincaré, Analyse non linéaire, 25 (2008), pp. 567–585. MR2422079 (2009c:35102)

[4] G. Barles, H. M. Soner, and P. E. Souganidis, *Front propagation and phase field theory*, SIAM J. Control Optim., 31 (1993), pp. 439–469. MR1205984 (94c:35005)

[5] G. Barles and P. E. Souganidis, *Convergence of approximation schemes for fully nonlinear second order equations*, Asymptotic Anal., 4 (1991), pp. 271–283. MR1115933 (92d:35137)

[6] L. Caffarelli, J.-M. Roquejoffre, and O. Savin, *Non local minimal surfaces*, 2009. arXiv 0905.1183.v1.

[7] L. Caffarelli and L. Silvestre, *Regularity theory for fully nonlinear integro-differential equations*, Comm. Pure Appl. Math. 62 (2009), pp. 597–638. MR2494809 (2010d:35376)

[8] L. Caffarelli and P. E. Souganidis, *Convergence of Nonlocal Threshold Dynamics Approximations to Front Propagation*, Arch. Rational Mech. Anal., 180 (2010), no. 2, 301–360. MR2609244

[9] Y. G. Chen, Y. Giga, and S. Goto, *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations*, J. Differential Geom., 33 (1991), pp. 749–786. MR1100211 (93a:35093)

[10] R. Cont and P. Tankov, *Financial modelling with jump processes*, Chapman & Hall/CRC Financial Mathematics Series, (2004), pp. xvi+535. MR2042661 (2004m:91004)

[11] M. G. Crandall, H. Ishii, and P.-L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (N.S.), 27 (1992), pp. 1–67. MR1118699 (92j:35050)

[12] M. G. Crandall and P.-L. Lions, *Condition d'unicité pour les solutions généralisées des équations de Hamilton-Jacobi du premier ordre*, C. R. Acad. Sci. Paris Sér. I Math., 292 (1981), pp. 183–186. MR610314 (82c:49020)

[13] F. Da Lio, N. Forcadel, and R. Monneau, *Convergence of a non-local eikonal equation to anisotropic mean curvature motion. Application to dislocation dynamics*, J. Eur. Math. Soc. (JEMS), 10 (2008), pp. 1061–1104. MR2443929 (2009j:35145)

[14] L. C. Evans and P. E. Souganidis, *Differential games and representation formulas for solutions of Hamilton-Jacobi-Isaacs equations*, Indiana Univ. Math. J., 33 (1984), pp. 773–797. MR756158 (86d:90185)

[15] L. C. Evans and J. Spruck, *Motion of level sets by mean curvature. I*, J. Differential Geom., 33 (1991), pp. 635–681. MR1100206 (92h:35097)

[16] N. Forcadel, C. Imbert, and R. Monneau, *Homogenization of some particle systems with two-body interactions and of the dislocation dynamics*, Discrete Contin. Dyn. Syst., 23 (2009), pp. 785–826. MR2461827 (2010h:35021)

[17] C. Imbert, *Level set approach for fractional mean curvature flows*, Interfaces Free Bound., 11 (2009), pp. 153–176. MR2487027 (2010g:35336)

[18] C. Imbert and P. E. Souganidis, *Phasefield theory for fractional reaction-diffusion equations and applications.* Preprint arXiv:0907.5524v1, 2009.

[19] E. R. Jakobsen and K. H. Karlsen, *A "maximum principle for semicontinuous functions" applicable to integro-partial differential equations*, NoDEA Nonlinear Differential Equations Appl. 13 (2006), no. 2, 137–165 MR2243708 (2007c:45020)

[20] R. V. Kohn and S. Serfaty, *A deterministic-control-based approach to motion by curvature*, Comm. Pure Appl. Math., 59 (2006), pp. 344–407. MR2200259 (2007h:53101)

[21] R. V. Kohn and S. Serfaty, *A deterministic-control-based approach to fully non-linear parabolic and elliptic equations*, Comm. Pure Appl. Math., 63 (2010), pp. 1298-1350

[22] P.-L. Lions, *Generalized solutions of Hamilton-Jacobi equations*, vol. 69 of Research Notes in Mathematics, Pitman (Advanced Publishing Program), Boston, Mass., 1982. MR667669 (84a:49038)

[23] S. Osher and J. A. Sethian, *Fronts propagating with curvature-dependent speed: algorithms based on Hamilton-Jacobi formulations*, J. Comput. Phys., 79 (1988), 1, pp. 12–49. MR0965860 (89h:80012)

[24] A. Sayah, *Équations de Hamilton-Jacobi du premier ordre avec termes intégro-différentiels. I. Unicité des solutions de viscosité*, Comm. Partial Differential Equations, 16 (1991), 6-7, pp. 1057–1074. MR1116853 (93b:35027a) & MR1116854 (93b:35027b)

[25] H. M. Soner, *Optimal control of jump-Markov processes and viscosity solutions*, in Stochastic differential systems, stochastic control theory and applications (Minneapolis, Minn., 1986), vol. 10 of IMA Vol. Math. Appl., Springer, New York, 1988, pp. 501–511. MR0934740 (89h:93055)

[26] P. E. Souganidis, *Front propagation: theory and applications*, in Viscosity solutions and applications (Montecatini Terme, 1995), vol. 1660 of Lecture Notes in Math., Springer, Berlin, 1997, pp. 186–242. MR1462703 (98g:35010)

[27] J. Spencer, *Balancing games*, J. Combinatorial Theory Ser. B, 23 (1977), pp. 68–74. MR0526057 (58#26058)

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