

Critical Points of Ambrosio-Tortorelli converge to critical points of Mumford-Shah in the one-dimensional Dirichlet case

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Abstract

Critical points of a variant of the Ambrosio-Tortorelli functional, for which non-zero Dirichlet boundary conditions replace the fidelity term, are investigated. They are shown to converge to particular critical points of the corresponding variant of the Mumford-Shah functional; those exhibit many symmetries. That Dirichlet variant is the natural functional when addressing a problem of brittle fracture in an elastic material.

1 Introduction

In the late 80's, D. Mumford & J. Shah proposed a new functional for image segmentation in their celebrated paper [13]. If $g \in L^\infty(\Omega; [0, 1])$ represents a continuous interpolation of the collected pixelated data over the image domain $\Omega \subset \mathbb{R}^2$, then the proposed segmentation consists in minimizing

$$(u, K) \mapsto \mathcal{MS}(u, K) := \int_{\Omega \setminus K} |\nabla u|^2 dx + 2\mathcal{H}^1(K) + \lambda \int_{\Omega} (u - g)^2 dx,$$

among all compact subsets $K \subset \Omega$ and all $u \in H^1(\Omega \setminus K)$. In that functional, λ is a positive weight left to the investigator's appreciation, K represents the contours of the image, and u the resulting grey contrast ($0 \leq u(x) \leq 1$).

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Proving existence for minimizers of that functional was not a trivial task and it gave rise to a abundant literature spearheaded by the work of E. De Giorgi and that of L. Ambrosio on the space $SBV(\Omega)$; see e.g [1]. The underlying idea was to view $\mathcal{MS}(u, K)$ as a one field functional

$$(1.1) \quad \mathcal{MS}(u) = \int_{\Omega} |\nabla u|^2 dx + 2\mathcal{H}^1(S(u)) + \lambda \int_{\Omega} (u - g)^2 dx,$$

over $SBV(\Omega)$, the space of functions $u \in L^1(\Omega)$ such that their distributional derivative is a Radon measure Du with finite total variation $|Du|(\Omega)$, jump set $S(u)$ (the complement of the set of Lebesgue points for u), and no Cantor part. The next step was to prove existence of a minimizer u_m of \mathcal{MS} in that space, and then to show that the pair $(u_m, \overline{S(u_m)})$ was actually a minimizer for \mathcal{MS} . That program was successfully completed, culminating in [8].

From a computational standpoint, the search for a minimizer of (1.1) is not easy, because the test fields exhibit discontinuities at unknown locations and the implementation of classical finite element methods becomes a perilous endeavor. A possible remedy consists in resorting to variational convergence, specifically Γ -convergence, so as to approximate \mathcal{MS} by a more regular functional – denoted henceforth by AT_{ε} – whose minimizers are easier to evaluate. For more information on Γ -convergence, we refer the interested reader to e.g. [7] and merely emphasize for now that an important property of Γ -convergence is that (approximate) minimizers of AT_{ε} that converge as $\varepsilon \searrow 0$ will converge to bona fide minimizers of \mathcal{MS} .

There is by now an abundant literature on the approximation of the Mumford-Shah functional and many approximating sequences have been proposed. The most computationally efficient in our opinion is that originally proposed by L. Ambrosio & V. Tortorelli in [3], [4], in the footsteps of the functional proposed by L. Modica and S. Mortola for the approximation of the perimeter [12]. Consider

$$AT_{\varepsilon}(u, v) = \int_{\Omega} \left((\eta_{\varepsilon} + v^2) |\nabla u|^2 + \varepsilon |\nabla v|^2 + \frac{(1 - v)^2}{\varepsilon} \right) dx + \lambda \int_{\Omega} (u - g)^2 dx,$$

with $0 < \eta_{\varepsilon} \ll \varepsilon$. It is proved in [3], [4] that AT_{ε} $\Gamma(\mathcal{B}(\Omega) \times \mathcal{B}(\Omega))$ -converges to \mathcal{MS} , suitably extended to a two-field functional as

$$\mathcal{MS}(u, v) = \begin{cases} \mathcal{MS}(u) & \text{if } v \equiv 1 \\ +\infty & \text{otherwise.} \end{cases}$$

Above, $\mathcal{B}(\Omega)$ stands for the set of all Borel functions on Ω , and the convergence is the convergence in measure. Actually, we can also view the convergence as taking place in $L^2(\Omega) \times L^2(\Omega)$.

The functional AT_{ε} is easily seen, through the direct method of the Calculus of Variations, to admit at least one minimizing pair $(u_{\varepsilon}, v_{\varepsilon}) \in H^1(\Omega) \times H^1(\Omega)$, for any fixed value of ε . The associated sequence is bounded in e.g. $L^{\infty}(\Omega) \times L^{\infty}(\Omega)$, and a subsequence can

be shown to converge in measure (and also strongly in $L^2(\Omega) \times L^2(\Omega)$) to $(u, v \equiv 1)$, which, by the already evoked property of Γ -convergence, will be a minimizer for \mathcal{MS} .

In an apparently disconnected context, recent years have witnessed the birth of a variational theory of brittle fracture evolution. One of its constitutive elements is that, at each time, the total energy of the system, the sum of the elastic and surface energies, is to be minimized among all admissible competitors [10]. That total energy is a close parent of the Mumford-Shah functional \mathcal{MS} for image segmentation. It is given – say in anti-plane shear, for which the displacement field is unidirectional, and for normalized shear modulus and fracture toughness – by

$$(1.2) \quad \mathcal{F}(u, v) = \begin{cases} \int_{\Omega} |\nabla u|^2 dx + 2\mathcal{H}^1(S(u)) & \text{if } u \in SBV(\Omega), \text{ and } v \equiv 1 \\ +\infty & \text{otherwise.} \end{cases}$$

In the context of fracture, the displacement field u is typically constrained by boundary values, say U on $\partial\Omega$, and the crack may go to the boundary of Ω . Thus we should impose that

$$u = U \text{ on } \partial\Omega \setminus S(u),$$

u being considered as an element of $SBV(\mathbb{R}^2)$ such that $u \equiv U$ on $\mathbb{R}^2 \setminus \bar{\Omega}$ with U defined on \mathbb{R}^2 (say $U \in H^1(\mathbb{R}^2)$). The relevant literature speaks of a hard device in this situation. In any case, the above quoted Γ -convergence result still applies in the current setting, so that \mathcal{F} , trivially extended to some $\Omega' \supset \bar{\Omega}$, can be variationally approximated by $\mathcal{E}_{\varepsilon}$, a close variant of AT_{ε} defined as

$$(1.3) \quad \mathcal{E}_{\varepsilon}(u, v) := \int_{\Omega'} \left((\eta_{\varepsilon} + v^2) |\nabla u|^2 + \varepsilon |\nabla v|^2 + \frac{(1-v)^2}{\varepsilon} \right) dx,$$

with $(u, v) \in H^1(\Omega') \times H^1(\Omega')$ and $u \equiv U$ on $\Omega' \setminus \bar{\Omega}$.

Remark 1.1 The extension to a larger domain Ω' permits the introduction of boundary jumps (boundary cracks) without modification of the resulting surface energy. In the remainder of the study, we prefer to restrict the functional to Ω , while imposing that the admissible fields u belong to $SBV(\mathbb{R}^2)$ with $u = U$ on $\mathbb{R}^2 \setminus \bar{\Omega}$. In that case, the correct surface energy for the $\Gamma(\mathcal{B}(\Omega) \times \mathcal{B}(\Omega))$ -limit of $\mathcal{E}_{\varepsilon}$ is

$$2\mathcal{H}^1(S(u) \cap \Omega) + \mathcal{H}^1(S(u) \cap \partial\Omega).$$

Although the functional $\mathcal{E}_{\varepsilon}$ is immediately seen to admit minimizers at fixed ε , those are not so easily determined computationally because $\mathcal{E}_{\varepsilon}$ is not convex in its two arguments, but only separately in each of them. This is a cause of major difficulties, as explained in [6]. The most expedient computational algorithm consists in performing alternate minimization in each variable at fixed ε . According to [6], that algorithm asymptotically converges to a critical point $(u_{\varepsilon}, v_{\varepsilon})$ of $\mathcal{E}_{\varepsilon}$. Thus, algorithmically, we should investigate the limit

behavior of critical pairs $(u_\varepsilon, v_\varepsilon)$ for \mathcal{E}_ε . Note that, at the expense of starting the alternate minimization with the same profile, say $v_\varepsilon = 1$, $u_\varepsilon = U$, we can easily enforce the additional assumption that

$$\mathcal{E}_\varepsilon(u_\varepsilon, v_\varepsilon) \leq \mathcal{C} < \infty,$$

for some ε -independent positive constant \mathcal{C} .

Critical points of \mathcal{E}_ε are not necessarily minimizers of \mathcal{E}_ε , and it is not so clear that they will converge toward even a critical point of \mathcal{F} . We recall, see [2], Chapter 7, that a critical point of \mathcal{F} is a couple (u, v) such that \mathcal{F} remains stationary under admissible inner variations, *i.e.*,

$$\frac{d\mathcal{F}(u \circ (id + t\phi)^{-1}, 1)}{dt} \Big|_{t=0} = 0, \text{ with } \phi \in \mathcal{C}_0^\infty(\Omega; \mathbb{R}^2).$$

If they do, then the Ambrosio-Tortorelli approximation scheme will prove even more fruitful, because fracture evolutions are more likely to be paths along critical (or maybe meta-stable) points for \mathcal{F} than those along global minimizers of \mathcal{F} , and the result would provide a theoretical, as well as a numerical tool for extending the variational theory of brittle fracture to a more realistic setting.

Unfortunately, criticality is not easily reconciled with variational convergence. Successful attempts have been made in other settings such as that of the Allen-Cahn functional in phase transitions, see [11], [16], [17], or that of the Ginzburg-Landau functional in superconductivity, see [5], [15], but, to our knowledge, nothing of the kind has been investigated in the framework of image segmentation via the Mumford-Shah functional.

This study is a first step in that direction. It investigates the one-dimensional case. Of course, the one-dimensional setting is of limited interest from the standpoint of applications to fracture, because one-dimensional fracture is primarily a textbook problem, except maybe when used in trusses. It is of marginal interest within the context of image segmentation, *i.e.*, for AT_ε and MS , although it may prove relevant for the de-blurring of bar codes [18]. Pursuing a similar analysis in a higher dimensional setting is quite a challenge for the time being. Among the many obstacles, the lack of explicit solutions for the Euler-Lagrange system associated with the criticality of the approximating fields $u_\varepsilon, v_\varepsilon$ for the Ambrosio-Tortorelli functional (see (2.4) below) makes the jump profile for v_ε less evident than in the Allen-Cahn, or Ginzburg-Landau settings. But, the knowledge of an explicit optimal profile in those settings is a precious ingredient in the analysis of critical points.

In the next section, the one-dimensional functional is introduced and the three main results are stated: the convergence of critical points of \mathcal{E}_ε to specific critical points of \mathcal{F} (see Theorem 2.2); conversely, any specific critical point of \mathcal{F} as described in Theorem 2.2 is actually a limit of critical points of \mathcal{E}_ε (see Theorem 2.4); finally, the convergence of the various terms in the energy \mathcal{E}_ε to their \mathcal{F} -analogues (see Theorem 2.5). The reader will note that, thanks to Theorem 2.2, the Ambrosio-Tortorelli approximation acts as a selection mechanism for the Mumford-Shah functional, in that only critical points with jumps that are symmetrically located on the interval of study can be obtained through that

approximation and that, thanks to Theorem 2.4, all of those are actually attained as limits of critical points of \mathcal{E}_ε . Also, Theorem 2.5 demonstrates that, generically, the Ambrosio-Tortorelli energy evaluated at one of its critical points converges to the Mumford-Shah energy, evaluated at the limit of that (sequence of) critical point(s). Section 3 establishes some general a priori estimates, and most notably bounds the discrepancy (see (3.3)), a pivotal quantity in the study of critical points because of its link to the energy momentum tensor (see e.g. [5], [14]). Section 4 is devoted to the proof of the first theorem; Section 5 carries out that of the second theorem, while Section 6 details that of the third theorem.

2 Statement of the results

Throughout, \mathcal{C} stands for a generic positive constant (so that e.g. $\mathcal{C} = 2\mathcal{C}$) and L is the length of the interval under consideration.

For $\varepsilon > 0$, we consider the following ε -indexed one-dimensional Ambrosio-Tortorelli type functional (see (1.3)):

$$(2.1) \quad \mathcal{E}_\varepsilon(u, v) = \int_0^L \left((\eta_\varepsilon + v^2)(u')^2 + \varepsilon(v')^2 + \frac{(1-v)^2}{\varepsilon} \right) dx.$$

In (2.1), η_ε is a positive number, and $(u_\varepsilon, v_\varepsilon)$ belongs to the space Y_ε defined by

$$Y_\varepsilon := \{u, v \in H^1(0, L), u(0) = 0, u(L) = a_\varepsilon\}$$

with $a_\varepsilon > 0$. Note that these boundary constraints are not really restrictive in one dimension (up to translation of u).

We assume that, as $\varepsilon \searrow 0$,

$$(2.2) \quad a_\varepsilon \rightarrow a > 0; \quad \eta_\varepsilon/\varepsilon \rightarrow 0, \text{ i.e., } \eta_\varepsilon \ll \varepsilon.$$

We also introduce, for $u \in SBV(\mathbb{R})$, the one dimensional Mumford-Shah functional (see (1.2) and Remark 1.1):

$$(2.3) \quad \mathcal{F}(u, v) = \begin{cases} \int_0^L (u')^2 dx + 2\#(S(u) \cap (0, L)) + \#(S(u) \cap \{0, L\}) & \text{if } v \equiv 1 \\ +\infty & \text{otherwise.} \end{cases}$$

In (2.3), u' denotes the approximate derivative of u , i.e., the density of the absolutely continuous part of the measure Du with respect to the Lebesgue measure, while $S(u)$ denotes the jump set of u , defined as the complement in \mathbb{R} of the set of Lebesgue points of u .

As explained in the introduction, we wish to impose Dirichlet type boundary conditions on the test fields. Thus, the pair (u, v) should lie in Y defined as

$$Y := \{u \in SBV(\mathbb{R}) : u \equiv 0 \text{ on } (-\infty, 0), u \equiv a \text{ on } (L, +\infty)\} \times L^\infty((0, L)),$$

so that, in particular, $S(u) \subset [0, L]$.

The spaces Y_ε and Y are endowed with the $L^2((0, L), \mathbb{R}^2)$ topology. We recall that \mathcal{E}_ε Γ -converges to \mathcal{F} hence minimizers of \mathcal{E}_ε converge to minimizers of \mathcal{F} . Those are very easy to identify: for $a < \sqrt{L}$ the only minimizer is $u \equiv ax/L$, while for $a > \sqrt{L}$ they are $u = a\chi_{(L, \infty)}$, or $u = a\chi_{(0, \infty)}$, where χ denotes the characteristic function of a set (for $a = \sqrt{L}$ all of the above are minimizers). Thus we see that fracture is indeed induced by this model, even for minimizers, by a boundary “tug” when a large enough. Minimization favors boundary cracks because the associated surface energy has a lesser weight (1 versus 2). In the context of Remark 1.1, an energy that would weigh equally $(0, L)$ and $\{0, L\}$ would produce, for $a > \sqrt{L}$, a minimizer of the form $u = a\chi_{(b, \infty)}$ for any $b \in [0, L]$. However, our results below would prove that not all of these minimizers are produced by this limit process.

As announced in the introduction, we propose to study the convergence property of critical points other than the minimizers. The critical points of the one dimensional Mumford-Shah functional are easily identified from (7.42) in [2], Ch. 7, as those pairs (u, v) with $v \equiv 1$, and u piecewise constant with a finite number of jumps, or $u \equiv ax/L$.

Let $(u_\varepsilon, v_\varepsilon)$ be critical points of the Ambrosio-Tortorelli functional (2.1), *i.e.*, pairs of functions $(u_\varepsilon, v_\varepsilon) \in Y_\varepsilon$ that satisfy the Euler-Lagrange equations

$$\begin{aligned}
 (2.4) \quad & -\varepsilon v_\varepsilon'' + v_\varepsilon (u_\varepsilon')^2 + \frac{v_\varepsilon - 1}{\varepsilon} = 0 \\
 & [u_\varepsilon'(\eta_\varepsilon + v_\varepsilon^2)]' = 0 \\
 & u_\varepsilon(0) = 0, \quad u_\varepsilon(L) = a_\varepsilon \\
 & v_\varepsilon'(0) = v_\varepsilon'(L) = 0.
 \end{aligned}$$

Our main goal is to study the limit properties of $(u_\varepsilon, v_\varepsilon)$ as ε goes to 0, provided additionally that

$$(2.5) \quad \mathcal{E}_\varepsilon(u_\varepsilon, v_\varepsilon) \leq \mathcal{C} < \infty.$$

The above bound is natural from a computational standpoint, as already emphasized in the introduction.

Note that the second equation of (2.4) implies that

$$(2.6) \quad u_\varepsilon'(\eta_\varepsilon + v_\varepsilon^2) = c_\varepsilon,$$

for some constant c_ε . It follows that u_ε' has a constant sign. The Dirichlet boundary conditions on u_ε imply that $c_\varepsilon > 0$ and thus that

$$(2.7) \quad u_\varepsilon \nearrow \text{ from } 0 \text{ to } a_\varepsilon.$$

One can substitute the relation (2.6) into the first equation of (2.4), and obtain

$$(2.8) \quad -\varepsilon v_\varepsilon'' + \frac{v_\varepsilon c_\varepsilon^2}{(\eta_\varepsilon + v_\varepsilon^2)^2} + \frac{v_\varepsilon - 1}{\varepsilon} = 0$$

$$v_\varepsilon'(0) = v_\varepsilon'(L) = 0.$$

It is a crucially convenient feature of the one-dimensional case that the system of ODE's can be reduced to this single second-order ODE (up to the unknown parameter c_ε though) with Neumann boundary conditions. We will use the properties, in particular symmetry properties, of solutions to this type of ODE's. However it is not our goal to completely classify the solutions to (2.8) or perform their stability analysis. Rather we focus on the $\varepsilon \rightarrow 0$ asymptotic analysis and we look to employ, as much as possible, arguments that are independent of the dimension and could be recast in dimensions higher than 1.

Remark 2.1 Equation (2.6) would still hold true if Neumann boundary conditions, namely $u_\varepsilon'(0) = u_\varepsilon'(L) = 0$, were imposed on u_ε , in lieu of the adopted Dirichlet boundary conditions. But then, $u_\varepsilon' \equiv 0$, $v_\varepsilon \equiv 1$, and the problem becomes trivial.

It is the presence of the fidelity term $\int_0^L |u - g|^2 dx$ of image segmentation that renders the Neumann problem non-trivial. As mentioned in the introduction, our present focus is the Dirichlet case, where no fidelity term is present.

Our main result is the following. It states on the one hand the symmetry properties of the solutions to (2.4), and on the other hand the more difficult fact that c_ε can only cluster to the two values 0 and a/L .

Theorem 2.2 *At the possible expense of extracting a subsequence of $\varepsilon \searrow 0$, $c_\varepsilon \rightarrow c_0$ where $c_0 \in \{0, a/L\}$ and $(u_\varepsilon, v_\varepsilon)$ converges to a critical point $(u, 1)$ of \mathcal{F} . In other words, $u_\varepsilon(x) \rightarrow u(x) (\in SBV(\mathbb{R}))$ and $v_\varepsilon(x) \rightarrow 1$, for a.e. $x \in (0, L)$.*

If $c_0 = a/L$, the limit critical point is $u \equiv ax/L$.

If $c_0 = 0$, there exists a fixed number n such that, at the possible expense of extracting a subsequence of $\varepsilon \searrow 0$, v_ε – extended by symmetry to $(-L, L)$ – is a juxtaposition of n identical graphs. The repeated subgraph exhibits either a strict minimum point (“well case”), for all ε 's, or a strict maximum point, for all ε 's (“bell case”). The limit critical point u – extended by anti-parity to $(-L, L)$ – is constant on $(-L, -L + L/n)$, with value $-a$ in the former case (see Figure 1), or on $(-L, -L + 2L/n)$, with value $-(n-1)a/n$ in the latter case (see Figure 2), then it jumps by a value of $2a/n$ at the end of each interval of length $2L/n$.

Remark 2.3 The Ambrosio-Tortorelli functional acts as a selector for the critical points of the Mumford-Shah functional, in that it asymptotically equi-distributes the possible jumps of u over the interval $[0, L]$. The graph of u extended by antiparity looks like a piece of a “perfect staircase”: all steps have the same height and the same width.

Our next main result is a converse of the above theorem in the sense that any “perfect

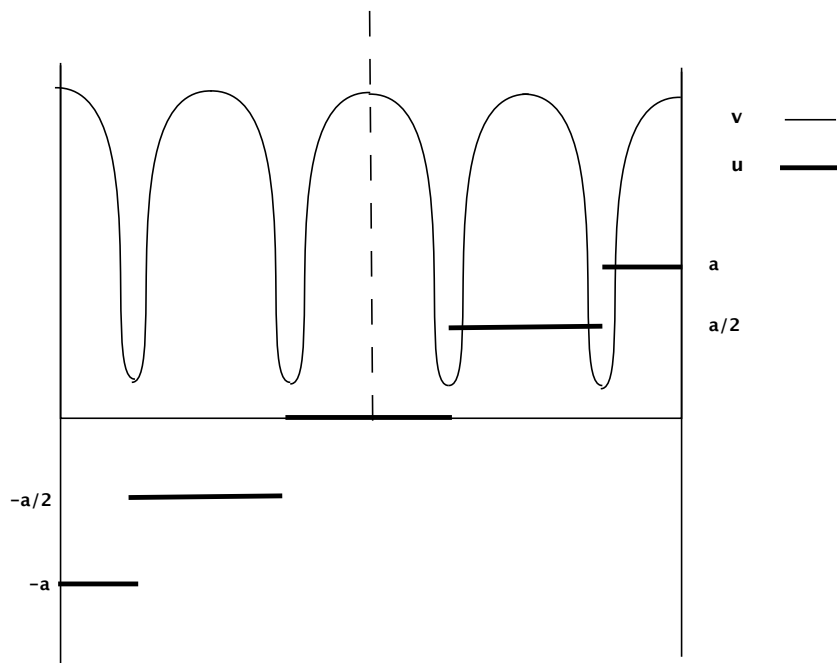


Figure 1: $n=4$, well case

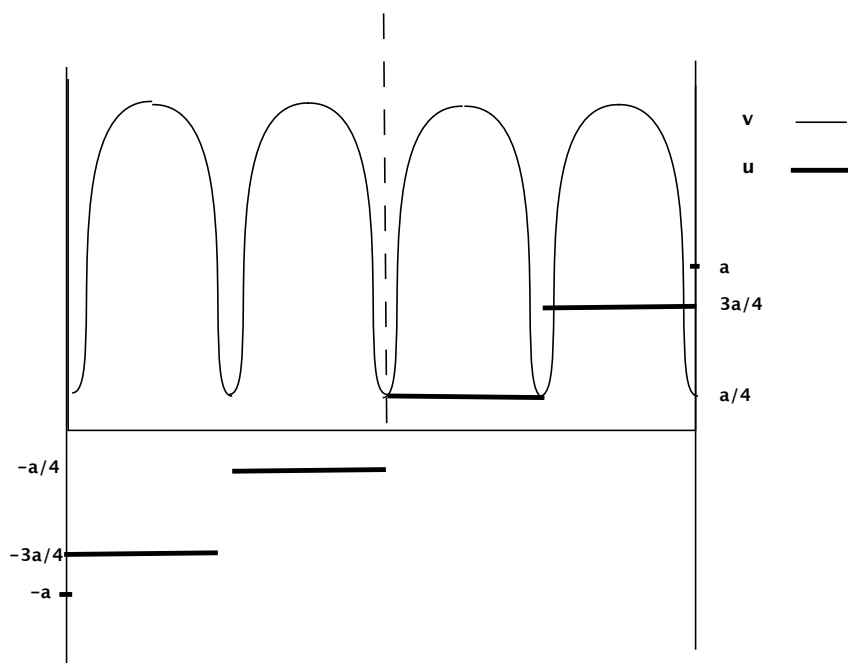


Figure 2: $n=4$, bell case

staircase” critical point of the Mumford-Shah functional \mathcal{F} as described in Theorem 2.2 is actually a limit of critical points of the Ambrosio-Tortorelli functional \mathcal{E}_ε .

Theorem 2.4 *Let u be a “perfect staircase” function on $[0, L]$ with n steps when viewed as a function on $[-L, L]$ as described in Theorem 2.2. Then, for all ε sufficiently small, there exists a critical point $(u_\varepsilon, v_\varepsilon)$ (i.e., with $v'_\varepsilon(0) = v'_\varepsilon(L) = 0$ and $u_\varepsilon(0) = 0$, $u_\varepsilon(L) = a_\varepsilon$), of \mathcal{E}_ε such that v_ε has exactly n isolated local maxima in $[-L, L]$ and*

$$(2.9) \quad \lim_{\varepsilon \rightarrow 0} \|u_\varepsilon - u\|_{L^2((0,L) \text{ or equiv. } (-L,L))} = 0.$$

We are also in a position to evaluate the measure limits of each of the terms entering the energy functional defined in (2.1). This is the object of the following

Theorem 2.5 *If $u_\varepsilon \rightarrow u$, a critical point for the Mumford-Shah functional, as given in Theorem 2.2, then*

- *The limit measure of $(\eta_\varepsilon + v_\varepsilon^2)(u'_\varepsilon)^2 dx$ is $(u')^2 dx$, u' , the approximate gradient of u , being 0 or a/L ;*
- *The limit measure of $\varepsilon(v'_\varepsilon)^2 dx$, which is also that of $(1 - v_\varepsilon)^2/\varepsilon dx$, is a finite sum of Dirac masses which, in the case that u is piecewise constant ($c_0 = 0$), are located at the end of each step of the “perfect staircase” that represents u . Each of those masses has weight 1 when the mass is located inside $(0, L)$ and $1/2$ if it is located at $x = 0, L$, if any.*

Remark 2.6 In the case $c_0 = a/L$, we expect that, for ε small enough, $u_\varepsilon(x) = a_\varepsilon x/L$ and $v_\varepsilon(x) = L^2/(L^2 + a_\varepsilon^2 \varepsilon)$, which would prove that v_ε has no v -jump in the sense of Definition 3.5 below, and thus that the resulting measure limit is *always* $\sum_{x \in S(u) \cap (0,L)} \delta_x + 1/2 \sum_{x \in S(u) \cap \{0,L\}} \delta_x$.

3 Preliminary Estimates

3.1 Classical a priori estimates

In this section, we establish a few canonical estimates that will prove instrumental in the proof of Theorems 2.2, 2.5. These estimates are completely standard but we include their proofs for convenience of the reader. The *a priori* bound (2.5) is essential in all that follows.

First note that, from (2.5) and (2.6),

$$\mathcal{C} \geq \mathcal{E}_\varepsilon(u_\varepsilon, v_\varepsilon) \geq \int_0^L (\eta_\varepsilon + v_\varepsilon^2)(u'_\varepsilon)^2 dx = \int_0^L c_\varepsilon u'_\varepsilon dx = a_\varepsilon c_\varepsilon,$$

so that

$$c_\varepsilon \leq \frac{\mathcal{C}}{a_\varepsilon}.$$

Hence, up to the possible expense of extracting a subsequence,

$$(3.1) \quad c_\varepsilon \rightarrow c_0, \quad \varepsilon \searrow 0.$$

The proof of Theorem 2.2 will hinge on the actual values that c_0 can take. This will be the object of Lemma 4.4 in the next section.

For now, we prove some elementary estimates on the critical points $(u_\varepsilon, v_\varepsilon)$ of (2.1), which, by the way, are smooth by elliptic regularity.

A first result is a maximum principle for v_ε , namely,

Lemma 3.1

$$0 \leq v_\varepsilon \leq 1.$$

Proof. Multiplying both sides of the first equation of (2.4) by $v_\varepsilon^- = \max(0, -v_\varepsilon)$, we get

$$\int_0^L -\varepsilon v_\varepsilon'' v_\varepsilon^- dx + \int_0^L v_\varepsilon (u_\varepsilon')^2 v_\varepsilon^- dx + \int_0^L \frac{v_\varepsilon - 1}{\varepsilon} v_\varepsilon^- dx = 0.$$

Because of the Neumann boundary conditions on v_ε , this yields

$$\int_0^L \varepsilon v_\varepsilon' (v_\varepsilon^-)' dx + \int_0^L v_\varepsilon (u_\varepsilon')^2 v_\varepsilon^- dx + \int_0^L \frac{v_\varepsilon - 1}{\varepsilon} v_\varepsilon^- dx = 0,$$

or still

$$(3.2) \quad - \int_0^L \varepsilon ((v_\varepsilon^-)')^2 dx - \int_0^L (v_\varepsilon^-)^2 (u_\varepsilon')^2 dx - \int_0^L \frac{(v_\varepsilon^- + 1)}{\varepsilon} v_\varepsilon^- dx = 0.$$

Each term on the right hand side of (3.2) is nonpositive. Thus,

$$\int_0^L \frac{(v_\varepsilon^- + 1)}{\varepsilon} v_\varepsilon^- dx = 0,$$

hence $v_\varepsilon^- \equiv 0$.

Multiplication of the first equation of (2.4) by $(v_\varepsilon - 1)^+ = \max(0, v_\varepsilon - 1)$ would yield the other inequality. \square

Next, we establish the convergence properties of the pair $(u_\varepsilon, v_\varepsilon)$.

Lemma 3.2

$$v_\varepsilon \rightarrow 1, \quad \text{strongly in } L^2((0, L)),$$

and, modulo extraction,

$$u_\varepsilon \rightarrow u \in BV((0, L)), \quad \text{strongly in } L^1((0, L)),$$

$$u_\varepsilon' \rightarrow c_0, \quad \text{a.e. in } (0, L).$$

Further, $|Du|((0, L)) \leq a$ and $c_0 \leq a/L$.

Proof. The energy bound (2.5) immediately implies the first convergence. The monotone character (2.7) of u_ε , together with (2.2), implies that u_ε is bounded in $BV((0, L))$. By the compactness of BV in L^1 (see [9]), a subsequence of u_ε converges in $L^1((0, L))$ to $u \in BV((0, L))$.

Because of the weak lower semi-continuous character of the total variation,

$$|Du|((0, L)) \leq \liminf_\varepsilon |Du_\varepsilon|((0, L)) = \liminf_\varepsilon a_\varepsilon = a,$$

hence the bound on $|Du|((0, L))$.

By virtue of (2.6),

$$u'_\varepsilon(x) \rightarrow c_0 \text{ as } \varepsilon \rightarrow 0 \text{ for a.e } x \in (0, L).$$

Fatou's lemma then yields the following refined bound on c_0 :

$$c_0 L = \int_0^L \liminf_{\varepsilon \rightarrow 0} u'_\varepsilon \leq \liminf_{\varepsilon \rightarrow 0} \int_0^L u'_\varepsilon = a.$$

□

It is also standard that, in such a context, a Noether type conservation law holds, as stated in the following

Proposition 3.3

$$\left\{ \frac{1}{2} \left(\frac{(1 - v_\varepsilon)^2}{\varepsilon} - (\eta_\varepsilon + v_\varepsilon^2)(u'_\varepsilon)^2 - \varepsilon(v'_\varepsilon)^2 \right) \right\}' = 0.$$

Proof. The left hand side of the previous expression also reads as

$$\begin{aligned} A_\varepsilon &:= \frac{(v_\varepsilon - 1)v'_\varepsilon}{\varepsilon} - \varepsilon v'_\varepsilon v''_\varepsilon - v_\varepsilon v'_\varepsilon (u'_\varepsilon)^2 - (v_\varepsilon^2 + \eta_\varepsilon) u'_\varepsilon u''_\varepsilon \\ &= v'_\varepsilon (-\varepsilon v''_\varepsilon - v_\varepsilon (u'_\varepsilon)^2 + \frac{v_\varepsilon - 1}{\varepsilon}) - (v_\varepsilon^2 + \eta_\varepsilon) u'_\varepsilon u''_\varepsilon. \end{aligned}$$

The first and second equation of (2.4) then imply that

$$A_\varepsilon = -v'_\varepsilon (2v_\varepsilon (u'_\varepsilon)^2) - (v_\varepsilon^2 + \eta_\varepsilon) u'_\varepsilon u''_\varepsilon = -u'_\varepsilon [u''_\varepsilon (\eta_\varepsilon + v_\varepsilon^2) + 2v_\varepsilon v'_\varepsilon u'_\varepsilon] = -u'_\varepsilon [u''_\varepsilon (\eta_\varepsilon + v_\varepsilon^2)]' = 0.$$

□

An immediate consequence of the proposition above is that

$$(3.3) \quad \frac{(1 - v_\varepsilon)^2}{\varepsilon} - (\eta_\varepsilon + v_\varepsilon^2)(u'_\varepsilon)^2 - \varepsilon(v'_\varepsilon)^2 = d_\varepsilon$$

for some constant d_ε . Furthermore, we can estimate this discrepancy constant d_ε as follows

$$\begin{aligned} |d_\varepsilon| L = \int_0^L |d_\varepsilon| dx &= \int_0^L \left| \frac{(1 - v_\varepsilon)^2}{\varepsilon} - (\eta_\varepsilon + v_\varepsilon^2)(u'_\varepsilon)^2 - \varepsilon(v'_\varepsilon)^2 \right| dx \\ &\leq \int_0^L \left\{ \frac{(1 - v_\varepsilon)^2}{\varepsilon} + (\eta_\varepsilon + v_\varepsilon^2)(u'_\varepsilon)^2 + \varepsilon(v'_\varepsilon)^2 \right\} dx \leq \mathcal{C}. \end{aligned}$$

Thus,

$$(3.4) \quad |d_\varepsilon| \leq \mathcal{C}.$$

This bound is key to the following gradient estimates.

Lemma 3.4 *For ε small enough,*

$$\|u'_\varepsilon\|_\infty \leq \frac{\mathcal{C}}{(\varepsilon\eta_\varepsilon)^{1/2}} \text{ and } \|v'_\varepsilon\|_\infty \leq \frac{\mathcal{C}}{\varepsilon}.$$

Whenever $c_0 > 0$, the following refined estimates hold true:

$$\|u'_\varepsilon\|_\infty \leq \frac{\mathcal{C}}{\varepsilon},$$

$$v_\varepsilon(x) \geq \mathcal{C}\sqrt{\varepsilon}, \quad \forall x \in (0, L).$$

Proof. From (3.3) and (3.4), we find that

$$(3.5) \quad \varepsilon(v'_\varepsilon)^2 = \frac{(1-v_\varepsilon)^2}{\varepsilon} - (\eta_\varepsilon + v_\varepsilon^2)(u'_\varepsilon)^2 - d_\varepsilon \leq \frac{(1-v_\varepsilon)^2}{\varepsilon} + \mathcal{C} \leq \frac{1}{\varepsilon} + \mathcal{C},$$

from which the L^∞ -estimate of v'_ε follows. Moreover,

$$(3.6) \quad c_\varepsilon u'_\varepsilon = (\eta_\varepsilon + v_\varepsilon^2)(u'_\varepsilon)^2 = \frac{(1-v_\varepsilon)^2}{\varepsilon} - \varepsilon(v'_\varepsilon)^2 - d_\varepsilon \leq \frac{(1-v_\varepsilon)^2}{\varepsilon} + \mathcal{C} \leq \frac{1}{\varepsilon} + \mathcal{C} \leq \frac{2}{\varepsilon}$$

from which the refined estimate on the L^∞ -norm of u'_ε follows if $c_0 > 0$. The lower bound for v_ε is in turn immediate from (3.6), (2.6) and (2.2).

Finally, because $u'_\varepsilon = c_\varepsilon/(\eta_\varepsilon + v_\varepsilon^2)$, we deduce from (3.6) that

$$(u'_\varepsilon)^2 \leq \frac{2}{\varepsilon c_\varepsilon} \frac{c_\varepsilon}{\eta_\varepsilon + v_\varepsilon^2} \leq \frac{2}{\varepsilon \eta_\varepsilon}$$

and thus obtain the first estimate on the L^∞ -norm of u'_ε , independently of the value of c_0 . \square

3.2 Definition of v -jump

We start by the following remark: recalling that $u'_\varepsilon = c_\varepsilon/(\eta_\varepsilon + v_\varepsilon^2)$, we may rewrite (3.3) in the form

$$(3.7) \quad \frac{(1-v_\varepsilon)^2}{\varepsilon} - \frac{c_\varepsilon^2}{\eta_\varepsilon + v_\varepsilon^2} - \varepsilon(v'_\varepsilon)^2 = d_\varepsilon.$$

Consequently, if x_ε is a critical point of v_ε , i.e. $v'_\varepsilon(x_\varepsilon) = 0$ and, using the fact that $v_\varepsilon \leq 1$, we have

$$(\eta_\varepsilon + v_\varepsilon^2(x_\varepsilon))(1 - v_\varepsilon(x_\varepsilon))^2 \leq \varepsilon(c_\varepsilon^2 + |d_\varepsilon|).$$

It easily follows that either $v_\varepsilon(x_\varepsilon) > 1 - 2\sqrt{\varepsilon}\sqrt{c_\varepsilon^2 + |d_\varepsilon|}$ or $v_\varepsilon(x_\varepsilon) < 2\sqrt{\varepsilon}\sqrt{c_\varepsilon^2 + |d_\varepsilon|}$. Recalling that c_ε and $|d_\varepsilon|$ are both bounded independently of ε , we can write that there exists a constant \mathcal{C} such that the **extremal** values of v_ε are either $> 1 - \mathcal{C}\sqrt{\varepsilon}$ or $< \mathcal{C}\sqrt{\varepsilon}$. Let us denote by $m_\varepsilon = \min_{[0,L]} v_\varepsilon$ and $M_\varepsilon = \max_{[0,L]} v_\varepsilon$ it follows that two cases are possible: either

$$(3.8) \quad m_\varepsilon > 1 - \mathcal{C}\sqrt{\varepsilon},$$

or

$$(3.9) \quad m_\varepsilon < \mathcal{C}\sqrt{\varepsilon} \quad \text{and} \quad 1 - \mathcal{C}\sqrt{\varepsilon} < M_\varepsilon.$$

Indeed, the case $M_\varepsilon < \mathcal{C}\sqrt{\varepsilon}$ would violate the energy bound (2.5).

This motivates the

Definition 3.5 We call $x_\varepsilon \in [0, L]$ a *v-jump* if x_ε is a critical point of v_ε with $v_\varepsilon(x_\varepsilon) \leq \mathcal{C}\sqrt{\varepsilon}$.

In view of the above discussion, it is equivalent (for ε small enough) to define a *v-jump* as a critical point of v_ε such that $v_\varepsilon(x_\varepsilon) \leq a_\varepsilon$ with $a_\varepsilon \leq \alpha$ for any threshold value $\alpha < 1$. Moreover case (3.8) happens if and only if there is no *v-jump*, and case (3.9) if and only if there is at least a *v-jump*.

Note that the pair

$$(u_\varepsilon, v_\varepsilon) = (a_\varepsilon x/L, L^2/(L^2 + \varepsilon a_\varepsilon^2))$$

is always a solution of (2.4).

We now show that in the case (3.8), or the case of no *v-jump*, it is the only possible solution.

Lemma 3.6 If, for *some* ε sufficiently small, v_ε has no *v-jump* in $(0, L)$ then the only solution to (2.4) is $(u_\varepsilon, v_\varepsilon) = (a_\varepsilon x/L, L^2/(L^2 + \varepsilon a_\varepsilon^2))$.

Proof. Differentiating equation (2.6), we get $u_\varepsilon''(\eta_\varepsilon + v_\varepsilon^2) + 2v_\varepsilon v_\varepsilon' u_\varepsilon' = 0$ and therefore

$$(3.10) \quad u_\varepsilon'' = \frac{-2v_\varepsilon u_\varepsilon' v_\varepsilon'}{\eta_\varepsilon + v_\varepsilon^2}.$$

Differentiating the first equation in (2.4) gives

$$(3.11) \quad -\varepsilon v_\varepsilon''' + v_\varepsilon'(u_\varepsilon')^2 + 2v_\varepsilon u_\varepsilon' u_\varepsilon'' + \frac{v_\varepsilon'}{\varepsilon} = 0.$$

Substituting (3.10) into (3.11), we find that

$$-\varepsilon v_\varepsilon''' + v_\varepsilon' \left[(u_\varepsilon')^2 + \frac{1}{\varepsilon} - 4 \frac{(v_\varepsilon u_\varepsilon')^2}{\eta_\varepsilon + v_\varepsilon^2} \right] = 0.$$

With $w := v'_\varepsilon$, the above equation becomes

$$(3.12) \quad \begin{cases} -\varepsilon w'' + w e_\varepsilon = 0 \\ w(0) = 0, w(L) = 0, \end{cases}$$

where

$$e_\varepsilon = (u'_\varepsilon)^2 + \frac{1}{\varepsilon} - 4 \frac{(v_\varepsilon u'_\varepsilon)^2}{\eta_\varepsilon + v_\varepsilon^2}$$

and substituting (2.6)

$$e_\varepsilon = \frac{1}{\varepsilon} + \frac{c_\varepsilon^2}{(\eta_\varepsilon + v_\varepsilon^2)^2} - 4 \frac{c_\varepsilon^2 v_\varepsilon^2}{(\eta_\varepsilon + v_\varepsilon^2)^3} = \frac{(\eta_\varepsilon + v_\varepsilon^2)^3 + \varepsilon c_\varepsilon^2 (\eta_\varepsilon - 3v_\varepsilon^2)}{\varepsilon (\eta_\varepsilon + v_\varepsilon^2)^3}$$

When there is no v -jump then, as remarked above, $v_\varepsilon \geq \frac{1}{2}$ and one can easily show that then $e_\varepsilon > 0$. Multiplying (3.12) by w and after suitable integration by parts, it follows that (3.12) has a unique solution $w \equiv 0$. Thus v_ε is a constant. Hence the result. \square

An obvious corollary of Lemma 3.6 is

Remark 3.7 Whenever $c_0 = 0$, then there exists a v -jump for a subsequence of $\varepsilon \searrow 0$.

4 Proof of Theorem 2.2

4.1 Symmetry properties

We start by stating some relatively easy symmetry properties of the solutions to (2.4). These follow from the equation (2.8) which we recall here

$$(4.1) \quad \begin{cases} -\varepsilon v_\varepsilon'' + \frac{v_\varepsilon c_\varepsilon^2}{(\eta_\varepsilon + v_\varepsilon^2)^2} + \frac{v_\varepsilon - 1}{\varepsilon} = 0 \\ v'_\varepsilon(0) = v'_\varepsilon(L) = 0. \end{cases}$$

Observe that that equation is of the form

$$(4.2) \quad \begin{cases} v_\varepsilon'' = f_\varepsilon(v_\varepsilon) \\ v'_\varepsilon(0) = v'_\varepsilon(L) = 0, \end{cases}$$

with f_ε of class C^2 . Symmetry properties **follow**.

Lemma 4.1 *The graph of v_ε is symmetric with respect to all the vertical lines passing through its critical points, which in turn are all (including boundary points) absolute maxima or minima of v_ε .*

Proof. If x_ε^c is a critical point of v_ε , then we can symmetrize the graph of v_ε through the vertical line $x = x_\varepsilon^c$. The uniqueness provided by the Cauchy-Lipschitz theorem for (4.2) imply the symmetry of the graph of v_ε around this line. In particular the graph of v_ε can be symmetrized with respect to $x = 0$ and v_ε can thus be extended into an even function on $[-L, L]$. Given x_ε^c a critical point of v_ε , the above mentioned symmetry implies the maximality or minimality of x_ε^c on $(-L, 2x_\varepsilon^c + L)$, and ultimately, by reiteration, on $(-L, L)$. \square

In the sequel we will often consider this extension of v_ε to $[-L, L]$, still denoting it v_ε .

With the same symmetry argument, we obtain the following **more precise description** of the graph of v_ε .

Proposition 4.2 *Given ε , there exists an integer n_ε such that the graph of v_ε in $(-L, L)$ is made of n_ε identical symmetric subgraphs. Moreover, if there is a v -jump, then each subgraph is a symmetric well with a unique interior critical point which is a v -jump, or a symmetric bell with a v -jump at each end (see Figure 3).*

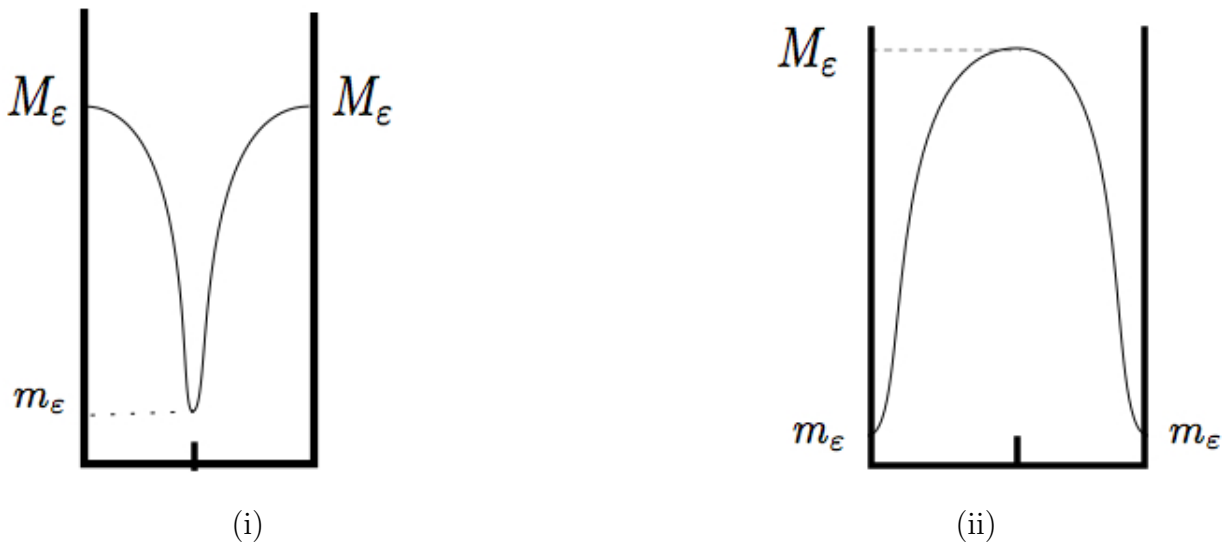


Figure 3: (i) the well, (ii) the bell.

It remains to show that n_ε may be chosen independently of ε . To this effect, we calculate the cost of each v -jump for ε sufficiently small. When there is a v -jump we are in case (3.9). Thus we can find points $\alpha < \beta$ on each subgraph satisfying say $v_\varepsilon(\alpha) \leq 1/10$ and $v_\varepsilon(\beta) \geq 9/10$. Consequently, each v -jump costs at least

$$\begin{aligned}
 \int_\alpha^\beta \left(\varepsilon (v'_\varepsilon)^2 + \frac{(1 - v_\varepsilon)^2}{\varepsilon} \right) dx &\geq \int_\alpha^\beta \left| 2v'_\varepsilon(1 - v_\varepsilon) \right| dx \\
 (4.3) \qquad \qquad \qquad &\geq \left| \int_\alpha^\beta (2v_\varepsilon - v_\varepsilon^2)' dx \right| = (2v_\varepsilon - v_\varepsilon^2)(\beta) - (2v_\varepsilon - v_\varepsilon^2)(\alpha) \geq 3/4.
 \end{aligned}$$

Thus, because of the energy bound (2.5), n_ε must be bounded.

So, up to possible subsequence extraction, we can consider that n_ε is a constant n for all ε sufficiently small.

Remark 4.3 The uniqueness of the “well” and “bell” profiles is not clear to us at this time. If it is the case, then solutions to (2.4) would be completely determined by their number of jumps (+ boundary values).

4.2 Characterisation of the possible limiting slopes

We prove that c_ε can only converge to two possible values.

Lemma 4.4 $c_0 \in \{0, a/L\}$.

Proof. Note that, by Lemma 3.2, $c_0 \leq a/L$. Assume by contradiction that $0 < c_0 < a/L$.

We first explain the idea of the proof. By Lemma 3.6, the difficulty in the proof can only come from the smallness of v_ε . In other words, there must be a v -jump. Then, let $x_\varepsilon \in [0, L]$ be the minimal point of v_ε on $[0, L]$; according to (3.9), $v_\varepsilon(x_\varepsilon) \leq \mathcal{C}\varepsilon^{1/2}$. Combining this with the lower bound on v in Lemma 3.4, we obtain that

$$\min_{[0, L]} v_\varepsilon \sim \sqrt{\varepsilon}.$$

Our proof is then based on the estimate on the size of the set $\{v_\varepsilon \leq M\sqrt{\varepsilon}\}$, for M large enough.

Recalling that $u'_\varepsilon = c_\varepsilon/(\eta_\varepsilon + v_\varepsilon^2)$, we rewrite the first equation of (2.4) as

$$-\varepsilon v_\varepsilon'' + \frac{v_\varepsilon c_\varepsilon^2}{(\eta_\varepsilon + v_\varepsilon^2)^2} + \frac{v_\varepsilon - 1}{\varepsilon} = 0.$$

Integrate this equation over $\{v_\varepsilon \leq M\sqrt{\varepsilon}\}$ to obtain

$$(4.4) \quad \int_{\{v_\varepsilon \leq M\sqrt{\varepsilon}\}} \varepsilon v_\varepsilon'' dx = \int_{\{v_\varepsilon \leq M\sqrt{\varepsilon}\}} \frac{v_\varepsilon c_\varepsilon^2}{(\eta_\varepsilon + v_\varepsilon^2)^2} dx + \int_{\{v_\varepsilon \leq M\sqrt{\varepsilon}\}} \frac{v_\varepsilon - 1}{\varepsilon} dx.$$

We now recall that

(4.5)

The number of connected components of $D_\varepsilon := \{v_\varepsilon \leq M\sqrt{\varepsilon}\}$ is bounded by a constant \mathcal{C} .

Indeed the study of the previous subsection implies that the number of connected components of D_ε is precisely the number of periods of v_ε , hence that it is bounded by $n_\varepsilon + 1 \leq \mathcal{C}$.

On each connected component (a_i, b_i) of D_ε , we obtain, by virtue of the gradient bound of Lemma 3.4,

$$\left| \int_{a_i}^{b_i} \varepsilon v_\varepsilon'' \right| = \left| \varepsilon v_\varepsilon'(b_i) - \varepsilon v_\varepsilon'(a_i) \right| \leq \mathcal{C}.$$

Then, with (4.5), the left hand side of (4.4) is bounded from above by \mathcal{C} . Because $c_0 > 0$, Lemmata 3.1, 3.4 imply that $1 \geq v_\varepsilon(x) \geq \mathcal{C}\sqrt{\varepsilon}$, $\forall x \in [0, L]$. It follows that, for ε sufficiently small, $v_\varepsilon^2(x) \gg \eta_\varepsilon$, $\forall x \in [0, L]$. Thus the right hand-side is bounded from below by

$$\int_{\{v_\varepsilon \leq M\sqrt{\varepsilon}\}} \frac{\mathcal{C}v_\varepsilon}{v_\varepsilon^4} dx - \frac{|\{v_\varepsilon \leq M\sqrt{\varepsilon}\}|}{\varepsilon} \geq \frac{\mathcal{C}|\{v_\varepsilon \leq M\sqrt{\varepsilon}\}|}{M^3\varepsilon^{3/2}} - \frac{|\{v_\varepsilon \leq M\sqrt{\varepsilon}\}|}{\varepsilon} \geq \frac{\mathcal{C}|\{v_\varepsilon \leq M\sqrt{\varepsilon}\}|}{M^3\varepsilon^{3/2}}.$$

Therefore, for ε sufficiently small,

$$\mathcal{C} \geq \frac{\mathcal{C}|\{v_\varepsilon \leq M\sqrt{\varepsilon}\}|}{M^3\varepsilon^{3/2}},$$

implying in turn that

$$(4.6) \quad |\{v_\varepsilon \leq M\sqrt{\varepsilon}\}| \leq \mathcal{C}M^3\varepsilon^{3/2}.$$

Using this inequality and the refined gradient bound for u_ε in Lemma 3.4 yields

$$(4.7) \quad \begin{aligned} a_\varepsilon &= \int_0^L u'_\varepsilon dx = \int_{\{v_\varepsilon \leq M\sqrt{\varepsilon}\}} u'_\varepsilon dx + \int_{\{M\sqrt{\varepsilon} \leq v_\varepsilon \leq \frac{1}{2}\}} u'_\varepsilon dx + \int_{\{v_\varepsilon \geq \frac{1}{2}\}} u'_\varepsilon dx \\ &\leq \mathcal{C}M^3\varepsilon^{1/2} + \int_{\{M\sqrt{\varepsilon} \leq v_\varepsilon \leq \frac{1}{2}\}} u'_\varepsilon dx + \int_0^L u'_\varepsilon \chi_{\{v_\varepsilon \geq \frac{1}{2}\}} dx =: \mathcal{C}M^3\varepsilon^{1/2} + J_\varepsilon + K_\varepsilon. \end{aligned}$$

Next, we bound J_ε and K_ε from above.

Because $u'_\varepsilon(x) \rightarrow c_0$ a.e $x \in (0, L)$ and $\chi_{\{v_\varepsilon \geq \frac{1}{2}\}}(x) \rightarrow \chi_{(0,L)}(x)$ a.e $x \in (0, L)$, it follows that $w_\varepsilon(x) := u'_\varepsilon(x)\chi_{\{v_\varepsilon \geq \frac{1}{2}\}}(x) \rightarrow c_0\chi_{(0,L)}(x)$ a.e. $x \in (0, L)$. On the other hand, for all $x \in (0, L)$,

$$|w_\varepsilon(x)| = \frac{c_\varepsilon}{\eta_\varepsilon + v_\varepsilon^2(x)} \chi_{\{v_\varepsilon \geq \frac{1}{2}\}}(x) \leq 4c_\varepsilon \leq \mathcal{C}.$$

Hence, by Lebesgue's dominated convergence theorem,

$$(4.8) \quad K_\varepsilon = \int_0^L w_\varepsilon dx \rightarrow \int_0^L c_0 \chi_{(0,L)} dx = c_0 L.$$

From the energy bound (2.5), it follows that

$$\mathcal{C} \geq \int_0^L \frac{(1-v_\varepsilon)^2}{\varepsilon} dx \geq \int_{\{v_\varepsilon \leq 1/2\}} \frac{(1-v_\varepsilon)^2}{\varepsilon} dx \geq \int_{\{v_\varepsilon \leq 1/2\}} \frac{1}{4\varepsilon} dx = \frac{1}{4\varepsilon} |\{v_\varepsilon \leq 1/2\}|,$$

yielding the estimate

$$(4.9) \quad |\{M\sqrt{\varepsilon} \leq v_\varepsilon \leq 1/2\}| \leq |\{v_\varepsilon \leq 1/2\}| \leq \mathcal{C}\varepsilon.$$

On $\{M\sqrt{\varepsilon} \leq v_\varepsilon \leq 1/2\}$, we recover, for ε small enough, the refined estimate on u'_ε from Lemma 3.4, that is

$$(4.10) \quad u'_\varepsilon = \frac{c_\varepsilon}{\eta_\varepsilon + v_\varepsilon^2(x)} \leq \frac{c_\varepsilon}{v_\varepsilon^2} \leq \frac{c_\varepsilon}{M^2\varepsilon} \leq \frac{2c_0}{M^2\varepsilon}.$$

Inserting inequalities (4.9), (4.10) into the expression for J_ε produces the following uniform upper bound:

$$(4.11) \quad J_\varepsilon = \int_{\{M\sqrt{\varepsilon} \leq v_\varepsilon \leq \frac{1}{2}\}} u'_\varepsilon \leq \frac{2c_0}{M^2\varepsilon} \mathcal{C}\varepsilon = \frac{\mathcal{C}}{M^2}.$$

Coalescing (4.7), (4.8) and (4.11) and letting ε tend to 0 finally leads to

$$a \leq \frac{\mathcal{C}}{M^2} + c_0L.$$

We let M tend to ∞ and obtain a contradiction since $a > c_0L$. Thus $c_0 \in \{0, a/L\}$ as desired. \square

Remark 4.5 In this proof we have tried again to **make minimal use of one-dimensional arguments**. The only place where the symmetry of the solution is used for simplicity is in the proof of (4.5). But this can easily be avoided: we can show instead directly from (2.8) that the number of connected components of D_ε is bounded by $\mathcal{C}\varepsilon^{-1/3}$ and this suffices in the proof.

4.3 Form of u

It remains for us to establish the form of the limit critical point. Note that u_ε is also extended by reflection about 0 so that

$$(4.12) \quad u_\varepsilon(-L) = -a_\varepsilon.$$

Proposition 4.2 immediately implies the following quantization property for the function u_ε :

$$(4.13) \quad u_\varepsilon\left(-L + \frac{2kL}{n}\right) = -a_\varepsilon + \frac{2ka_\varepsilon}{n} \text{ for } 0 \leq k \leq n.$$

Indeed, recalling (2.6), (4.12),

$$2a_\varepsilon = \int_{-L}^L \frac{c_\varepsilon}{\eta_\varepsilon + v_\varepsilon^2} dx = n \int_{-L}^{-L+2L/n} \frac{c_\varepsilon}{\eta_\varepsilon + v_\varepsilon^2} dx = n \left(u_\varepsilon\left(-L + \frac{2L}{n}\right) + a_\varepsilon \right).$$

Now, denote by \bar{n} the number of times v_ε reaches its minimal value ($\bar{n} = n$ in the well case and $\bar{n} = n + 1$ in the bell case). Because of the quantization property (4.13), it suffices to consider the case $\bar{n} = 1$ in the well case, and $\bar{n} = 2$ in the bell case.

Assume first – well case – that $v_\varepsilon(-L) = v_\varepsilon(L) = M_\varepsilon$ and that v_ε reaches its minimum (a v -jump) at $x = 0$ and that 0 is the only critical point for v_ε on $(-L, L)$.

Fix $\delta < L$. Then, for ε sufficiently small, v_ε converges to 1 uniformly on $K := [\delta, L]$. Indeed, the closed set $A_\varepsilon := \{x \in (-L, L) \mid v_\varepsilon(x) \leq 1 - \varepsilon^{1/4}\}$ is centered around 0, as immediately seen from the assumption that there is only one critical point at 0. Further,

$$\mathcal{C} \geq \int_{-L}^L \frac{(1 - v_\varepsilon)^2}{\varepsilon} \geq \int_{A_\varepsilon} \frac{(\varepsilon^{1/4})^2}{\varepsilon} = \frac{|A_\varepsilon|}{\varepsilon^{1/2}},$$

hence $\text{diam}(A_\varepsilon) \leq \mathcal{C}\varepsilon^{1/2} < \delta$ for $\varepsilon \leq \varepsilon_0$.

Consequently, $u'_\varepsilon = c_\varepsilon/(\eta_\varepsilon + v_\varepsilon^2)$ converges uniformly to $c_0(= 0)$ on K . Thus for any $x \in K$

$$(4.14) \quad u_\varepsilon(x) = u_\varepsilon(L) - \int_x^L u'_\varepsilon(t)dt = a_\varepsilon - \int_x^L u'_\varepsilon(t)dt \longrightarrow a - c_0(L - x).$$

Using the arbitrariness of δ , we conclude, since $c_0 = 0$, that $u = a$ on $(0, L]$. Similarly, we would find that $u = -a$ on $(-L, 0)$.

Assume now – bell case – that $v_\varepsilon(-L) = v_\varepsilon(L) = m_\varepsilon$ and that v_ε reaches its maximum at $x = 0$ and that 0 is the only critical point for v_ε on $(-L, L)$. An argument identical to that above would demonstrate that u'_ε converges uniformly to 0 on $K := [-L + \delta, L - \delta]$, for δ small. But we know that $u_\varepsilon(0) = 0$, so that, for any $x \in (-L, L)$,

$$(4.15) \quad u_\varepsilon(x) = u_\varepsilon(0) + \int_0^x u'_\varepsilon(t)dt = 0 + \int_0^x u'_\varepsilon(t)dt \longrightarrow c_0x.$$

Using the arbitrariness of δ , we conclude, since $c_0 = 0$, that $u = 0$ on $(-L, L)$.

Remark 4.6 Note that all results of [this subsection](#) hold true in the case $c_0 \neq 0$, in particular (4.14), (4.15), provided that v_ε admits a v -jump.

Finally we examine the case $c_0 = a/L$. If there is no v -jump then in view of Lemma 3.6, the result is that announced in Theorem 2.2. If there is a v -jump, then according to Remark 4.6, all results of the previous case hold true and, in particular, (4.14), (4.15). Hence the result upon replacing c_0 by a/L .

5 Proof of Theorem 2.4

In this section, we prove Theorem 2.4. To this end, we will use the Γ -convergence of the Ambrosio-Tortorelli functional \mathcal{E}_ε to the Mumford-Shah functional \mathcal{F} together with the equi-distribution of the v -jumps of any critical point $(u_\varepsilon, v_\varepsilon)$ of \mathcal{E}_ε .

Suppose now that u is a piecewise constant function with n steps in $[0, L]$ as described in Theorem 2.2. Because our method is general enough to handle all possible configurations of u , we can assume from now on that, with $N := n - 1$, $u \equiv 0$ on $(0, L/2N)$, jumps by a/N at $L/2N$, and then jumps by a value of a/N at the end of each interval of length L/N . We will construct critical points of \mathcal{E}_ε – points that satisfy the Euler-Lagrange equations (2.4) – that converge to $(u, 1)$.

Before going into the details of the proof, we briefly explain the ideas. Suppose we have found $(u_\varepsilon, v_\varepsilon)$. Then, from Theorem 2.2, we know that v_ε has a periodic structure with an equi-distribution of v -jumps. Furthermore, the function u_ε enjoys a quantization property: its values at v -jumps are explicit, depending only on a_ε, L and the number of jumps. By our assumption on the structure of u , the graph of v_ε is expected to consist

of N symmetric wells on $[0, L]$. Then, we have $u_\varepsilon(0) = 0$ and $u_\varepsilon(kL/2N) = ka_\varepsilon/2N$ for $0 \leq k \leq 2N$. Furthermore, the graph of v_ε on $[0, L/2N]$ is a half-well with a sink at $x = L/2N$. This sink is clearly a v -jump. If we are able to construct a critical point $(u_\varepsilon, v_\varepsilon)$ of \mathcal{E}_ε on $[0, L/2N]$ having one v -jump with $u_\varepsilon(0) = 0$ and $u_\varepsilon(L/2N) = a_\varepsilon/2N$, then we can glue appropriately identical pieces of this critical point to produce a critical point of \mathcal{E}_ε on $[0, L]$, still denoted by $(u_\varepsilon, v_\varepsilon)$, such that $u_\varepsilon(0) = 0$, $u_\varepsilon(L) = a_\varepsilon$ and also that v_ε has N v -jumps and $n = N + 1$ local maxima. The gluing process is always possible because the first equation of (2.4) is satisfied on each interval of length $L/2N$ and also, since $v'_\varepsilon = 0$, at the gluing points. We can thus assume without loss of generality that $N = 1$ and that the v -jump is at $x = L$. In this case, for ε small, the graph of the function v_ε , if it exists, is a half-well and $u = 0$ in $[0, L)$ with $u(L) = a$. We now investigate the details.

The sought functions v_ε clearly belong to

$$\mathcal{U} := \{v \in H^1(0, L) : \mu \leq v(0) \leq 2 - \mu; v(L) \leq \alpha\}.$$

In the above definition, $0 < \alpha < \mu < 1$, both are independent of ε . Further conditions on μ and α will be added later on whenever necessary.

The heuristic argument above suggests that we seek local minimizers $(u_\varepsilon, v_\varepsilon)$ in the following set

$$B_\varepsilon := \{(u, v) \in (H^1(0, L))^2 : u(0) = 0, u(L) = a_\varepsilon, v \in \mathcal{U}\}.$$

We use the following notation:

$$\mathcal{F}_\varepsilon(v, r, s) := \int_r^s \left(\varepsilon(v')^2 + \frac{(1-v)^2}{\varepsilon} \right) dx$$

for $0 \leq r \leq s \leq L$ and note that, for $f(x) = x - x^2/2$, then, for all $v \in H^1(0, L)$ and for all $0 \leq x_1 \leq x_2 \leq x_3 \leq L$,

$$(5.1) \quad \mathcal{F}_\varepsilon(v, x_1, x_3) \geq 2|f(v(x_1)) + f(v(x_3)) - 2f(v(x_2))|.$$

Indeed, assume with no loss of generality that $v \in C^1(0, L)$. Then, by Cauchy's inequality,

$$\begin{aligned} \mathcal{F}_\varepsilon(v, x_1, x_3) &= \int_{x_1}^{x_3} \left(\varepsilon(v')^2 + \frac{(1-v)^2}{\varepsilon} \right) dx \geq \int_{x_1}^{x_3} 2|v'(1-v)| dx \\ &= 2 \int_{x_1}^{x_2} \left| \left(v - \frac{v^2}{2} \right)' \right| dx + 2 \int_{x_2}^{x_3} \left| \left(v - \frac{v^2}{2} \right)' \right| dx \\ &\geq 2 \int_{x_1}^{x_2} \left(v - \frac{v^2}{2} \right)' dx + 2 \int_{x_2}^{x_3} - \left(v - \frac{v^2}{2} \right)' dx \\ &= 2(2f(v(x_2)) - f(v(x_1)) - f(v(x_3))). \end{aligned}$$

Arguing similarly, we also obtain

$$\mathcal{F}_\varepsilon(v, x_1, x_3) \geq 2(f(v(x_1)) + f(v(x_3)) - 2f(v(x_2)))$$

and thus (5.1) follows.

In a first step, we establish a universal lower bound for \mathcal{E}_ε over B_ε as well as an upper bound for the infimum of \mathcal{E}_ε over B_ε . In fact, for all $(u, v) \in B_\varepsilon$, we have

$$(5.2) \quad \mathcal{E}_\varepsilon(u, v) \geq \mathcal{F}_\varepsilon(v, 0, L) \geq 2(f(\mu) - f(\alpha)),$$

while there exists $(\bar{u}_\varepsilon, \bar{v}_\varepsilon) \in B_\varepsilon$ such that, for ε small,

$$(5.3) \quad \mathcal{E}_\varepsilon(\bar{u}_\varepsilon, \bar{v}_\varepsilon) \leq 1 + o(1).$$

To see (5.2), let $(u, v) \in B_\varepsilon$. Then, using (5.1) with $x_1 = x_2 = 0$ and $x_3 = L$, we get

$$\mathcal{F}_\varepsilon(v, 0, L) \geq 2(f(v(0)) - f(v(L))) \geq 2\left(\min_{x \in [\mu, 2-\mu]} f(x) - \max_{x \leq \alpha} f(x)\right) = 2(f(\mu) - f(\alpha)).$$

Assertion (5.3) is derived upon constructing a sequence $(\bar{u}_\varepsilon, \bar{v}_\varepsilon)$ which is very similar to that used in the proof of the Γ -limsup of the Ambrosio-Tortorelli functional in [4]. We omit the details.

The direct method of the calculus of variations immediately implies that the minimum of \mathcal{E}_ε over B_ε is achieved at, say $(u_\varepsilon, v_\varepsilon)$. By (5.3),

$$(5.4) \quad \mathcal{E}_\varepsilon(u_\varepsilon, v_\varepsilon) = \min_{(u, v) \in B_\varepsilon} \mathcal{E}_\varepsilon(u, v) \leq 1 + o(1).$$

For a suitable choice of (μ, α) , a minimizer $(u_\varepsilon, v_\varepsilon)$ of \mathcal{E}_ε over B_ε is actually a critical point of \mathcal{E}_ε for ε sufficiently small. This is the object of the following

Lemma 5.1 *There exists (μ, α) , independent of ε , such that if $(u_\varepsilon, v_\varepsilon)$ is a minimizer of \mathcal{E}_ε over B_ε then it is a critical point of \mathcal{E}_ε if ε is small enough.*

Proof. To prove the criticality of $(u_\varepsilon, v_\varepsilon)$, it suffices to prove the following inequalities with a suitable choice of (μ, α) :

- (i) $\mu < v_\varepsilon(0) < 2 - \mu$, and
- (ii) $v_\varepsilon(L) < \alpha$.

Indeed, then $(u_\varepsilon, v_\varepsilon)$ is not on the boundary of B_ε but rather in its interior, hence it is a local minimizer of \mathcal{E}_ε and thus a critical point.

Our proof is by contradiction. We assume that there exists a sequence $\varepsilon_j \rightarrow 0$ such that either (i), or (ii) is not satisfied. Then, for each j , we have at least an equality in either (i), or (ii). However, through relabelling, we can assume that either (i), or (ii) is never satisfied.

Suppose that, for all ε , either $v_\varepsilon(0) = \mu$, or $v_\varepsilon(0) = 2 - \mu$. It is enough to consider the first case because $f(\mu) = f(2 - \mu)$ and only the value of $f(v(0))$ enters the proof.

We introduce the following special points x_ε . From (5.4), we know that

$$\int_{L/4}^{3L/4} \frac{(1 - v_\varepsilon)^2}{\varepsilon} dx \leq \mathcal{E}_\varepsilon(u_\varepsilon, v_\varepsilon) \leq 1 + o(1).$$

Thus, we can find points $x_\varepsilon \in (L/4, 3L/4)$ such that $v_\varepsilon(x_\varepsilon) \rightarrow 1$ as $\varepsilon \rightarrow 0$. Then, using (5.1),

$$\begin{aligned} \mathcal{F}_\varepsilon(v_\varepsilon, 0, L) &\geq 2(2f(v_\varepsilon(x_\varepsilon)) - f(v_\varepsilon(0)) - f(v_\varepsilon(L))) \geq 2(2f(v_\varepsilon(x_\varepsilon)) - f(\mu) - f(\alpha)) \\ &\geq 2(1 - f(\mu) - f(\alpha)) - o(1). \end{aligned}$$

Therefore

$$(5.5) \quad 1 + o(1) \geq \mathcal{E}_\varepsilon(u_\varepsilon, v_\varepsilon) \geq \mathcal{F}_\varepsilon(v_\varepsilon, 0, L) \geq 2(1 - f(\alpha) - f(\mu)) - o(1).$$

Thus, if we require additionally that

$$(5.6) \quad f(\alpha) + f(\mu) < 1/2,$$

we reach a contradiction in (5.5).

Were (ii) not true, then $v_\varepsilon(L) = \alpha$. Set

$$(5.7) \quad \alpha_\varepsilon^* = \min_{x \in [0, L]} v_\varepsilon(x).$$

Then, for all ε , one has $\alpha_\varepsilon^* \leq \alpha$. We first improve the lower bound for $\mathcal{F}_\varepsilon(v_\varepsilon, 0, L)$ as follows

$$(5.8) \quad \mathcal{F}_\varepsilon(v_\varepsilon, 0, L) \geq 2[1/2 + \min\{f(\alpha) - 2f(\alpha_\varepsilon^*), 1/2 - 2f(\alpha)\}] - o(1).$$

Indeed, if $y_\varepsilon \in (0, L]$ is such that $v_\varepsilon(y_\varepsilon) = \alpha_\varepsilon^*$, then either $x_\varepsilon > y_\varepsilon$, or $x_\varepsilon < y_\varepsilon$. If $x_\varepsilon \in (y_\varepsilon, L]$ then, applying (5.1) with $x_1 = y_\varepsilon$, $x_2 = x_\varepsilon$, $x_3 = L$, and recalling that $\alpha^* \leq \alpha < 1$, yields

$$\begin{aligned} \mathcal{F}_\varepsilon(v_\varepsilon, 0, L) &\geq 2(2f(v_\varepsilon(x_\varepsilon)) - f(v_\varepsilon(y_\varepsilon)) - f(v_\varepsilon(L))) \geq 2(1 - f(\alpha_\varepsilon^*) - f(\alpha)) - o(1) \\ &\geq 2(1 - 2f(\alpha)) - o(1). \end{aligned}$$

If $x_\varepsilon < y_\varepsilon \leq L$, then, applying (5.1) with $x_1 = x_\varepsilon$, $x_2 = y_\varepsilon$, $x_3 = L$ yields

$$\begin{aligned} \mathcal{F}_\varepsilon(v_\varepsilon, 0, L) &\geq 2(f(v_\varepsilon(x_\varepsilon)) + f(v_\varepsilon(L)) - 2f(v_\varepsilon(y_\varepsilon))) \\ &\geq 2(1/2 + f(\alpha) - 2f(\alpha_\varepsilon^*)) - o(1). \end{aligned}$$

Hence (5.8).

If $f(\alpha) - 2f(\alpha_\varepsilon^*)$ is strictly positive, say,

$$(5.9) \quad f(\alpha) - 2f(\alpha_\varepsilon^*) \geq \frac{1}{2}f(\alpha) > 0$$

then we obtain a contradiction with (5.4) since

$$(5.10) \quad 1/2 - 2f(\alpha) > 0,$$

provided α is small enough. Thus, the only case left is $f(\alpha) - 2f(\alpha_\varepsilon^*) < \frac{1}{2}f(\alpha)$. But this case never occurs, thanks to the following

Proposition 5.2 Take $\alpha_1 \in (0, \alpha)$, independent of ε , such that $f(\alpha) \geq 4f(\alpha_1)$. Then,

$$(5.11) \quad \min_{x \in [0, L]} v_\varepsilon(x) = \alpha_\varepsilon^* \leq \alpha_1,$$

and thus (5.9) is always satisfied.

Proof. Assume that (5.11) is false, i.e.,

$$(5.12) \quad v_\varepsilon(x) \geq \alpha_1 \quad \forall x \in [0, L].$$

We show that the energy $\mathcal{E}_\varepsilon(u_\varepsilon, v_\varepsilon)$ is actually greater than 1 and thus reach a new contradiction. Since $(u_\varepsilon, v_\varepsilon)$ minimizes \mathcal{E}_ε over B_ε , $(u_\varepsilon, v_\varepsilon)$ is, at the least, a critical point of \mathcal{E}_ε on $(0, L)$ with respect to *compactly supported* variations in both v_ε and u_ε . Thus, on $(0, L)$,

$$(5.13) \quad \begin{aligned} -\varepsilon v_\varepsilon'' + v_\varepsilon (u_\varepsilon')^2 + \frac{v_\varepsilon - 1}{\varepsilon} &= 0 \\ [u_\varepsilon'(\eta_\varepsilon + v_\varepsilon^2)]' &= 0. \end{aligned}$$

The second equation of (5.13) shows that, on $(0, L)$,

$$(5.14) \quad u_\varepsilon'(x)(\eta_\varepsilon + v_\varepsilon^2(x)) = c_\varepsilon \text{ a.e. } x$$

for some constant c_ε . From (5.4), (5.14), and, since u_ε is in particular in $W^{1,1}(0, L)$,

$$o(1) + 1 \geq \mathcal{E}_\varepsilon(u_\varepsilon, v_\varepsilon) \geq \int_0^L (\eta_\varepsilon + v_\varepsilon^2)(u_\varepsilon')^2 = \int_0^L c_\varepsilon u_\varepsilon' = a_\varepsilon c_\varepsilon,$$

so that

$$c_\varepsilon \leq \frac{C}{a_\varepsilon}.$$

Hence, at the possible expense of extracting a subsequence,

$$(5.15) \quad c_\varepsilon \rightarrow c_0, \quad \varepsilon \searrow 0.$$

On the other hand, the energy bound $\mathcal{E}_\varepsilon(u_\varepsilon, v_\varepsilon) \leq 1 + o(1)$ implies that v_ε converges to 1 in $L^2(0, L)$ and thus $v_\varepsilon(x) \rightarrow 1$ a.e $x \in (0, L)$. Therefore

$$u_\varepsilon'(x) = \frac{c_\varepsilon}{\eta_\varepsilon + v_\varepsilon^2(x)} \rightarrow c_0 \text{ as } \varepsilon \rightarrow 0 \text{ for a. e. } x \in (0, L).$$

In view of (5.12), (5.14), u_ε' is bounded. Therefore, by Lebesgue's dominated convergence theorem,

$$a_\varepsilon = \int_0^L u_\varepsilon' dx \rightarrow \int_0^L c_0 dx = c_0 L.$$

Hence $c_0 = a/L > 0$. Now,

$$\int_0^L (\eta_\varepsilon + v_\varepsilon^2)(u'_\varepsilon)^2 dx = \int_0^L c_\varepsilon u'_\varepsilon dx = c_\varepsilon a_\varepsilon \geq c_0 a - o(1) \geq \frac{a^2}{L} - o(1).$$

This, together with inequality (5.2), allows us to deduce that

$$1 + o(1) \geq \mathcal{E}_\varepsilon(u_\varepsilon, v_\varepsilon) = \int_0^L (\eta_\varepsilon + v_\varepsilon^2)(u'_\varepsilon)^2 dx + \mathcal{F}_\varepsilon(v_\varepsilon, 0, L) \geq \frac{a^2}{L} + 2(f(\mu) - f(\alpha)) - o(1).$$

This is again impossible if we initially choose

$$(5.16) \quad \frac{a^2}{L} + 2(f(\mu) - f(\alpha)) > 1.$$

□

Summing up and recalling (5.6), (5.10) and (5.16), any minimizer $(u_\varepsilon, v_\varepsilon)$ of \mathcal{E}_ε over B_ε is actually a critical point of \mathcal{E}_ε when ε is small enough, provided the following conditions on μ and α are satisfied

$$(5.17) \quad 0 < \alpha < \mu < 1, f(\mu) + f(\alpha) < 1/2, 1/2 - 2f(\alpha) > 0, \frac{a^2}{L} + 2(f(\mu) - f(\alpha)) > 1.$$

The set of (μ, α) meeting the above requirements is not empty. □

The third and final step in the proof of Theorem 2.4 is to describe the behavior of $(u_\varepsilon, v_\varepsilon)$ as $\varepsilon \rightarrow 0$. By Lemma 5.1, we easily see that $(u_\varepsilon, v_\varepsilon)$ is a local minimizer of \mathcal{E}_ε . To complete the proof, we have to prove that

- (i) v_ε has only one v -jump at $x = L$, and
- (ii) u , the limit function of u_ε in $L^2(0, L)$, is identically 0 on $(0, L)$.

Let us prove (i). From the definition of v_ε , we know that $v_\varepsilon(L) \leq \alpha < 1$. We also know from Lemma 5.1 that $(u_\varepsilon, v_\varepsilon)$ is a critical point of \mathcal{E}_ε , hence $v'_\varepsilon(0) = v'_\varepsilon(L) = 0$. Now, we invoke the remarks of Section 3.5 to conclude that v_ε has a v -jump at $x = L$. If v_ε has another v -jump, either at $x = 0$ or at an interior point of $(0, L)$, then, using (4.3), we find that the limit energy of \mathcal{E}_ε is at least $2 \times 3/4 = 3/2$. However, from (5.4), we know that the limit energy of $\mathcal{E}_\varepsilon(u_\varepsilon, v_\varepsilon)$ is not greater than 1, and thus v_ε has only one v -jump at $x = L$.

It remains to establish (ii). From (i), we know that v_ε is a half-well with the sink at $x = L$. So, from Theorem 2.2, we see that either $u \equiv 0$ or $u(x) = ax/L$ for $x \in (0, L)$. The latter case cannot happen, because if it does, then by lower semicontinuity and recalling (5.2) and the fact that $v_\varepsilon \rightarrow 1$ in $L^2(0, L)$, one finds that

$$\begin{aligned} 1 &\geq \liminf_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon(u_\varepsilon, v_\varepsilon) \geq \liminf_{\varepsilon \rightarrow 0} \int_0^L (\eta_\varepsilon + v_\varepsilon^2(x))(u'_\varepsilon(x))^2 dx + \liminf_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(v_\varepsilon, 0, L) \\ &\geq \int_0^L (u'(x))^2 dx + 2(f(\mu) - f(\alpha)) = \frac{a^2}{L} + 2(f(\mu) - f(\alpha)) > 1. \end{aligned}$$

which is impossible in view of (5.17). The proof of Theorem 2.4 is now complete.

Remark 5.3 If $N = 1$ and the v -jump is at the boundary of $(0, L)$ then the critical points $(u_\varepsilon, v_\varepsilon)$ of \mathcal{E}_ε constructed in Theorem 2.4 are also local minimizers of \mathcal{E}_ε . We conjecture that if $N = 1$ and the v -jump is at $L/2$ or if $N \geq 2$, then the critical points $(u_\varepsilon, v_\varepsilon)$ found in Theorem 2.4 are also local minimizers of \mathcal{E}_ε .

6 Proof of Theorem 2.5

The measure limit of $(\eta_\varepsilon + v_\varepsilon^2)(u'_\varepsilon)^2 dx$ is immediately computed upon remarking that, thanks to (2.6),

$$(\eta_\varepsilon + v_\varepsilon^2)(u'_\varepsilon)^2 = c_\varepsilon u'_\varepsilon,$$

so that the measure limit of $(\eta_\varepsilon + v_\varepsilon^2)(u'_\varepsilon)^2 dx$ is that of $c_\varepsilon u'_\varepsilon dx$. Testing with a smooth compactly supported function φ , we obtain

$$\int_0^L c_\varepsilon u'_\varepsilon \varphi dx = -c_\varepsilon \int_0^L u_\varepsilon \varphi' dx \longrightarrow -c_0 \int_0^L u \varphi' dx,$$

so that, upon recalling Theorem 2.2, we obtain the desired result. The reader will have not failed to note the fortuitousness of the value of c_0 , *i.e.*, 0, when u jumps; the result would be false if u could jump for non-zero values of c_0 !

The computation of the measure limit of the other two terms is more delicate. We first establish that there is no concentration of energy for those two terms away from the jump points. We will only use the property that *there is a finite ε -independent number j of v -jumps on $[0, L]$, denoted by $x_\varepsilon^1, \dots, x_\varepsilon^j$, with $x_\varepsilon^k \rightarrow x_k$, $k = 1, \dots, j$.*

Lemma 6.1 *For any compact subset $K \subset [0, L] \setminus \cup_{k=1, \dots, j} \{x_k\}$, we have*

$$\int_K (\varepsilon(v'_\varepsilon(x))^2 + (1 - v_\varepsilon(x))^2/\varepsilon) dx \leq \mathcal{C}_K \varepsilon^{1/4},$$

where \mathcal{C}_K may depend only on K .

Proof. Consider $A_\varepsilon := \{x \in [0, L] : v_\varepsilon(x) \leq 1 - \varepsilon^{1/4}\}$. Then, from the energy bound (2.5), $|A_\varepsilon| \leq \mathcal{C}\sqrt{\varepsilon}$, while $\{x_\varepsilon^1, \dots, x_\varepsilon^j\} \subset A_\varepsilon$. For a given compact subset K of $[0, L] \setminus \cup_{k=1, \dots, j} \{x_k\}$, set $\delta := 1/2 \min_{k=1, \dots, j} \text{dist}(x_k, K)$. If ε is small enough, $\{x_1, \dots, x_j\} \subset A_\varepsilon \subset U_\delta := \cup_{k=1, \dots, j} [x_k - \delta, x_k + \delta]$, so that, $K \cap A_\varepsilon = \emptyset$. Because $K \subset [0, L] \setminus U_\delta$, it suffices to prove that

$$(6.1) \quad \int_{[0, L] \setminus U_\delta} (\varepsilon(v'_\varepsilon(x))^2 + (1 - v_\varepsilon(x))^2/\varepsilon) dx \leq \mathcal{C}_K \varepsilon^{1/4}.$$

Let us denote $V_\delta = [0, L] \setminus U_\delta$. Multiplying both sides of the first equation of (2.4) by $v_\varepsilon - 1$ and integrating over V_δ , we get

$$(6.2) \quad \int_{V_\delta} -\varepsilon v''_\varepsilon(x)(v_\varepsilon(x) - 1) dx + \int_{V_\delta} v_\varepsilon(x)(u'_\varepsilon(x))^2(v_\varepsilon(x) - 1) dx + \int_{V_\delta} \frac{(v_\varepsilon(x) - 1)^2}{\varepsilon} dx = 0.$$

Note that V_δ is a union of a finite ε -independent number J ($\leq j + 1$) of intervals on $[0, L]$: $V_\delta = \cup_{k=1, \dots, J} [a_\varepsilon^k, b_\varepsilon^k]$. Now, integrating by parts the first term of (6.2), recalling (2.6) and rearranging, one obtains

$$(6.3) \quad \int_{V_\delta} \varepsilon (v'_\varepsilon(x))^2 dx + \int_{V_\delta} \frac{(v_\varepsilon(x) - 1)^2}{\varepsilon} dx = \sum_{k=1}^J \varepsilon \left(v'_\varepsilon(b_\varepsilon^k) (v_\varepsilon(b_\varepsilon^k) - 1) - v'_\varepsilon(a_\varepsilon^k) (v_\varepsilon(a_\varepsilon^k) - 1) \right) + \int_{V_\delta} \frac{c_\varepsilon^2}{(\eta_\varepsilon + v_\varepsilon^2(x))^2} v_\varepsilon(x) (1 - v_\varepsilon(x)) dx.$$

By the definitions of A_ε and V_δ , we have $|1 - v_\varepsilon| \leq \varepsilon^{1/4}$ on V_δ . Combining this fact with the gradient bound for v_ε in Lemma 3.4 yields that the right hand side of (6.3) is bounded from above by $\mathcal{C}_K \varepsilon^{1/4}$ for some constant \mathcal{C}_K which may depend only on K . Hence the desired result stated in (6.1) follows. \square

Remark 6.2 The previous lemma shows that the measure limits of $\varepsilon (v'_\varepsilon(x))^2 dx$, and of $(v_\varepsilon(x) - 1)^2 / \varepsilon dx$ are Dirac masses concentrated at x_1, \dots, x_j . We will evaluate their respective weight in the fourth and final step below.

Also note that, thanks to Lemma 3.6, those limits are immediately computed (and found to be 0!) in the absence of v -jumps.

The second step consists in computing the limit d_0 of the discrepancy d_ε defined in (3.3), which exists, at least for a well chosen subsequence, by the boundedness (3.4) of d_ε . To this effect, we prove the following

Lemma 6.3 $d_0 + c_0^2 = 0$.

Proof. Recalling (3.3), (2.6), we obtain

$$(6.4) \quad d_\varepsilon + c_\varepsilon u'_\varepsilon(x) = \frac{(v_\varepsilon(x) - 1)^2}{\varepsilon} - \varepsilon (v'_\varepsilon(x))^2,$$

and thus

$$(6.5) \quad |d_\varepsilon + c_\varepsilon u'_\varepsilon(x)| = \left| \frac{(v_\varepsilon(x) - 1)^2}{\varepsilon} - \varepsilon (v'_\varepsilon(x))^2 \right|.$$

Let A_ε, K with $|K| > 0$ and δ be as in the proof of Lemma 6.1. Then, on K , one has $1 - \varepsilon^{1/4} \leq v_\varepsilon \leq 1$. By virtue of (2.6), u'_ε is bounded on K . Upon integrating (6.5) over K , recalling Lemma 3.2 and letting $\varepsilon \searrow 0$, we have by Lebesgue's dominated convergence theorem

$$(6.6) \quad |K| |d_0 + c_0^2| = \lim_{\varepsilon \rightarrow 0} \int_K |d_\varepsilon + c_\varepsilon u'_\varepsilon(x)| dx = \lim_{\varepsilon \rightarrow 0} \int_K \left| \frac{(v_\varepsilon(x) - 1)^2}{\varepsilon} - \varepsilon (v'_\varepsilon(x))^2 \right| dx.$$

In view of Lemma 6.1, the last term of the above equation is 0. Thus $d_0 + c_0^2 = 0$ as claimed. \square

The third step consists of the following equi-partition result

Lemma 6.4 *For all x in $[0, L]$,*

$$|v'_\varepsilon(x)| \leq \frac{1 - v_\varepsilon(x)}{\varepsilon}.$$

Further,

$$\lim_{\varepsilon \rightarrow 0} \int_0^L |\varepsilon(v'_\varepsilon(x))^2 - (v_\varepsilon(x) - 1)^2/\varepsilon| dx = 0.$$

Proof. According to (6.4), the term $(v_\varepsilon(x) - 1)^2/\varepsilon - \varepsilon(v'_\varepsilon(x))^2$ attains, for a fixed ε , its minimum on $[0, L]$ precisely where $u'_\varepsilon = c_\varepsilon/(\eta_\varepsilon + v_\varepsilon^2)$ attains its minimum, or still, where v_ε attains its maximum. But, at such points, v'_ε cancels, so that the minimum of that term is non-negative. Thus,

$$\begin{aligned} \int_0^L \left| \frac{(v_\varepsilon(x) - 1)^2}{\varepsilon} - \varepsilon(v'_\varepsilon(x))^2 \right| dx &= \int_0^L \left(\frac{(v_\varepsilon(x) - 1)^2}{\varepsilon} - \varepsilon(v'_\varepsilon(x))^2 \right) dx \\ &= \int_0^L (d_\varepsilon + c_\varepsilon u'_\varepsilon(x)) dx = Ld_\varepsilon + c_\varepsilon a_\varepsilon \\ &\rightarrow Ld_0 + c_0 a = L(d_0 + c_0^2) = 0, \end{aligned}$$

in view of Lemma 6.3. \square

The fourth and final step consists in evaluating the respective weights of the Dirac masses making up the limit of $\varepsilon(v'_\varepsilon(x))^2 dx$ and of $(v_\varepsilon(x) - 1)^2/\varepsilon dx$.

To this effect, we first remark that, at the possible expense of extending v_ε by reflection around $x = 0$ and/or $x = L$, we may always compute the measure limit μ of $(\varepsilon(v'_\varepsilon(x))^2 + (v_\varepsilon(x) - 1)^2/\varepsilon) dx$ over some $I \supset [0, L]$, so that $\mu(\partial I) = 0$, in which case $\mu(I) = \lim_\varepsilon \int_I (\varepsilon(v'_\varepsilon(x))^2 + (v_\varepsilon(x) - 1)^2/\varepsilon) dx$, while Lemma 6.4 still applies over I .

Note that

$$\begin{aligned} \int_I \left(\varepsilon(v'_\varepsilon(x))^2 + \frac{(v_\varepsilon(x) - 1)^2}{\varepsilon} - 2|v'_\varepsilon(x)|((1 - v_\varepsilon(x))) \right) dx &= \int_I \left(\varepsilon^{1/2}|v'_\varepsilon(x)| - \frac{(1 - v_\varepsilon(x))}{\varepsilon^{1/2}} \right)^2 dx \\ &\leq \int_I \left| \varepsilon(v'_\varepsilon(x))^2 - \frac{(1 - v_\varepsilon(x))^2}{\varepsilon} \right| dx, \end{aligned}$$

which goes to 0 with ε , according to Lemma 6.4 above. Thus, the total mass of the measure limit of $(\varepsilon(v'_\varepsilon(x))^2 + (v_\varepsilon(x) - 1)^2/\varepsilon) dx$, is also that of $2|v'_\varepsilon(x)|((1 - v_\varepsilon(x))) dx$. But, we know, according to Proposition 4.2, that the graph of v_ε is symmetric around each v -jump, so that it suffices to compute the mass of measure limit of $2|v'_\varepsilon(x)|((1 - v_\varepsilon(x))) dx$ over a half well, that is,

$$\int_{\{x: v_\varepsilon(x) \in [m_\varepsilon, M_\varepsilon]\}} 2|v'_\varepsilon(x)|((1 - v_\varepsilon(x))) dx,$$

or still, since $v'_\varepsilon(x) > 0$ on $\{x : v_\varepsilon(x) \in [m_\varepsilon, M_\varepsilon]\}$,

$$\int_{m_\varepsilon}^{M_\varepsilon} 2(1-y) dy = (2M_\varepsilon - M_\varepsilon^2) - (2m_\varepsilon - m_\varepsilon^2) \longrightarrow 1,$$

as $\varepsilon \searrow 0$, since $M_\varepsilon \nearrow 1$, while $m_\varepsilon \searrow 0$. Hence the measure limit μ is given by

$$\mu = 2 \sum_{\{x: x \text{ is a } v\text{-jump}\}} \delta_x.$$

The proof of Theorem 2.5 is now complete.

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