LARGE DEVIATIONS FOR THE TWO-DIMENSIONAL TWO-COMPONENT PLASMA

THOMAS LEBLÉ, SYLVIA SERFATY, OFER ZEITOUNI, WITH AN APPENDIX BY WEI WU

ABSTRACT. We derive a large deviations principle for the two-dimensional two-component plasma in a box. As a consequence, we obtain a variational representation for the free energy, and also show that the macroscopic empirical measure of either positive or negative charges converges to the uniform measure. An appendix, written by Wei Wu, discusses applications to the supercritical complex Gaussian multiplicative chaos and to the XY model.

1. INTRODUCTION

1.1. **General setting.** The two-dimensional two-component plasma is a standard ensemble of statistical mechanics, in which N particles of positive charge and N particles of negative charge interact logarithmically in the plane, cf [Frö76, GP77, DL74, For10]. The associated Gibbs measure at inverse temperature $\beta > 0$ is given by

$$d\mathbb{P}_N^\beta(\vec{X}_N, \vec{Y}_N) := \frac{1}{Z_{N,\beta}} e^{-\frac{\beta}{2} w_N(\vec{X}_N, \vec{Y}_N)} d\vec{X}_N \, d\vec{Y}_N, \tag{1.1}$$

where $Z_{N,\beta}$ is the normalizing constant, i.e. the *partition function*

$$Z_{N,\beta} := \int_{\Lambda^{2N}} e^{-\frac{\beta}{2} w_N(\vec{X}_N, \vec{Y}_N)} d\vec{X}_N d\vec{Y}_N, \tag{1.2}$$

and we have written

$$w_N(\vec{X}_N, \vec{Y}_N) := \sum_{1 \le i \neq j \le N} -\log|x_i - x_j| - \log|y_i - y_j| + \sum_{1 \le i, j \le N} \log|x_i - y_j| \tag{1.3}$$

for any $N \geq 1$ and any N-tuples $\vec{X}_N = (x_1, \dots, x_N)$ and $\vec{Y}_N = (y_1, \dots, y_N)$ of points in, say, the unit cube $\Lambda := [0,1]^2$ of $\mathbb{R}^2$. The notation $d\vec{X}_N \, d\vec{Y}_N$ refers to the Lebesgue measure on Λ^{2N}. The choice of $\beta/2$ instead of β in the exponent of (1.1) is made in order to match the existing literature. In physical terms, $w_N(\vec{X}_N, \vec{Y}_N)$ computes the two-dimensional electrostatic (or logarithmic) interaction of the point charges $(x_1, \dots, x_N)$ and $(y_1, \dots, y_N)$, the former carrying a $+1$ charge and the latter a -1 charge.

We are interested in proving a Large Deviation Principle (LDP) on the Gibbs measure $\mathbb{P}_N^\beta$, which is inspired by [LS15], where such a result was obtained for the one-component plasma in arbitrary dimension. The (say, two-dimensional) one-component plasma corresponds to a system of point charges which all have *same sign* and interact logarithmically, but that need to be confined by some external potential, acting in effect like a slowly varying neutralizing (opposite) charge distribution.

Motivations for studying two-component plasmas are numerous. Besides its intrinsic interest as a toy model for classical electrons and ions, it is also related to the so-called Sine-Gordon model: the grand canonical partition function of the two component plasma can be related to the Euclidean version of the sine-Gordon partition function, and the Coulomb gas on a lattice itself related to the XY model and the Kosterlitz-Thouless phase transition (see the review [Spe97] and references therein). Another motivation, which will be described in more details in the appendix, is the connection with the partition function of "complex multiplicative Gaussian chaos" (cf. [LRV15]), which may be formally written as $\int e^{i\beta h(x)}\,dx$, where $h(x)$ is the Gaussian Free Field. This question is itself related to height functions of dimer models and to the Lee-Yang theorem for the XY model. It turns out that computing moments of $e^{i\beta h}$ makes the Gibbs measure of the two-component plasma appear and, as described in the appendix, our results yield a rate of decay in terms of β of the tails of this partition function.

Due to the presence of point charges of opposite signs, the system is unstable at low temperature because the thermal excitation does not compensate the energetical trend for configurations with $-\infty$ energy, in which at least two particles of opposite signs collide. The domain of stability of the system (i.e. the range of the parameter β for which the integral in (1.2) converges, so that (1.1) makes sense) was found to be $\beta < 2$ in [DL74], together with a first bound on $Z_{N,\beta}$. A more accurate estimate and the existence of the thermodynamic limit were proven in [Frö76] by Euclidean quantum field techniques, and [GP77] by classical methods (the question of obtaining it by classical methods was apparently first raised in [SM76]). The result can be summarized as follows.

Proposition 1.1 (Gunson-Panta, [GP77])**.** *For any $\beta < 2$,*

$$\log Z_{N,\beta} = \frac{\beta}{2} N \log N + C_\beta N + o(N), \tag{1.4}$$

with a constant C_β and an error $o(N)$ both depending on β.

Later, a number of other results, such as the asymptotics of two-point correlation functions and the thermodynamic properties of the two-component plasma, were obtained in the physics literature. We refer to the review [Šam03] and references therein.

For any $\vec{X}_N, \vec{Y}_N$, let μ_N^+ and μ_N^- be the empirical measures associated to the positive and negative charges

$$\mu_N^+ := \frac{1}{N}\sum_{i=1}^N \delta_{x_i}, \quad \mu_N^- := \frac{1}{N}\sum_{i=1}^N \delta_{y_i}.$$

A natural question is to ask for the large N behavior of these *macroscopic* quantities in the space $\mathcal{P}(\Lambda)$ of probability measures on Λ. At the microscopic level, we may also ask whether the point process induced by $\mathbb{P}_N^\beta$ has a typical behavior. In this paper, we give an LDP for a spatially averaged microscopic behavior and as a consequence we show that both empirical measures μ_N^+, μ_N^- converge a.s. to the uniform measure on Λ.

1.2. **Main result.** Before stating our main result, we need to introduce some notation and concepts, which are all "signed" versions of those introduced in [SS15, LS15] (in the latter papers, all the charges have the same sign). We denote by $\mathcal{X}$ the set of locally finite signed point configurations with the topology of local convergence (for more details see Section 2). If $(\vec{X}_N, \vec{Y}_N)$ is a pair of N-tuples of points in the square Λ, we may see it as an element of the space $\mathcal{X}$ by associating to $\vec{X}_N$ (resp. $\vec{Y}_N$) the point configuration $\nu_N^+ := \sum_{i=1}^N \delta_{x_i}$ (resp. $\nu_N^- := \sum_{i=1}^N \delta_{y_i}$). When starting from $(\vec{X}_N, \vec{Y}_N)$, we first rescale the associated finite signed configurations by a factor $\sqrt{N}$ to get

$$\hat{\nu}_N^+ := \sum_{i=1}^N \delta_{\sqrt{N}x_i} \quad \hat{\nu}_N^- := \sum_{i=1}^N \delta_{\sqrt{N}y_i},$$

and we then define the map

$$\begin{aligned} i_N : (\mathbb{R}^2)^N \times (\mathbb{R}^2)^N \quad &\longrightarrow \quad \mathcal{P}(\Lambda \times \mathcal{X}) \\ (\vec{X}_N, \vec{Y}_N) \quad &\longmapsto \quad \bar{P}_{(\vec{X}_N,\vec{Y}_N)} := \int_\Lambda \delta_{(x, \theta_{\sqrt{N}x} \cdot (\hat{\nu}_N^+, \hat{\nu}_N^-))} dx \end{aligned} \tag{1.5}$$

where for any Borel space X, $\mathcal{P}(X)$ denotes the set of Borel probability measures on X and θ_λ denotes the action of translation by a vector $\lambda \in \mathbb{R}^2$, that is, for $\nu \in \mathcal{P}(\mathbb{R}^2)$, we have $\theta_\lambda \nu(A) = \nu(A + \lambda)$ for any measurable set A. The variable x is a "tag" that is keeping track of the point $x \in \Lambda$ around which the configuration was blown-up, and this way we build from any signed point configuration the law of a "tagged signed point process", $\bar{P}_{(\vec{X}_N,\vec{Y}_N)}$. The laws of these signed point processes will easily be shown to be tight, and any accumulation point as $N \to \infty$ is a stationary probability measure on $\Lambda \times \mathcal{X}^0$ (i.e. the law of a stationary tagged point process) whose first marginal is the Lebesgue measure on Λ. We will generally denote with bars the quantities corresponding to tagged point processes and without bars the quantitites corresponding to non-tagged point processes.

Throughout we will always consider the subset

$$\left\{ \bar{P} \in \mathcal{P}(\Lambda \times \mathcal{X}), \bar{P}(A \times \mathcal{X}) = \mathbf{Leb}(A), \forall A \text{ Borel} \right\},$$

and continue, with some abuse of notation, to denote it by $\mathcal{P}(\Lambda \times \mathcal{X})$. This assumption allows us to consider the disintegration probability measures $\bar{P}^x \in \mathcal{P}(\mathcal{X})$ for any $x \in \Lambda$. We denote by $\mathcal{P}_{\text{inv}}(\mathcal{X})$ the set of stationary laws of signed point processes, and we denote by $\mathcal{P}_{\text{inv}}(\Lambda \times \mathcal{X})$ the set of stationary laws of tagged signed point processes, that is those $\bar{P} \in \mathcal{P}(\Lambda \times \mathcal{X})$ so that the corresponding disintegration measure $\bar{P}^x$ is stationary for Lebesgue-a.e. $x \in \Lambda$. Finally we denote by $\mathcal{P}_{\text{inv},1}(\Lambda \times \mathcal{X})$ the set of $\bar{P} \in \mathcal{P}(\Lambda \times \mathcal{X})$ such that $\bar{P}$ has total intensity 1 (i.e. there is, in average, one point of each sign per unit volume).

In Section 2.4 we will define an interaction energy functional $\widetilde{\mathbb{W}}$ on the space $\mathcal{P}_{\text{inv}}(\mathcal{X})$. It can be understood as the expectation of the infinite-volume limit of the logarithmic interaction in the system of charges described by the signed configurations. We then define the interaction energy of $\bar{P} \in \mathcal{P}_{\text{inv}}(\Lambda \times \mathcal{X})$ as

$$\overline{\mathbb{W}}(\bar{P}) := \int_\Lambda \widetilde{\mathbb{W}}(\bar{P}^x) dx. \tag{1.6}$$

Next, we define the *specific relative entropy* of the law of a signed point process as the infinite-volume limit of the usual relative entropy with respect to a reference measure.

Definition 1.2. *Let $P \in \mathcal{P}_{\text{inv}}(\mathcal{X})$. The relative specific entropy* $\mathsf{ent}[P]$ *with respect to the signed Poisson point process of uniform intensity* 1 *is given by*

$$\mathsf{ent}[P] := \lim_{R \to \infty} \frac{1}{R^2} \mathrm{Ent}\left(P_R | \mathbf{\Pi}_R^s\right), \tag{1.7}$$

where P_R denotes the restriction of P to $C_R := [-R/2, R/2]^2$, and

$$\mathrm{Ent}(\mu|\nu) = \begin{cases} \int \log \frac{d\mu}{d\nu} d\mu & \textit{if } \mu \textit{ is absolutely continuous with respect to } \nu, \\ +\infty & \textit{otherwise} \end{cases}$$

is the usual relative entropy. The reference measure is the law of a signed Poisson point process

$$\mathbf{\Pi}^s := \mathbf{\Pi}^1 \otimes \mathbf{\Pi}^1,$$

which is nothing but the law of two independent Poisson point processes of intensity 1.

The good definition of such an infinite-volume relative entropy ent is known in the "non-signed" case where one deals with standard point processes (see e.g. [RAS09], and [LS15] for an extension to the case of tagged point processes), and we recast its properties in the setting of signed point processes in Section 2.6. We may then define the specific relative entropy of $\bar{P} \in \mathcal{P}_{\mathrm{inv}}(\Lambda \times \mathcal{X})$ as

$$\overline{\mathsf{ent}}(\bar{P}) := \int_\Lambda \mathsf{ent}[\bar{P}^x] dx. \tag{1.8}$$

Using (1.6) and (1.8), we introduce the function $\overline{\mathcal{F}}_\beta$ defined on the space $\mathcal{P}_{\mathrm{inv},1}(\Lambda \times \mathcal{X})$,

$$\overline{\mathcal{F}}_\beta(\bar{P}) := \begin{cases} \frac{\beta}{2}\overline{\mathbb{W}}(\bar{P}) + \overline{\mathsf{ent}}[\bar{P}] & \text{if } \overline{\mathsf{ent}}[\bar{P}] < +\infty, \\ +\infty & \text{otherwise.} \end{cases} \tag{1.9}$$

We let $\overline{\mathcal{F}}_\beta^{\mathrm{sc}}$ be the lower semi-continuous regularization of $\overline{\mathcal{F}}_\beta$ on $\mathcal{P}_{\mathrm{inv},1}(\Lambda \times \mathcal{X})$ i.e.

$$\overline{\mathcal{F}}_\beta^{\mathrm{sc}}(\bar{P}) := \lim_{\varepsilon \to 0} \inf_{B(\bar{P},\varepsilon)} \overline{\mathcal{F}}_\beta. \tag{1.10}$$

In particular it is standard that $\overline{\mathcal{F}}_\beta$ and $\overline{\mathcal{F}}_\beta^{\mathrm{sc}}$ have the same infimum on $\mathcal{P}_{\mathrm{inv},1}(\Lambda \times \mathcal{X})$. We remark that we do not know whether $\overline{\mathcal{F}}_\beta$ is lower semi-continuous, and in particular we do not rule out the possibility that $\overline{\mathcal{F}}_\beta = \overline{\mathcal{F}}_\beta^{\mathrm{sc}}$.

When $\beta < 2$ is fixed, we let $\bar{P}_N$ be the random variable $i_N(\vec{X}_N, \vec{Y}_N)$ (as in (1.5)) when $(\vec{X}_N, \vec{Y}_N)$ are sampled according to $\mathbb{P}_N^\beta$, and we let $\overline{\mathfrak{P}}_N^\beta$ be its law. In other terms $\overline{\mathfrak{P}}_N^\beta$ is the push-forward of $\mathbb{P}_N^\beta$ by i_N.

We may now state our main result.

Theorem 1. *The sequence $\{\overline{\mathfrak{P}}_N^\beta\}_N$ satisfies a Large Deviations Principle at speed N with good rate function given by*

$$\overline{\mathcal{F}}_\beta^{\mathrm{sc}} - \inf_{\mathcal{P}_{\mathrm{inv},1}(\Lambda \times \mathcal{X})} \overline{\mathcal{F}}_\beta.$$

As a consequence we obtain the following expansion for $\log Z_{N,\beta}$ with $\beta < 2$:

Corollary 1.3. *For any $\beta < 2$ it holds*

$$\log Z_{N,\beta} = \frac{\beta}{2} N \log N - \left(\inf_{\mathcal{P}_{\mathrm{inv},1}(\Lambda, \mathcal{X})} \overline{\mathcal{F}}_\beta \right) N + o(N), \tag{1.11}$$

where the term $o(N)$ depends on β.

In comparison with Proposition 1.1, we now have a characterization of the constant C_β in front of the term of order N. We also have information on the asymptotic behavior of the empirical measures.

Theorem 2. *The sequence $\{(\mu_N^+, \mu_N^-)\}_N$ converges $\mathbb{P}_N^\beta$-a.s. to $\mathbf{Leb}_\Lambda \otimes \mathbf{Leb}_\Lambda$, where $\mathbf{Leb}_\Lambda$ is the uniform probability measure on Λ.*

We emphasize that in the present case, in contrast with the one-component case, the optimal macroscopic distribution of the points cannot be deduced from the leading order behavior of the system. Indeed, a leading order LDP (see Section 7) only shows that μ_N^+ and μ_N^- have the same limit. The next order analysis of Theorem 1 allows to identify *at the same time* the macroscopic distribution of the particles, and their microscopic behavior.

1.3. Interpretation and method.

Comparison with the one-component case. To explain these results and their proof, it is useful to return to the case of the one-component plasma, as was studied in [LS15] using tools introduced in [SS15,RS15,PS15] (see also [Ser15]). We recall that in the one-component plasma, the particles have the same (positive) sign and are confined by an external potential, which can be shown to act like a neutralizing negative diffuse background charge. It is well-known that the macroscopic distribution of the particles can be identified by a leading order LDP (at speed N^2) to be the so-called *equilibrium measure*, uniquely determined by the confining potential (cf. [HP00, AZ98]). Then the next order LDP analysis performed in [LS15] allows to identify the behavior of the particles at the microscopic scale as minimizing a certain rate function, which is of similar nature to (1.9). This analysis relied on expressing the logarithmic interaction energy via the *electric field*, or *electric potential* that the set of charges generated. A crucial tool was then the so-called *screening result*, which allows, via the electric field formulation, to localize the interaction energy in microscopic boxes which can then be seen as (essentially) independent and non-interacting.

The main difference between the one-component and two-component cases is that in the one-component case, the interaction energy has a sign and is bounded below, hence no configuration can give a large contribution to the partition function. In the two-component case, the interaction energy is not bounded below, and it is only thanks to the *entropy term*, corresponding to the volume in phase-space, that configurations with very negative energy do not weigh too much in the partition function. Another heuristic way of saying this is that the Lebesgue measure (in phase-space) behaves like a "Lebesgue repulsion" which prevents particle of opposite signs from getting too close to each other too often, and this is only true because $\beta < 2$, i.e. when this Lebesgue repulsion is strong enough. In other words, energy and volume considerations always have to be worked with jointly, and we always need to exploit the fact that $\beta < 2$.

Positive part of the energy and dipole contributions. Because the number of positively charged particles and the number of negatively charged particles are the same, it is natural to see a configuration as a set of *dipoles* of particles of opposite sign, matched by nearest neighbor pairing (or minimal matching, also called minimal connection). It is tempting to try to prove directly an LDP on the pairing, but we have not been able to do so. Instead, we exploit the idea of matching (borrowed from [GP77]) in conjunction with simple computations originating in [SS15, RS15], which relies on the expression of the interaction energy via the electric potential generated by the system of charges. We rewrite the interaction energy as the sum of a positive part and a part corresponding only to nearest neighbor interactions, which can be thought of as a "dipole contribution". More precisely for each x_i or y_i, we define

$$r(x_i) = \min\left(1, \frac{1}{2}\min_{j\neq i}|x_i - x_j|, \frac{1}{2}\min_{j}|x_i - y_j|\right)$$

to be the (half) nearest neighbor distance truncated at 1. We then prove the identity, valid for every pair of N-tuples $\vec{X}_N, \vec{Y}_N$ with distinct coordinates:

$$w_N(\vec{X}_N, \vec{Y}_N) = \frac{1}{2\pi}\int_{\mathbb{R}^2} |\nabla V_{N,r}|^2 + \sum_{i=1}^N \log r(x_i) + \sum_{i=1}^N \log r(y_i). \tag{1.12}$$

where

$$V_{N,r} = \log * \left(\sum_{i=1}^{N} \delta_{x_i}^{(r(x_i))} - \sum_{i=1}^{N} \delta_{y_i}^{(r(y_i))} \right)$$

and where $\delta_x^{(\eta)}$ denotes the uniform measure of mass 1 on the sphere of center x and radius η (for $x \in \mathbb{R}^2$ and $\eta > 0$).

The identity (1.12) is similar to the "electrostatic inequality" found in [GP77] (however we do not discard the positive term as they do), and it allows us to use the method of [GP77], which controls the negative (and possibly unbounded) dipole contributions. The analysis of [GP77] exploits in a quantitative way the fact that $\beta < 2$ and that the Lebesgue repulsion dominates the dipole attraction.

Similarly, the interaction energy $\overline{\mathbb{W}}$ is the sum of two terms, one positive part corresponding to the large N limit (after blow-up at the scale $\sqrt{N}$) of the positive quantity $\frac{1}{2\pi}\int_{\mathbb{R}^2} |\nabla V_{N,r}|^2$, and a negative part corresponding to the large N limit of the nearest-neighbor contributions $\sum_{i=1}^N \log r(x_i) + \sum_{i=1}^N \log r(y_i)$.

Interpretation of Theorem 1. The way to read our result at the microscopic scale is to say that the Gibbs measure must concentrate on minimizers of $\overline{\mathcal{F}}_\beta$. Unfortunately, we do not know whether a minimizer can be shown to be unique, but in any case it reduces to a minimization problem (with the structure of a *free energy* as in statistical physics) which should identify some optimal random signed point processes. For comparison, in the two-dimensional one-component case, the analogous result allows to say that the law of the well-known Ginibre point process minimizes the rate function for a certain value of β. We may interpret the minimization of $\overline{\mathcal{F}}_\beta$ heuristically as follows. The term $\overline{\mathbb{W}}(\bar{P})$ in (1.9) favors signed configurations which minimize the logarithmic interaction, hence we expect that it favors short dipoles (and as such, is clearly not bounded below). On the contrary, the *specific relative entropy* in (1.9) favors disorder, and thus tend to "separate" the dipole points. When $\beta < 2$, the sum of the two terms can be shown to be bounded below. The competition between the two terms depends of course on the value of β. When β is small (i.e. the temperature is large) then the entropy term (or "thermal agitation") dominates, whereas as β gets larger and approaches 2 the dipole attraction gets stronger, until the system can no longer sustain (or spontaneously generate) dipoles.

Plan of the paper. The rest of the paper is organized as follows: in Section 2, we gather all the definitions and notation that we use, and we present the rewriting of the energy which isolates the dipole contribution. In Section 3 we use the results of [GP77] to prove that $\overline{\mathcal{F}}_\beta^{\mathrm{sc}}$ is a good rate function and we establish the main result, postponing some of the main proofs. In Section 4 we recall, for the reader's convenience, the main computations of [GP77] (on which we rely heavily). In Section 5 we prove the LDP lower bound, and in Section 6 we prove the LDP upper bound. Section 7 is devoted to the proof of the large deviations principle for the empirical measures (μ_N^+, μ_N^-), at the leading order speed (which is N^2).

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2. Definitions, notation, and preliminary results

2.1. **General notation.** If X is a topological space we denote by $\mathcal{P}(X)$ the set of Borel probability measures on X, and if P is in $\mathcal{P}(X)$ we denote by $\mathbf{E}_P[\cdot]$ the expectation under P. We endow the space $\mathcal{P}(X)$ of Borel probability measures on X with the Dudley distance:

$$d_{\mathcal{P}(X)}(P_1, P_2) = \sup \left\{ \int F(dP_1 - dP_2) |\ F \in \mathrm{Lip}_1(X) \right\} \tag{2.1}$$

where $\mathrm{Lip}_1(X)$ denotes the set of functions $F : X \to \mathbb{R}$ that are 1-Lipschitz with respect to d_X and such that $\|F\|_\infty \leq 1$. It is well-known that the distance $d_{\mathcal{P}(X)}$ metrizes the topology of weak convergence on $\mathcal{P}(X)$. We denote by $C^0(X)$ (resp. $C^0_c(X)$) the space of continuous functions (resp. with compact support on X), and by $C^0_b(X)$ the set of continuous, bounded functions on X.

If $R > 0$ we let $C_R := [-R/2, R/2]^2$ be a square of center 0 and sidelength R. If U is a Borel subset of $\mathbb{R}^2$ we denote by $|U|$ its Lebesgue measure (or area). If (X, d_X) is a metric space, $x \in X$ and $r > 0$ we denote by $B(x, r)$ the closed ball of center x and radius r for d_X. In $\mathbb{R}^2$ we will use the notation $D(x, r)$ for the closed disk of center x and radius r. If $A \subset X$ we denote by $\mathring{A}$ its interior and by $\bar{A}$ its closure.

(Signed) point configurations. If A is a Borel set of $\mathbb{R}^2$ we denote by $\mathcal{X}^0(A)$ the set of locally finite point configurations in A or equivalently the set of non-negative, purely atomic Radon measures on A giving an integer mass to singletons (see [DVJ88]). We let $\mathcal{X}^0 := \mathcal{X}^0(\mathbb{R}^2)$. We endow the sets $\mathcal{X}^0(A)$ (for A Borel) with the topology induced by the topology of weak convergence of Radon measure (also known as vague convergence or convergence against compactly supported continuous functions).

If B is a compact subset of $\mathbb{R}^2$ we endow $\mathcal{X}^0(B)$ with the following distance:

$$d_{\mathcal{X}^0(B)}(\mathcal{C}_1, \mathcal{C}_2) := \sup \left\{ \int F(d\mathcal{C}_1 - d\mathcal{C}_2) |\ F \in \mathrm{Lip}_1(B) \right\}. \tag{2.2}$$

Similarly we endow $\mathcal{X}^0 := \mathcal{X}^0(\mathbb{R}^2)$ with the following distance:

$$d_{\mathcal{X}^0}(\mathcal{C}_1, \mathcal{C}_2) := \sum_{k \geq 1} \frac{1}{2^k} \left(\frac{d_{\mathcal{X}^0(C_k)}(\mathcal{C}_1, \mathcal{C}_2)}{(\mathcal{C}_1(C_k) + \mathcal{C}_2(C_k)) \vee 1} \right). \tag{2.3}$$

We now define the analogue of $\mathcal{X}^0(A)$ and $\mathcal{X}^0$ in the setting of signed point configuration.

Definition 2.1. *A signed point configuration in $\mathbb{R}^2$ (resp. in A) is defined as an element $\mathcal{C} = (\mathcal{C}^+, \mathcal{C}^-)$ of $\mathcal{X} := \mathcal{X}^0 \times \mathcal{X}^0$ (resp. $\mathcal{X}(A) := \mathcal{X}^0(A) \times \mathcal{X}^0(A)$). We say that $\mathcal{C}$ is simple if the mass of each singleton is exactly one and the supports of $\mathcal{C}^+$ and $\mathcal{C}^-$ are disjoint.*

We endow these product spaces with the product topology and the usual 1-product metric (the sum of distances componentwise). We will sometimes abuse notation and write $\int f d\mathcal{C}$ as the integral of a test function f against the signed measure $d\mathcal{C}^+ - d\mathcal{C}^-$. For $A \subset \mathbb{R}^2$ a measurable subset we let $|\mathcal{C}|(A)$ be the total number of points in A i.e. $|\mathcal{C}|(A) := \mathcal{C}^+(A) + \mathcal{C}^-(A)$.

Definition 2.2. *We define the "pruning" map $\mathrm{Pr} : \mathcal{X} \to \mathcal{X}$ by associating to any signed point configuration $\mathcal{C} = (\mathcal{C}^+, \mathcal{C}^-)$ the Jordan decomposition of the signed measure $\mathcal{C}^+ - \mathcal{C}^-$.*

The effect of Pr is to remove any dipole which would not be felt at the level of the signed measure. For example, if $\mathcal{C} = (\delta_{x_0} + \delta_{x_1}, \delta_{x_0} + \delta_{x_2})$ with $x_i \in \mathbb{R}^2$ distinct, we have $\mathrm{Pr}(\mathcal{C}) = (\delta_{x_1}, \delta_{x_2})$.

The additive group $\mathbb{R}^2$ acts on $\mathcal{X}^0$ by translations $\{\theta_t\}_{t\in\mathbb{R}^2}$: if $\mathcal{C} = \{x_i, i \in I\} \in \mathcal{X}^0$ we let

$$\theta_t \cdot \mathcal{C} := \{x_i - t, i \in I\}. \tag{2.4}$$

We extend this action (while keeping the same notation) to the setting of signed configurations by acting on both components $\mathcal{C}^+$ and $\mathcal{C}^-$ of a given $\mathcal{C} \in \mathcal{X}(\mathbb{R}^2)$.

For any integer N we identify a configuration $\mathcal{C}$ with N points with an unordered N-tuples of points in $\mathbb{R}^2$, which we still denote by $\mathcal{C}$. Denoting by π_N the projection from $(\mathbb{R}^2)^N$ to unordered N-tuples in $\mathbb{R}^2$, for a set A of configurations with N points we write $\mathbf{Leb}^{\otimes N}(A) = \mathbf{Leb}^{\otimes N}(\pi_N^{-1}(A))$.

Random tagged signed point configurations. The space $\mathcal{P}(\Lambda \times \mathcal{X})$ can be viewed as the space of laws of tagged signed point configurations, where we keep as a tag the point $x \in \Lambda$ around which the signed configuration was blown up. It is equipped with the topology of weak convergence of measures on $\Lambda \times \mathcal{X}$. Throughout we will always consider the subset

$$\left\{\bar{P} \in \mathcal{P}(\Lambda \times \mathcal{X}), \bar{P}(A \times \mathcal{X}) = \mathbf{Leb}(A), \forall A \text{ Borel}\right\},$$

and continue, with some abuse of notation, to denote it by $\mathcal{P}(\Lambda \times \mathcal{X})$. This assumption allows us to consider the disintegration probability measures $\bar{P}^x \in \mathcal{P}(\mathcal{X})$ for any $x \in \Lambda$, which satisfy by definition

$$\int F(x, \mathcal{C}) d\bar{P}(x, \mathcal{C}) = \int_\Lambda \left(\int F(x, \mathcal{C}) d\bar{P}^x(\mathcal{C}) \right) dx,$$

for any function $F \in C^0_b(\Lambda \times \mathcal{X})$. We refer to [AGS05, Section 5.3] for a proof of the existence of disintegration measures.

. Intensity. To any $P \in \mathcal{P}(\mathcal{X})$ we may associate two probability measures P^+, P^- on $\mathcal{X}^0$ as the push-forwards of P by the two canonical projections of $\mathcal{X}$ on $\mathcal{X}^0$, namely $(\mathcal{C}^+, \mathcal{C}^-) \mapsto \mathcal{C}^+$ and $(\mathcal{C}^+, \mathcal{C}^-) \mapsto \mathcal{C}^-$.

Let $P \in \mathcal{P}(\mathcal{X}^0)$. If there exists a measurable function $\rho_{1,P}$ such that for any function $\varphi \in C^0_c(\mathbb{R}^2)$ we have

$$\mathbf{E}_P \left[\sum_{x \in \mathcal{C}} \varphi(x) \right] = \int_{\mathbb{R}^2} \rho_{1,P}(x) \varphi(x) dx, \tag{2.5}$$

then we say that $\rho_{1,P}$ is the one-point correlation function (or *intensity*) of P. For $m \geq 0$ we say that P is of intensity m when the function $\rho_{1,P}$ of (2.5) exists and satisfies $\rho_{1,P} \equiv m$.

When $\bar{P} \in \mathcal{P}(\Lambda \times \mathcal{X}^0)$ we let $\rho_{\bar{P}}$ be the intensity measure of $\bar{P}$ defined by $\rho_{\bar{P}}(x) := \rho_{1,\bar{P}^x}$. If $\bar{P} \in \mathcal{P}(\Lambda \times \mathcal{X})$ we let $\rho^+_{\bar{P}}$, $\rho^-_{\bar{P}}$ be the respective intensity measure of $\bar{P}^+$, $\bar{P}^-$. We denote by $\mathcal{P}_{\mathrm{inv},1}(\Lambda \times \mathcal{X})$ the set of all $\bar{P} \in \mathcal{P}_{\mathrm{inv}}(\Lambda \times \mathcal{X})$ such that

$$\int_\Lambda \rho^+_{\bar{P}} = \int_\Lambda \rho^-_{\bar{P}} = 1.$$

2.2. **Rewriting the interaction energy.** Here we adapt computations from [RS15,PS15] to rewrite the interaction energy in terms of the electric potential generated by the points, seen as charges (this is also the analogue of the "electrostatic inequality" of [GP77]). Comparing with [RS15, PS15], instead of using a fixed (small) truncation distance we use the nearest neighbor distance.

Truncation of the logarithmic interaction. Following [PS15] we define $\delta_p^{(\eta)}$ to be the normalized surface measure on $\partial D(p,\eta)$ (it coincides with the Dirac mass at p if $\eta = 0$). We will also need the notion of truncated logarithmic kernel defined for $1 > \eta > 0$ and $x \in \mathbb{R}^2$ by

$$f_\eta(x) = (-\log|x| - \log(\eta))_+\,, \tag{2.6}$$

and by $f_\eta \equiv 0$ if $\eta = 0$. We note that the function f_η vanishes outside the disk $D(0,\eta)$ and satisfies that

$$\frac{1}{2\pi}\mathrm{div}\,(\nabla f_\eta) + \delta_0 = \delta_0^{(\eta)}. \tag{2.7}$$

Nearest-neighbor distance. If $\vec{X}_N, \vec{Y}_N$ are two N-tuples of points, we define the nearest-neighbor (half-)distance as

$$r(x_i) = \min\left(1, \frac{1}{2}\min_{j\neq i}|x_i - x_j|, \frac{1}{2}\min_{j}|x_i - y_j|\right), \tag{2.8}$$

for any $i = 1,\dots,N$, and similarly for $r(y_i)$.
Rewriting of the energy functional. Let $\vec{X}_N, \vec{Y}_N$ be two N-tuples of points in Λ. We let $V_{N,r}$ be the electric potential generated by $\mathcal{C}$, after “smearing out” the charges on a distance r. More precisely,

$$V_{N,r} := 2\pi(-\Delta)^{-1}\left(\sum_{i=1}^N \delta_{x_i}^{(r(x_i))} - \sum_{i=1}^N \delta_{y_i}^{(r(y_i))}\right), \tag{2.9}$$

where $2\pi(-\Delta)^{-1}$ is the convolution by log. By the properties of $\delta_p^{(\eta)}$, this is the same as setting

$$V_{N,r}(x) = \sum_{i=1}^N \left(-\log|x - x_i| - f_{r(x_i)}(x - x_i)\right) + \sum_{i=1}^N \left(\log|x - y_i| + f_{r(y_i)}(x - y_i)\right). \tag{2.10}$$

We also write

$$V_{N,0} := \sum_{i=1}^N \left(-\log|x - x_i| + \log|x - y_i|\right). \tag{2.11}$$

The next crucial identity expresses the fact that the interaction energy $w_N(\vec{X}_N, \vec{Y}_N)$ can be computed using the electric potential $V_{N,r}$ (more precisely its gradient, the truncated electric field) even if the interaction has been truncated at distance r.

Lemma 2.3. *Let $\vec{X}_N, \vec{Y}_N$ be such that the associated global point configuration $\mathcal{C}$ is simple. Then,*

$$w_N(\vec{X}_N, \vec{Y}_N) = \frac{1}{2\pi}\int_{\mathbb{R}^2}|\nabla V_{N,r}|^2 + \sum_{i=1}^N \log r(x_i) + \sum_{i=1}^N \log r(y_i). \tag{2.12}$$

Proof. This can be seen as a simple application of Newton's theorem. Since the system is globally neutral (there are N positive charges and N negative charges), the electric potential

$V_{N,r}$ decays like $1/|x|$ as $|x| \to \infty$, and $\nabla V_{N,r}$ decays like $1/|x|^2$. We may thus integrate by parts and find that the boundary terms tend to zero, and therefore, using (2.10), we obtain

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} |\nabla V_{N,r}|^2 = \int_{\mathbb{R}^2} -\frac{1}{2\pi}\Delta V_{N,r} V_{N,r} = \sum_{i=1}^N \int_{\mathbb{R}^2} V_{N,r}\left(\delta_{x_i}^{(r(x_i))} - \delta_{y_i}^{(r(y_i))}\right)$$
$$= \sum_{i,j=1}^N \Big(-\log|x-x_j| - f_{r(x_j)}(x-x_j) + \log|x-y_j| + f_{r(y_j)}(x-y_j)\Big)\left(\delta_{x_i}^{(r(x_i))} - \delta_{y_i}^{(r(y_i))}\right).$$

Next, we use the fact that the disks $D(x_i, r(x_i))$ and $D(y_i, r(y_i))$ are disjoint by the definition of r, and that for any p, η, the measure $\delta_p^{(\eta)}$ is supported on $\partial D(p,\eta)$ where $f_\eta(x-p)$ vanishes, to obtain

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} |\nabla V_{N,r}|^2 = \sum_{i\neq j}\left(-\log|x-x_j|\delta_{x_i}^{(r(x_i))} - \log|x-y_j|\delta_{y_i}^{(r(y_i))}\right)$$
$$+ \sum_{i,j}\left(\log|x-x_j|\delta_{y_i}^{(r(y_i))} + \log|x-y_j|\delta_{x_i}^{(r(x_i))}\right) - \sum_{i=1}^N (\log r(x_i) + \log r(y_i)). \tag{2.13}$$

In addition, by Newton's theorem (or by the mean-value property), the average of $\log|x-p|$ over any disk $D(q,\eta)$ not containing p is $\log|x-q|$. Since $\delta_p^{(\eta)}$ is precisely the uniform measure of mass 1 on $\partial D(p,\eta)$, and using again the fact that the disks $D(x_i, r(x_i))$ and $D(y_i, r(y_i))$ are disjoint, we conclude that

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} |\nabla V_{N,r}|^2 = \sum_{i\neq j}(-\log|x_i-x_j| - \log|y_i-y_j|)$$
$$+ \sum_{i,j}(\log|y_i-x_j| + \log|x_i-y_j|) - \sum_{i=1}^N(\log r(x_i) + \log r(y_i)), \tag{2.14}$$

which is the desired result. □

2.3. **Blow-up coordinates.** In view of Lemma 2.3, and since the finite point configurations $\vec{X}_N, \vec{Y}_N$ are simple $\mathbb{P}_N^\beta$-a.s., we may rewrite the Gibbs measure as the probability measure whose density with respect to the Lebesgue measure on Λ^{2N} is given by

$$\frac{1}{Z_{N,\beta}}\exp\left(-\frac{\beta}{2}\left(\frac{1}{2\pi}\int_{\mathbb{R}^2}|\nabla V_{N,r}|^2 + \sum_{i=1}^N \log r(x_i) + \log r(y_i)\right)\right). \tag{2.15}$$

We rescale the finite configurations by a factor $\sqrt{N}$ and use a prime symbol for the new quantities. In particular we let $\vec{X}'_N = (x'_1, \dots, x'_N)$ with $x'_i = \sqrt{N}x_i$ (and the same for $\vec{Y}'_N$), and we have of course $r(x'_i) = \sqrt{N}r(x_i)$. We let $V'_{N,r}$ be the electric potential generated by the rescaled point configuration after truncation

$$V'_{N,r} := V_{N,r}\left(\frac{\cdot}{\sqrt{N}}\right),$$

and

$$V'_N := V_{N,0}\left(\frac{\cdot}{\sqrt{N}}\right). \tag{2.16}$$

We have, by a change of variables

$$\int_{\mathbb{R}^2} |\nabla V'_{N,r}|^2 = \int_{\mathbb{R}^2} |\nabla V_{N,r}|^2,$$

whereas the nearest-neigbor distance term scales as

$$\sum_{i=1}^N \log r(x_i) + \log r(y_i) = \sum_{i=1}^N \log r'(x_i) + \log r'(y_i) + N \log N.$$

Combining these identities with (1.4) and (2.15) we may write the Gibbs measure as the probability measure whose density with respect to the Lebesgue measure on Λ^{2N} is given by

$$\frac{1}{K_{N,\beta}} \exp\left(-\frac{\beta}{2}\left(\frac{1}{2\pi}\int_{\mathbb{R}^2} |\nabla V'_{N,r}|^2 + \sum_{i=1}^N \log r(x'_i) + \log r(y'_i)\right)\right), \tag{2.17}$$

where the new normalizing constant $K_{N,\beta}$ satisfies $\log K_{N,\beta} = C_\beta N + o(N)$, with C_β as in (1.4).

The computations in the last two subsections serve as a motivation for the upcoming definition of the interaction energy for the infinite configurations which arise after taking the limit $N \to \infty$. We note in particular that $V_{N,r}$ solves

$$-\Delta V_{N,r} = 2\pi \left(\sum_{i=1}^N \delta_{x_i}^{(r(x_i))} - \sum_{i=1}^N \delta_{y_i}^{(r(y_i))}\right),$$

a relation which will pass to the limit (in the sense of distributions) as $N \to \infty$. The electric field associated to the potential $V_{N,r}$ is $\nabla V_{N,r}$ and its divergence is $\Delta V_{N,r}$.

2.4. Interaction energy for signed configurations.

Electric fields and electric processes. We may thus define the class Elec of "electric" vector fields on $\mathbb{R}^2$ to be the set of vector fields E which belong to $L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ for some $p < 2$ and satisfy

$$-\operatorname{div} E = 2\pi(\mathcal{C}^+ - \mathcal{C}^-) \quad \text{in } \mathbb{R}^2 \tag{2.18}$$

for some signed point configuration $\mathcal{C} = (\mathcal{C}^+, \mathcal{C}^-)$. When E satisfies (2.18) we write $E \in \mathsf{Elec}(\mathcal{C})$ and say that E is *compatible* with $\mathcal{C}$. We note that two elements of $\mathsf{Elec}(\mathcal{C})$ differ by a divergence-free vector field. Unless stated otherwise, we endow the space $L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ with the weak topology. If $E \in \mathsf{Elec}$ we let $\mathrm{Conf}(E)$ be the underlying signed point configuration i.e. the signed point configuration $\mathcal{C} = (\mathcal{C}^+, \mathcal{C}^-)$ where $(\mathcal{C}^+, \mathcal{C}^-)$ is the Jordan decomposition of $\frac{-1}{2\pi}\operatorname{div} E$. In particular, if $\mathcal{C}$ is a signed point configuration with $\mathcal{C} = \mathrm{Pr}(\mathcal{C})$ (the latter is implied if $\mathcal{C}$ is simple), and such that $E \in \mathsf{Elec}(\mathcal{C})$, then $\mathcal{C} = \mathrm{Conf}(E)$.

We define an electric field law as an element of $\mathcal{P}(L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2))$ where $p < 2$, concentrated on Elec. It will usually be denoted by P^{elec}. We say that P^{elec} is stationary when it is invariant under the (push-forward by) translations $\theta_x \cdot E = E(\cdot - x)$ for any $x \in \mathbb{R}^2$. A tagged electric field law is an element of $\mathcal{P}(\Lambda \times L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2))$ whose first marginal is the normalized Lebesgue measure on Λ and whose disintegration slices are electric field laws. It will be denoted by $\bar{P}^{\text{elec}}$. We say that $\bar{P}^{\text{elec}}$ is stationary if for a.e. $z \in \Lambda$, the disintegration measure $\bar{P}^{\text{elec},z}$ is stationary.

Nearest-neighbor distance and truncation. If $\mathcal{C} = (\mathcal{C}^+, \mathcal{C}^-)$ is a signed point configuration we define the nearest-neighbor (half-)distance as

$$r(x) = \min\left(1, \frac{1}{2} \min_{x' \in \mathcal{C}^+, x' \neq x} |x - x'|, \frac{1}{2} \min_{y \in \mathcal{C}^-} |x - y|\right), \tag{2.19}$$

for any $x \in \mathcal{C}^+$ and we define similarly $r(y)$ for $y \in \mathcal{C}^-$.

For any electric field E we define its truncation

$$E_r := E - \sum_{p \in \mathcal{C}^+} \nabla f_{r(p)}(x - p) + \sum_{p \in \mathcal{C}^-} \nabla f_{r(p)}(x - p), \tag{2.20}$$

where $\mathcal{C} = \mathrm{Conf}(E)$ and r is defined above, computed with respect to $\mathrm{Conf}(E)$. This is the "infinite configuration" equivalent of the truncated electric field $\nabla V_{N,r}$ as in (2.12). We may now define the interaction energy of an admissible electric field in a similar fashion as what arises in Lemma 2.3.

A control on the L^2-norm of E_r can be translated into a bound of the L^p-norm on E as follows.

Lemma 2.4. *Let $\mathcal{C}$ be a point configuration, let $E \in \mathsf{Elec}(\mathcal{C})$ and let $R > 0$. We have*

$$\|E\|_{L^p(C_R)} \leq L_1 \|E_r\|_{L^2(C_R)} + L_2 |\mathcal{C}|(C_{R+1}) \tag{2.21}$$

for some $L_1 > 0$ depending only on R and p and some universal constant L_2.

Proof. From the definition (2.20) we get

$$|E| \leq |E_r| + \sum_{q \in \mathcal{C}^+} |\nabla f_{r(q)}(x - q)| + \sum_{q \in \mathcal{C}^-} |\nabla f_{r(q)}(x - q)|,$$

and (2.21) follows by using the triangle inequality for the L^p-norm, Hölder's inequality which yields $\|E_r\|_{L^p(C_R)} \leq L_1 \|E_r\|_{L^2(C_R)}$ for some L_1 depending on R and p, and by observing that the terms $\|\nabla f_{r(q)}\|_{L^p}$ are uniformly bounded by some $L_2 > 0$ and that the number of such terms is bounded by the total number of points of $\mathcal{C}$ in C_{R+1}. □

2.4.1. *Positive part of the energy.* For any $E \in \mathsf{Elec}$ we define $\mathcal{W}^o(E)$ as

$$\mathcal{W}^o(E) := \limsup_{R \to \infty} \frac{1}{R^2} \int_{C_R} |E_r|^2, \tag{2.22}$$

and we call it the "positive part" of the energy of E. Recall that the truncation E_r of E at nearest-neighbor distance is defined with respect to the "minimal" underlying signed point configuration $\mathrm{Conf}(E)$.

Next, if $\mathcal{C}$ is a signed point configuration we let $\mathbb{W}^o(\mathcal{C})$ be the infimum of $\mathcal{W}^o(E)$ over the electric fields E compatible with $\mathcal{C}$

$$\mathbb{W}^o(\mathcal{C}) := \inf \{\mathcal{W}^o(E), E \in \mathsf{Elec}(\mathcal{C})\}. \tag{2.23}$$

We emphasize that the definition of $\mathbb{W}^o(\mathcal{C})$ proceeds by considering the energy of associated electric fields, and that the truncation of an electric field is defined with respect to the "minimal" underlying signed point configuration $\mathrm{Conf}(E)$. In particular if $\mathrm{Pr}(\mathcal{C}_1) = \mathrm{Pr}(\mathcal{C}_2)$ then $\mathbb{W}^o(\mathcal{C}_1) = \mathbb{W}^o(\mathcal{C}_2) = \mathbb{W}^o(\mathrm{Pr}(\mathcal{C}_1))$, where Pr is the pruning map of Definition 2.2.

2.4.2. *Negative part of the energy.* If $\chi : \mathbb{R}^2 \to \mathbb{R}$ is a nonnegative measurable function with compact support and $\mathcal{C}$ a signed point configuration we let

$$\mathbb{W}^\star(\chi, \mathcal{C}) := -\int \chi(x) \log(r(x))(d\mathcal{C}^+ + d\mathcal{C}^-)(x)$$
$$\mathbb{W}^\star(\mathcal{C}) := \limsup_{R\to\infty} \frac{1}{R^2} \mathbb{W}^\star(\mathbf{1}_{C_R}, \mathcal{C}).$$

Similarly, for any $0 < \tau < 1$, we let

$$\mathbb{W}^\star_\tau(\chi, \mathcal{C}) := -\int \chi \log(r(x) \vee \tau)(d\mathcal{C}^+ + d\mathcal{C}^-)(x)$$
$$\mathbb{W}^\star_\tau(\mathcal{C}) := \limsup_{R\to\infty} \frac{1}{R^2} \mathbb{W}^\star_\tau(\mathbf{1}_{C_R}, \mathcal{C}).$$

The function $\mathbb{W}^\star$ is a non-negative quantity, corresponding to the expected "dipole contribution" to the energy. We will call $-\mathbb{W}^\star$ the negative part of the energy. It can be obtained as the monotone limit of $\mathbb{W}^\star_\tau(\mathcal{C})$.

We could now try to define the interaction energy of a signed point configuration as the difference $\mathbb{W}^o(\mathcal{C}) - \mathbb{W}^\star(\mathcal{C})$. However it turns out that for good definition it is preferable to consider such an object at the level of signed point processes, as we do below.

2.4.3. *A compatibility lemma.*

Lemma 2.5. *Let $\{E_N\}_N$ be a sequence of electric fields, let $E \in \mathsf{Elec}$ and let $\mathcal{C} \in \mathcal{X}$. Assume that*

(1) $\{E^{(N)}\}_N$ converges to E weakly in $L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$.

(2) $\{\mathrm{Conf}(E^{(N)})\}_N$ converges to $\mathcal{C}$ in $\mathcal{X}$.

Then $\mathrm{Conf}(E) = \mathrm{Pr}(\mathcal{C})$*, and in particular E is an electric field. Moreover,*

$$\liminf_{N\to\infty} \int \chi |E_r^{(N)}|^2 \geq \int \chi |E_r|^2 \tag{2.24}$$

for any smooth, compactly supported, nonnegative function χ.

Proof. Let B_R be the ball of radius R, and let $f \in C^0_b(W^{-1,p}(B_R))$. One may check that $(C^0_0(B_R))^*$ embeds continuously into $W^{-1,p}(B_R)$ (indeed, $W^{-1,p}(B_R)$ is the dual of the Sobolev space $W^{1,q}_0(B_R)$ where q is the conjugate exponent to p, and the latter embeds continuously into $C^0_0(B_R)$ since $q > 2$). It thus follows that f is also bounded and continuous on $(C^0_0(B_R))^*$. The convergence of $\mathrm{Conf}(E^{(N)})$ to $\mathcal{C}$ and the fact that $\mathcal{C}$ is locally finite thus imply that

$$\lim_{N\to\infty} f\left(-\frac{1}{2\pi}\mathrm{div}\, E^{(N)}\right) = f(\mathcal{C}^+ - \mathcal{C}^-), \quad \forall f \in C^0_b(W^{-1,p}(B_R)). \tag{2.25}$$

Since the function $E \mapsto -\frac{1}{2\pi}\mathrm{div}\, E$ is continuous from $L^p_{\mathrm{loc}}(B_R)$ to $W^{-1,p}(B_R)$, we may use the first assumption to get

$$\lim_{N\to\infty} f\left(-\frac{1}{2\pi}\mathrm{div}\, E^{(N)}\right) = f\left(-\frac{1}{2\pi}\mathrm{div}\, E\right). \tag{2.26}$$

Since this is true for all $f \in C^0_b(W^{-1,p}(B_R))$ and for all $R > 1$, we conclude that $-\frac{1}{2\pi}\mathrm{div}\, E = \mathcal{C}^+ - \mathcal{C}^-$ and thus $\mathrm{Conf}(E) = \mathrm{Pr}(\mathcal{C})$.

We next prove (2.24). We may assume that the left-hand side is finite, otherwise there is nothing to prove. This implies that, up to an extraction of a subsequence, $\sqrt{\chi}E_r^{(N)}$ converges weakly in $L^2(\mathbb{R}^2, \mathbb{R}^2)$ to some vector field, which we claim can only be $\sqrt{\chi}E_{r'}$ where r' denotes the nearest-neighbour distance computed in $\mathcal{C}$ (and not in $\mathrm{Conf}(E)$). Indeed, it suffices to check that $\sqrt{\chi}E_r^{(N)}$ converges to $\sqrt{\chi}E_{r'}$ weakly in L^p. In view of the first assumption and of (2.20), it suffices to check that

$$\sum_{p\in\mathcal{C}^{(N),+}} \nabla f_{r(p)}(x-p) - \sum_{q\in\mathcal{C}^{(N),-}} \nabla f_{r(q)}(x-q) \rightharpoonup \sum_{p\in\mathcal{C}^{+}} \nabla f_{r(p)}(x-p) - \sum_{q\in\mathcal{C}^{-}} \nabla f_{r(q)}(x-q)$$

weakly in $L^p_{\mathrm{loc}}(\mathbb{R}^2)$, where $\mathcal{C}^{(N)} = (\mathcal{C}^{(N),+}, \mathcal{C}^{(N),-}) := \mathrm{Conf}(E^{(N)})$. But from (2.6), we have $\nabla f_\eta(x-p) = -\frac{x-p}{|x-p|^2}\mathbf{1}_{|x-p|<\eta}$ (and 0 if $\eta = 0$), which is continuous in L^p with respect to both p and η. So, the stated convergence follows from the second assumption and the definition of r', using also the fact that the point configurations are locally finite. We conclude that $\sqrt{\chi}E_r^{(N)}$ converges weakly in $L^2(\mathbb{R}^2)$ as claimed, so by the lower semi-continuity of the L^2 norm,

$$\liminf_{N\to\infty} \int \chi|E_r^{(N)}|^2 \geq \int \chi|E_{r'}|^2.$$

We may finally observe that since $r' < r$ (the nearest neighbor distances are smaller in $\mathcal{C}$ than in $\mathrm{Pr}(\mathcal{C})$) we have $|E_{r'}|^2 \geq |E_r|^2$ pointwise, which concludes the proof. □

At the level of laws of electric fields, the result of Lemma 2.5 translates into

Lemma 2.6. *Let $\{P_N\}_N$ be a sequence of random signed point processes and $\{P_N^{\mathrm{elec}}\}_N$ be a sequence of laws of electric fields. Let P be a random signed point process and P^{elec} be a law of electric fields. Let us assume that:*

1. *For any $N \geq 1$, the push-forward of P_N^{elec} by* Conf *coincides with P_N.*
2. *The sequence $\{P_N\}_N$ converges to P as $N \to \infty$ in $\mathcal{P}(\mathcal{X})$.*
3. *The sequence $\{P_N^{\mathrm{elec}}\}_N$ converges to P^{elec} as $N \to \infty$ in $\mathcal{P}(L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2))$.*

Then we have

1. *The push-forward of P^{elec} by $E \mapsto -\frac{1}{2\pi}\mathrm{div}\, E$ is concentrated on signed measures of the type $\mathcal{C}^+ - \mathcal{C}^-$ for some $(\mathcal{C}^+, \mathcal{C}^-)$ in $\mathcal{X}$. In particular P^{elec} is concentrated on* Elec.
2. *The push-forward of P^{elec} by* Conf *coincides with the push-forward of P by* Pr.

Moreover for any smooth, compactly supported, nonnegative function χ we have

$$\liminf_{N\to\infty} \mathbf{E}_{P_N^{\mathrm{elec}}}\left[\int \chi|E_r|^2\right] \geq \mathbf{E}_{P^{\mathrm{elec}}}\left[\int \chi|E_r|^2\right]. \tag{2.27}$$

2.5. **Process level energy.**

Energy of signed point processes. Let P (resp. $\bar{P}$) be the law of a signed point process (resp. of a tagged signed point process). We define

$$\widetilde{\mathbb{W}}^o(P) := \mathbf{E}_P\left[\mathbb{W}^o(\mathcal{C})\right], \quad \overline{\mathbb{W}}^o(\bar{P}) := \mathbf{E}_{\bar{P}}\left[\mathbb{W}^o(\mathcal{C})\right] \text{ (positive part of the energy)}$$

$$\widetilde{\mathbb{W}}^\star(P) := \mathbf{E}_P\left[\mathbb{W}^\star(\mathcal{C})\right], \quad \overline{\mathbb{W}}^\star(\bar{P}) := \mathbf{E}_{\bar{P}}\left[\mathbb{W}^\star(\mathcal{C})\right] \text{ (contribution of dipoles)}$$

$$\widetilde{\mathbb{W}}^\star_\tau(P) := \mathbf{E}_P\left[\mathbb{W}^\star_\tau(\mathcal{C})\right], \quad \overline{\mathbb{W}}^\star_\tau(\bar{P}) := \mathbf{E}_{\bar{P}}\left[\mathbb{W}^\star_\tau(\mathcal{C})\right] \text{ (dipoles at truncated distance} \geq \tau).$$

Finally we define the interaction energy of P (resp. $\bar{P}$)

$$\widetilde{\mathbb{W}}(P) := \widetilde{\mathbb{W}}^o(P) - \widetilde{\mathbb{W}}^\star(P) \text{ and } \overline{\mathbb{W}}(\bar{P}) := \overline{\mathbb{W}}^o(\bar{P}) - \overline{\mathbb{W}}^\star(\bar{P}). \tag{2.28}$$

It is yet unclear whether the right-hand sides in (2.28) have an actual meaning, because it could be the difference of two infinite quantities. However we will see in Section 3 that for a certain class of point processes, which are the only candidates for describing the microscopic behavior, the quantity $\widetilde{\mathbb{W}}^{\star}(\bar{P})$ is in fact finite.

Stationary lifting with minimal energy. The following useful lemma shows that we may associate to any stationary tagged point process the law of a *stationary* tagged electric field which is compatible with it and whose energy is minimal.

Lemma 2.7. *Assume $\bar{P} \in \mathcal{P}_{\mathrm{inv}}(\Lambda \times \mathcal{X})$ is such that $\overline{\mathbb{W}}^o(\bar{P})$ is finite, and such that signed point configuration are $\bar{P}$-a.s. simple. Then there exists a law $\bar{P}^{\mathrm{elec}}$ of tagged electric fields such that*

(1) *The push-forward of $\bar{P}^{\mathrm{elec}}$ by $(z, E) \mapsto (z, \mathrm{Conf}(E))$ is $\bar{P}$.*
(2) *We have*

$$\overline{\mathbb{W}}^o(\bar{P}) = \mathbf{E}_{\bar{P}^{\mathrm{elec}}}\left[\mathcal{W}^o\right]. \tag{2.29}$$

Proof. The proof of Lemma 2.7 is very similar to that of [LS15, Lemma 2.12] and we only sketch it here. For simplicity we present the argument in the non-tagged case, the extension to random tagged signed point processes being straightforward. Let $\varepsilon > 0$ be fixed. For any simple signed point configuration $\mathcal{C}$ such that $\mathbb{W}^o(\mathcal{C})$ is finite, by definition of $\mathbb{W}^o(\mathcal{C})$ we may find an electric field $E^{(\mathcal{C},\varepsilon)}$ such that $\mathcal{W}^o(E^{(\mathcal{C},\varepsilon)}) \le \mathbb{W}^o(\mathcal{C}) + \varepsilon$. For any $k \ge 1$ we let

$$P^{\mathrm{elec}}_{(\mathcal{C},\varepsilon,k)} := \frac{1}{|C_k|}\int_{C_k} \delta_{\theta_x \cdot E^{(\mathcal{C},\varepsilon)}}\, dx,$$

which is an electric field law (as defined in Section 2.4). For any $m \ge 1$ we have (using Fubini's theorem)

$$\int_{C_m} |E_r|^2 dP^{\mathrm{elec}}_{(\mathcal{C},\varepsilon,k)}(E) \le \frac{1}{|C_k|}\int_{C_{m+k}} |E_r^{(\mathcal{C},\varepsilon)}|^2,$$

and the right-hand side is bounded as $k \to \infty$ by the finiteness of $\mathcal{W}^o(E^{(\mathcal{C},\varepsilon)})$. Using Lemma 2.4 and a standard compactness result in L^p-spaces, it follows that the sequence $\{P^{\mathrm{elec}}_{(\mathcal{C},\varepsilon,k)}\}_k$ is tight in $\mathcal{P}(L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2))$ (for the weak L^p_{loc} topology). We let $P^{\mathrm{elec}}_{(\mathcal{C},\varepsilon)}$ be a limit point as $k \to \infty$. Set

$$P_{(\mathcal{C},k)} := \frac{1}{|C_k|}\int_{C_k} \delta_{\theta_x \cdot \mathcal{C}}\, dx$$

Since the signed point configurations are simple P-a.s. we see that $P_{(\mathcal{C},k)}$ is the push-forward of $P^{\mathrm{elec}}_{(\mathcal{C},\varepsilon,k)}$ by Conf for P-a.e. $\mathcal{C}$. Moreover the ergodic theorem implies that for P-a.e. $\mathcal{C}$, the sequence $\{P_{(\mathcal{C},k)}\}_k$ converges to a stationary signed point process $P_{\mathcal{C}}$. By Lemma 2.5 we conclude that the push-forward of $P^{\mathrm{elec}}_{(\mathcal{C},\varepsilon)}$ by Conf coincides with the push-forward of $P_{\mathcal{C}}$ by Pr and that the energy is lower semi-continuous in the following sense

$$\begin{aligned}\mathbf{E}_{P^{\mathrm{elec}}_{(\mathcal{C},\varepsilon)}}\left[\frac{1}{|C_m|}\int_{C_m}|E_r|^2\right] &\le \liminf_{k\to\infty} \mathbf{E}_{P^{\mathrm{elec}}_{(\mathcal{C},\varepsilon,k)}}\left[\frac{1}{|C_m|}\int_{C_m}|E_r|^2\right] \\ &\le \liminf_{k\to\infty}\frac{1}{|C_k|}\int_{C_{m+k}}|E_r^{(\mathcal{C},\varepsilon)}|^2 \le \mathcal{W}^o(E^{(\mathcal{C},\varepsilon)}).\end{aligned} \tag{2.30}$$

Define next $P^{\mathrm{elec}}_{\varepsilon} := \int P^{\mathrm{elec}}_{(\mathcal{C},\varepsilon)} dP(\mathcal{C})$, or by duality

$$\int f dP^{\mathrm{elec}}_{\varepsilon} := \int \left(\int f dP^{\mathrm{elec}}_{(\mathcal{C},\varepsilon)} \right) dP(\mathcal{C}),$$

for any test function $f \in C^0(L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2))$. It is not difficult to check that the electric field law $P^{\mathrm{elec}}_{\varepsilon}$ is stationary and that its push-forward by Conf is P (because $\mathcal{C} = \mathrm{Pr}(\mathcal{C})$, P-a.s.). Moreover we get from (2.30), for any $m \geq 1$,

$$\mathbf{E}_{P^{\mathrm{elec}}_{\varepsilon}} \left[\frac{1}{|C_m|} \int_{C_m} |E_r|^2 \right] \leq \mathbf{E}_P \left[\mathcal{W}^o(E^{(\mathcal{C},\varepsilon)}) \right] \leq \mathbb{W}^o(P) + \varepsilon.$$

Letting $\varepsilon \to 0$, we may thus find a limit point P^{elec} of $\{P^{\mathrm{elec}}_{\varepsilon}\}$ that is still compatible with P and such that $\mathbf{E}_{P^{\mathrm{elec}}}[\mathcal{W}^o] \leq \mathbb{W}^o(P)$. The converse inequality is always true by definition of $\mathbb{W}^o$. □

We also obtain the following useful lower semi-continuity property.

Lemma 2.8. *The map $\bar{P} \mapsto \overline{\mathbb{W}}^o(\bar{P})$ is lower semi-continuous on $\mathcal{P}_{\mathrm{inv}}(\Lambda \times \mathcal{X})$.*

Proof. Let $\{\bar{P}_k\}_k$ be a sequence in $\mathcal{P}_{\mathrm{inv}}(\Lambda \times \mathcal{X})$ converging to some stationary $\bar{P}$, and such that $\liminf_{k\to\infty} \overline{\mathbb{W}}^o(\bar{P}_k)$ is finite. Up to the extraction of a subsequence, we may assume that the lim inf is a limit. For any $k \geq 1$ we may apply Lemma 2.7 and obtain a stationary electric field law $\bar{P}^{\mathrm{elec}}_k$ such that (2.29) holds. Using the stationarity of $\bar{P}^{\mathrm{elec}}_k$ and Fubini's theorem we may write, for any k

$$\mathbf{E}_{\bar{P}^{\mathrm{elec}}_k} [\mathcal{W}^o(E)] = \mathbf{E}_{\bar{P}^{\mathrm{elec}}_k} \left[\int_{C_1} |E_r|^2 \right],$$

and in fact the left-hand side is equal to $\mathbf{E}_{\bar{P}^{\mathrm{elec}}_k} \left[\frac{1}{R^2} \int_{C_R} |E_r|^2 \right]$ for any $R > 0$. The sequence of the push-forwards of $\bar{P}^{\mathrm{elec}}_k$ by $E \mapsto E_r$ is thus tight in $\mathcal{P}(L^2_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2))$, and using Lemma 2.4 we see that $\{\bar{P}^{\mathrm{elec}}_k\}_k$ itself is tight in $\mathcal{P}(L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2))$. Using Lemma 2.6 we see that any limit point $\bar{P}^{\mathrm{elec}}$ is compatible with $\bar{P}$ and that

$$\liminf_{k\to\infty} \mathbf{E}_{\bar{P}^{\mathrm{elec}}_k} \left[\int_{C_1} |E_r|^2 \right] \geq \mathbf{E}_{\bar{P}^{\mathrm{elec}}} \left[\int_{C_1} |E_r|^2 \right],$$

and using again the stationarity we see that the right-hand side satisfies

$$\mathbf{E}_{\bar{P}^{\mathrm{elec}}} \left[\int_{C_1} |E_r|^2 \right] = \mathbf{E}_{\bar{P}^{\mathrm{elec}}}[\mathcal{W}^o(E)] \geq \overline{\mathbb{W}}^o(\bar{P}),$$

which concludes the proof. □

2.6. **Specific relative entropy and large deviations.** Let $\mathbf{\Pi}^1$ denote the law of the Poisson point process of intensity 1 on $\mathbb{R}^2$, and let $\mathbf{\Pi}^s := \mathbf{\Pi}^1 \otimes \mathbf{\Pi}^1$.

Existence and main properties. The following proposition is an adaptation of classical results concerning the existence and properties of the so-called specific relative entropy for stationary point processes.

Proposition 2.9. *Let $P \in \mathcal{P}_{\mathrm{inv}}(\mathcal{X})$. The following limit exists in $[0, +\infty]$*

$$\mathsf{ent}[P] := \lim_{R\to\infty} \frac{1}{R^2} \mathrm{Ent}\left(P_R | \mathbf{\Pi}^s_R\right), \tag{2.31}$$

moreover the functional $P \mapsto \mathsf{ent}[P]$ is affine and lower semi-continuous on $\mathcal{P}_{\mathrm{inv}}(\mathcal{X})$.

Proof. The proofs of the corresponding results in the non-signed setting extend readily to our context, see e.g. [RAS09, Section 7.2]. □

Large deviations for signed empirical fields. Let $\bar{\mathfrak{Q}}_N^\beta$ be the "reference" empirical field defined as the push-forward by i_N of the Lebesgue measure $\mathbf{Leb}^N \otimes \mathbf{Leb}^N$ on $\Lambda^N \times \Lambda^N$, where we recall that i_N was defined in (1.5).

Proposition 2.10. *For any $A \subset \mathcal{P}_{\mathrm{inv}}(\Lambda \times \mathcal{X})$, we have (with $\overline{\mathsf{ent}}$ defined in (1.8))*

$$-\inf_{\bar{P} \in \mathring{A} \cap \mathcal{P}_{\mathrm{inv},1}} \overline{\mathsf{ent}}[\bar{P}] \leq \liminf_{N \to \infty} \frac{1}{N} \log \bar{\mathfrak{Q}}_N^\beta(A) \leq \limsup_{N \to \infty} \frac{1}{N} \log \bar{\mathfrak{Q}}_N^\beta(A) \leq -\inf_{\bar{P} \in \bar{A}} \overline{\mathsf{ent}}[\bar{P}]. \tag{2.32}$$

The proof of Proposition 2.10 is almost identical to that of [LS15, Proposition 1.6], see [LS15, Section 7].

2.7. Tightness and discrepancy estimates.

Compactness and exponential tightness.

Lemma 2.11. *The sequence $\{\overline{\mathfrak{P}}_N^\beta\}_N$ is exponentially tight.*

Proof. Let $\mathcal{N}_R : (\Lambda \times \mathcal{X}^0) \to \mathbb{R}_+$ be the map $\mathcal{N}_R(x, \mathcal{C}) := \mathcal{C}^+(D(0,R)) + \mathcal{C}^-(D(0,R))$ which gives the total number of points in the disk $D(0,R)$ of a signed point configuration $\mathcal{C} = (\mathcal{C}^+, \mathcal{C}^-)$. (Although it may seem not to depend on x, in applications we will always use $\mathcal{N}_R(x, \mathcal{C}_x)$ where $\mathcal{C}_x$ is the blow-up around x of a signed point configuration.) By construction it is clear that $\overline{\mathfrak{P}}_N^\beta$ is concentrated on

$$\left\{ \bar{P}_N \in \mathcal{P}(\Lambda \times \mathcal{X}), \mathbf{E}_{\bar{P}_N}[\mathcal{N}_R] \leq 2\pi R^2 \right\}.$$

This set is easily seen to be compact in $\mathcal{P}(\Lambda \times \mathcal{X})$, see e.g. [LS15, Lemma 4.1]. □

Discrepancy estimate, equality of intensities. Here we prove that we may control the difference between the number of positive and negative charges in terms of the two-component interaction energy of the signed point configuration. In particular, we show that if P is stationary, the finiteness of $\overline{\mathcal{F}}_\beta(P)$ implies that the intensities of positive and negative charges coincide.

Lemma 2.12. *Let $P \in \mathcal{P}_{\mathrm{inv}}(\mathcal{X})$ be such that $\overline{\mathbb{W}}^o(P)$ and $\mathsf{ent}[P]$ are finite. Then we have $\rho_P^+ = \rho_P^-$.*

Proof. In this proof C denotes a constant depending only on P. Let $\mathcal{C}$ be a signed point configuration and let $\mathcal{D}_R$ be the discrepancy in the square C_R, i.e. the difference between the number of positive and negative charges in C_R

$$\mathcal{D}_R := \int_{C_R} (d\mathcal{C}^+ - d\mathcal{C}^-).$$

Assume that E is an electric field compatible with $\mathcal{C}$. Using the relation (2.20) and (2.18) and integrating by parts over C_R, we have

$$\int_{\partial C_R} E_r \cdot \vec{n} = 2\pi \left(\mathcal{D}_R + d_R \right),$$

with $\vec{n}$ the outer unit normal, where the error term d_R is bounded by the number of points of $\mathcal{C}$ in a 1-neighborhood of ∂C_R. Let $\psi : \mathbb{R}_+ \to \mathbb{R}$ be a map such that $\psi(x) = x \log\log x$ for x large and such that ψ is convex, nonnegative, nondecreasing. We have

$$\psi(|\mathcal{D}_R|) = \psi\left(|\frac{1}{2\pi}\left(\int_{\partial C_R} E_r \cdot \vec{n}\right) - d_R|\right) \leq C\psi\left(\left|\int_{\partial C_R} E_r \cdot \vec{n}\right|\right) + C\psi(|d_R|) + C.$$

Using Jensen's inequality and the stationarity of P we get

$$\mathbf{E}_P\left[\psi\left(\int_{\partial C_R} |E_r|\right)\right] \leq \mathbf{E}_P\left[\psi\left(4R\,|E_r|\right)\right].$$

By stationarity of P, $\mathbf{E}_P\left[|E_r|^2\right]$ is equal to the positive part energy of P, which is finite by assumption. We may thus bound

$$\mathbf{E}_P\left[\psi\left(4R\,|E_r|\right)\right] \leq CR\log\log R + CR + C. \tag{2.33}$$

On the other hand, again by stationarity of P and using the convexity of ψ we have

$$\mathbf{E}_P\left(\psi\left(|d_R|\right)\right) \leq CR\mathbf{E}_P\left(d_1 \log\log(Rd_1)\right) \leq CR\log\log R\,\mathbf{E}_P[d_1] + CR\,\mathbf{E}_P[d_1 \log\log d_1].$$

The exponential moments of d_1 and $d_1 \log\log d_1$ under a Poisson point process are finite, and P has finite entropy, hence both expectations under P are finite. We obtain

$$\mathbf{E}_P\left(\psi(|d_R|)\right) \leq CR\log\log R + CR + C\,, \tag{2.34}$$

where C depends on the entropy of P. Combining (2.33) and (2.34) we get $\mathbf{E}_P\left[\psi(|\mathcal{D}_R|)\right] = o(\psi(R^2))$ as $R \to \infty$. It implies by Jensen's inequality (and the fact that $\psi(x) = x\log\log x$ for x large) that $\mathbf{E}_P[|\mathcal{D}_R|] = o(R^2)$, but since P is stationary we have $\mathbf{E}_P[|\mathcal{D}_R|] = R^2\mathbf{E}_P[|\mathcal{D}_1|]$, hence we deduce that for all $R > 0$ we have $\mathbf{E}_P[|\mathcal{D}_R|] = 0$. In particular the mean density of positive and negative charges are equal. □

2.8. **Scaling relations and optimal intensity.** Let P be a stationary signed point process such that the intensity of P^+ and P^- are both equal to ρ. Let $\sigma_\rho(P)$ be the stationary point processes obtained as the push-forward of P by $\mathcal{C} \mapsto \sqrt{\rho}\mathcal{C}$. It is easy to see that both component of $\sigma_\rho(P)$ now have intensity 1.

Lemma 2.13. *We have the following scaling relations*

$$\begin{aligned}\widetilde{\mathbb{W}}^o(P) &= \rho(\widetilde{\mathbb{W}}^o(\sigma_\rho(P)))\\ \widetilde{\mathbb{W}}^\star(P) &= \rho(\widetilde{\mathbb{W}}^\star(\sigma_\rho(P)) - \log\rho)\\ \mathsf{ent}[P] &= \rho\,\mathsf{ent}[\sigma_\rho(P)] + 1 - \rho + \rho\log\rho\end{aligned}$$

Proof. The first two relations are deduced by a change of variable in the definitions, see also the scaling relations in [SS15, Equation (2.4)]. The scaling relation for the entropy is proven in a elementary way as in [LS15, Lemma 4.2]. □

Lemma 2.14. *Let $\bar{P} \in \mathcal{P}_{\mathrm{inv},1}(\Lambda \times \mathcal{X})$ be a stationary tagged signed point process such that the intensity measures $\rho_{\bar{P}^+}$ and $\rho_{\bar{P}^-}$ are equal to the same function ρ on Λ with $\int_\Lambda \rho = 1$. Then*

$$\overline{\mathcal{F}}_\beta(\bar{P}) \geq \inf_{\mathcal{P}_{\mathrm{inv},1}} \overline{\mathcal{F}}_\beta,$$

with equality only if $\rho(x) = 1$ for Lebesgue-a.e. $x \in \Lambda$.

Proof. From Lemma 2.13 we deduce a scaling relation for $\overline{\mathcal{F}}_\beta$:

$$\overline{\mathcal{F}}_\beta(\bar{P}) = \frac{\beta}{2}\int_\Lambda \rho(x)\widetilde{\mathbb{W}}(\sigma_{\rho(x)}(\bar{P}^x))dx + \int_\Lambda \rho(x)\mathsf{ent}[\sigma_{\rho(x)}(\bar{P}^x)]dx + \int_\Lambda \left[(1-\frac{\beta}{2})\rho\log\rho - \rho + 1\right] dx.$$

In particular we have

$$\overline{\mathcal{F}}_\beta(\bar{P}) \geq \inf_{\mathcal{P}_{\mathrm{inv},1}} \overline{\mathcal{F}}_\beta + (1-\frac{\beta}{2})\int_\Lambda \rho\log\rho.$$

The total intensity being fixed, since $\frac{\beta}{2} < 1$ the expression above is minimized only if $\rho = 1$ Lebesgue-a.e. and we get

$$\overline{\mathcal{F}}_\beta(\bar{P}) \geq \inf_{\mathcal{P}_{\mathrm{inv},1}} \overline{\mathcal{F}}_\beta.$$

□

3. Study of the Gibbs measure and main conclusions

3.1. **Bounds on the partition function.** We rely on the work of Gunson-Panta [GP77] which gives a "classical" (in contrast to the Quantum Field Theory techniques of [Frö76]) approach to the study of the Gibbs measure $\mathbb{P}_N^\beta$ and of the partition function $Z_{N,\beta}$. For the reader's reference, a rewriting of their results can be found in Section 4 below.

In this subsection we mostly re-phrase the key points of their analysis in our notation.

Dipole contribution. The analysis in [GP77], recalled in Section 4, see (4.1) and (4.11), yields the following lemma.

Lemma 3.1. *For any integer N and any $\beta < 2$ we have*

$$\log\int_{\Lambda^{2N}} \exp\left(-\frac{\beta}{2}\left(\sum_{i=1}^N \log r(x_i) + \sum_{i=1}^N \log r(y_i)\right)\right) d\vec{X}_N d\vec{Y}_N \leq \frac{\beta}{2}N\log N + C_\beta N, \tag{3.1}$$

with a constant C_β depending only on β.

Exponential moments. We give another consequence of the analysis in [GP77]. For any pair of integers N_+, N_-, and real $R > 0$ we denote by $\mathbf{B}_{N_+,N_-,R}$ the law of the signed point process on C_R obtained from two independent Bernoulli processes with N_+ and N_- points. The following lemma gives a bound on (the exponential moments of) the dipole contribution in the interaction energy.

Lemma 3.2. *For any $\beta < 2$ and any $R > 0$ we have*

$$\log \mathbf{E}_{\mathbf{B}_{N_+,N_-,R}}\left[e^{\frac{\beta}{2}\mathbb{W}^\star(\mathbf{1}_{C_R},\mathcal{C})}\right] \leq \frac{\beta}{4}(N_+ + N_-)\log(N_+ + N_-) + (N_+ + N_-)C_\beta - \frac{\beta}{2}(N_+ + N_-)\log R. \tag{3.2}$$

Proof. Scaling the configuration by a factor R^{-1} changes the left-hand side by $\frac{\beta}{2}(N_+ + N_-)\log R$ and then we are left to prove the inequality for $R = 1$. With our notation, it reduces to the upper bound on [GP77, (2.4)] as expressed in [GP77, (2.9)] (cf. (4.11) and (4.1)). Let us emphasize that although the analysis of [GP77] initially deals with a system such that $N_+ = N_- = N$, the bound on [GP77, (2.4)] is not affected by the actual sign of each charge, as it is merely a bound on some given integral on $(\mathbb{R}^2)^{2N}$. We may thus follow the lines of [GP77, Section 2.2.] with $N_+ + N_-$ instead of $2N$ and [GP77, (2.9)] yields (3.2). □

3.2. **Study of the rate function.** In this subsection we show that $\overline{\mathcal{F}}_\beta$ is bounded below and is well-defined as a functional $\overline{\mathcal{F}}_\beta : \mathcal{P}_{\mathrm{inv}}(\Lambda \times \mathcal{X}^0) \to \mathbb{R} \cup \{+\infty\}$.

Lemma 3.3. *For any $\beta < 2$, any $\tau, R > 0$ and any $P \in \mathcal{P}_{\mathrm{inv}}(\mathcal{X})$ such that $\mathsf{ent}[P]$ is finite, it holds that*

$$-\frac{\beta}{2}\mathbf{E}_P\left[\mathbb{W}^\star_\tau(\mathbf{1}_{C_R}, \mathcal{C})\right] + \mathrm{Ent}[P_R|\mathbf{\Pi}^s_R] \geq -L_\beta R^2 \tag{3.3}$$

where L_β is a constant depending only on β. Consequently we get as $\tau \to 0$

$$-\frac{\beta}{2}\mathbf{E}_P\left[\mathbb{W}^\star(\mathbf{1}_{C_R}, \mathcal{C})\right] + \mathrm{Ent}[P_R|\mathbf{\Pi}^s_R] \geq -L_\beta R^2 \tag{3.4}$$

and finally in the limit $R \to +\infty$,

$$-\mathbf{E}_P\left[\frac{\beta}{2}\mathbb{W}^\star(\mathcal{C})\right] + \mathsf{ent}[P] \geq -L_\beta. \tag{3.5}$$

Proof. By the variational characterization of the relative entropy, we know that

$$-\log \mathbf{E}_{\mathbf{\Pi}^s_R}\left[e^{\frac{\beta}{2}\mathbb{W}^\star_\tau(\mathbf{1}_{C_R}, \mathcal{C})}\right] \leq \mathrm{Ent}[P_R|\mathbf{\Pi}^s_R] - \mathbf{E}_P\left[\frac{\beta}{2}\mathbb{W}^\star_\tau(\mathbf{1}_{C_R}, \mathcal{C})\right]. \tag{3.6}$$

We evaluate the left-hand side of (3.6). Combining the definition of a Poisson point process with Lemma 3.2 we get

$$\log \mathbf{E}_{\mathbf{\Pi}^s_R}\left[e^{\frac{\beta}{2}\mathbb{W}^\star_\tau(\mathbf{1}_{C_R}, \mathcal{C})}\right] = \log \sum_{N_+, N_-=0}^{+\infty} \mathbf{\Pi}^s_R(N_+, N_-)\mathbf{E}_{\mathbf{B}_{N_+,N_-,R}}\left[e^{\frac{\beta}{2}\mathbb{W}^\star_\tau(\mathbf{1}_{C_R}, \mathcal{C})}\right]$$
$$\leq \log \sum_{N_+, N_-=0}^{+\infty} e^{-2R^2}\frac{R^{2(N_+ + N_-)}}{N_+! N_-!} e^{\frac{\beta}{4}(N_+ + N_-)\log(N_+ + N_-) + (N_+ + N_-)C_\beta - \frac{\beta}{2}(N_+ + N_-)\log R}.$$

Using the elementary inequality

$$(N_+ + N_-)\log(N_+ + N_-) \leq (N_+ \log N_+ + N_- \log N_- + N_+ + N_-),$$

we may separate the variables N_+ and N_- (which play a symmetric role) and write, using the fact that $\frac{1}{N!} \leq e^{-N\log N + (C+1)N}$ for a certain constant C,

$$\log \mathbf{E}_{\mathbf{\Pi}^s_R}\left[e^{\frac{\beta}{2}\mathbb{W}^\star_\tau(\mathbf{1}_{C_R}, \mathcal{C})}\right] \leq 2\log \sum_{N=0}^{+\infty} e^{-R^2}\frac{R^{2N}}{N!}e^{\frac{\beta}{4}(N\log N + N) + N(C_\beta + C) - \frac{\beta}{2}N\log R}$$
$$\leq 2\log \sum_{N=0}^{+\infty} e^{-R^2} R^{2(1-\frac{\beta}{4})N} e^{(\frac{\beta}{4}-1)N\log N + (C_\beta + C + 1)N} \leq L_\beta R^2$$

for a certain constant L_β depending only on β. Inserting this estimate into (3.6) yields (3.3), (3.4) follows by sending $\tau \to 0$ and (3.5) is obtained by dividing (3.4) by R^2 and then sending $R \to +\infty$ (together with the definition (1.7) of ent). □

In particular if $\overline{\mathsf{ent}}[\bar{P}]$ is finite, then for Lebesgue-a.e. $x \in \Lambda$ the disintegration measure $\bar{P}^x$ has finite entropy and satisfies (3.5), hence the functional $\overline{\mathcal{F}}_\beta$ is well-defined.

Conclusion.

Lemma 3.4. *The functional* $\overline{\mathcal{F}}_\beta^{\mathrm{sc}}$ *is a good rate function.*

Proof. Lemma 3.3 shows that $\overline{\mathcal{F}}_\beta$ is well-defined as a functional $\overline{\mathcal{F}}_\beta : \mathcal{P}_{\mathrm{inv}}(\Lambda\times\mathcal{X}) \to \mathbb{R}\cup\{+\infty\}$. It also implies that the sub-level sets of $\overline{\mathcal{F}}_\beta$ are included in sub-level sets of $\bar{P} \mapsto \overline{\mathsf{ent}}[\bar{P}]$, which are compact.

Thus $\overline{\mathcal{F}}_\beta^{\mathrm{sc}}$ is well-defined and it is lower semi-continuous by definition. Since the sub-level sets of $\overline{\mathcal{F}}_\beta$ are pre-compact, those of its lower semi-continuous regularization are compact. It proves that $\overline{\mathcal{F}}_\beta^{\mathrm{sc}}$ is a good rate function. □

3.3. **Properties of the limit objects.** One of the crucial points in order to get Theorem 2 from Theorem 1 is to show, by entropy arguments, that the intensities of the underlying point processes coincide with the limits of the empirical measures, while by the scaling argument of Lemma 2.14 the rate function is minimized only when these intensities equal 1.

In this subsection we use the preliminary bounds on $Z_{N,\beta}$ available for $\beta < 2$ thanks to the analysis of Gunson-Panta to derive some *a priori* properties of the possible limits of μ_N^+ and $\bar{P}_N$. In particular we wish to show that the intensity of the limits of $\bar{P}_N$ equals, most of the time, the density of the limits of μ_N^+. This is not obvious since, with the topology that we use for the convergence of $\bar{P}_N$, there can be a loss of mass when taking the limit.

To overcome this, we will show below that with overwhelming probability (i.e. up to neglecting events of $\mathbb{P}_N^\beta$-probability less than e^{-NT} with T arbitrarily large) the limiting objects must have finite entropy, which will yield a uniform integrability of the densities of points, which in turn ensures that no loss of mass occurs in the limit.

3.3.1. *A priori bounds on the entropy.*

Lemma 3.5. *For any* $\beta < 2$*, the following holds with a constant* C_β *depending only on* β*.*

(1) For any $\mu^+ \in \mathcal{P}(\Lambda)$ *we have*

$$\lim_{\varepsilon\to 0}\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}_N^\beta\left(\mu_N^+ \in B(\mu^+,\varepsilon)\right) \le C_\beta - \frac{1}{C_\beta}\mathrm{Ent}[\mu^+|\mathbf{Leb}_\Lambda]. \tag{3.7}$$

(2) For any $\bar{P} \in \mathcal{P}_{\mathrm{inv}}(\Lambda\times\mathcal{X})$ *we have*

$$\lim_{\varepsilon\to 0}\limsup_{N\to\infty}\frac{1}{N}\log\overline{\mathfrak{P}}_N^\beta\left(B(\bar{P},\varepsilon)\right) \le C_\beta - \frac{1}{C_\beta}\overline{\mathsf{ent}}[\bar{P}]. \tag{3.8}$$

(3) For any R, N *let* $\{C_i\}_{i\in I}$ *be a partition of* Λ *by squares of sidelength in* $(\frac{R}{2\sqrt{N}}, \frac{3R}{2\sqrt{N}})$*, and let* $n_i = \mu_N^+(C_i)$ *be the number of positive charges in the square* C_i*. When* $R > 0$ *is fixed we have*

$$\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}_N^\beta\left(\left(\frac{1}{\#I}\sum_{i\in I}\frac{n_i}{R^2}\max\left(1,\left(\log\frac{n_i}{R^2}\right)^{\frac12}\right)\right)\ge M\right)\le f_\beta(M,R), \tag{3.9}$$

with $\lim f_\beta(M,R) = -\infty$ *as* $M\to\infty$*.*

Proof. For any $\beta < 2$, let us fix some $p > 1$ such that $p\beta < 2$ and let q be the conjugate exponent of p.

Let $\mu \in \mathcal{P}(\Lambda)$. Let $A \subset \mathcal{P}(\Lambda)$ be measurable. For any N we obtain using Hölder's inequality that

$$
\begin{aligned}
(3.10)\quad \mathbb{P}_N^\beta\left(\Lambda^{2N} \cap \{\mu_N^+ \in A\}\right) &= \frac{1}{Z_{N,\beta}} \int_{\Lambda^{2N}\cap\{\mu_N^+\in A\}} e^{-\frac{\beta}{2} w_N(\vec{X}_N, \vec{Y}_N)} d\vec{X}_N d\vec{Y}_N \\
&\le \frac{1}{Z_{N,\beta}} \left(\int_{\Lambda^{2N}} e^{-p\frac{\beta}{2} w_N(\vec{X}_N,\vec{Y}_N)} d\vec{X}_N d\vec{Y}_N \right)^{\frac{1}{p}} \left(\int_{\Lambda^{2N}\cap\{\mu_N^+\in A\}} d\vec{X}_N d\vec{Y}_N \right)^{\frac{1}{q}} \\
&= \frac{Z_{N,p\beta}^{\frac{1}{p}}}{Z_{N,\beta}} \left(\int_{\Lambda^{2N}\cap\{\mu_N^+\in A\}} d\vec{X}_N d\vec{Y}_N \right)^{\frac{1}{q}}
\end{aligned}
$$

where p, q are as above. By (1.4) we have

$$
(3.11)\quad \log Z_{N,p\beta} = p\frac{\beta}{2} N \log N + C_{p\beta} N + o(N) \text{ and } \log Z_{N,\beta} = \frac{\beta}{2} N \log N + C_\beta N + o(N)
$$

where $C_\beta, C_{p\beta}$ depend only on β. On the other hand we have by Sanov's theorem

$$
(3.12)\quad \lim_{\varepsilon\to 0} \limsup_{N\to\infty} \frac{1}{N} \log \int_{\Lambda^{2N}\cap\{\mu_N^+\in B(\mu^+,\varepsilon)\}} d\vec{X}_N d\vec{Y}_N = -\mathrm{Ent}[\mu^+|\mathbf{Leb}_\Lambda].
$$

Combining (3.10) (with $A = B(\mu^+, \varepsilon)$), (3.11) and (3.12) yields (3.7). The proof of (3.8) is similar, using Proposition 2.10 instead of Sanov's theorem in the last step, where (3.12) is replaced by

$$
\lim_{\varepsilon\to 0} \limsup_{N\to\infty} \frac{1}{N} \log \int_{\Lambda^{2N}\cap \bar{P}_N \in B(\bar{P},\varepsilon)} d\vec{X}_N d\vec{Y}_N = -\overline{\mathsf{ent}}[\bar{P}].
$$

To see (3.9), we take $A = A(M, R)$ in (3.10) as the event inside the probability in (3.9). Using (3.11), the proof of (3.9) reduces to proving

$$
(3.13)\quad \limsup_{N\to\infty} \frac{1}{N} \log \int_{\Lambda^{2N}\cap A} d\vec{X}_N d\vec{Y}_N \le f_\beta(M, R).
$$

The proof of (3.13) is simplified if one uses comparison to a Poisson process of intensity N on Λ, denoted $\mathbf{\Pi}_N$. Indeed, extend the event A in a natural way to apply to any collection of integers $\{n_i\}_{i\in I}$, and note that A is monotone increasing with respect to $K = \sum_{i\in I} n_i$. Since there exists a constant $\eta > 0$ independent of N so that $\mathbf{\Pi}_N(K \ge N) > \eta$, and since conditioned on K the points of the Poisson process are independent and uniformly distributed in Λ, the proof of (3.13) reduces to proving that

$$
(3.14)\quad \limsup_{N\to\infty} \frac{1}{N} \log \mathbf{\Pi}_N(A) \le f_\beta(M, R).
$$

The advantage of working with $\mathbf{\Pi}_N$ is that the random variables n_i are now independent Poisson of parameter in $(R^2/4, 9R^2/4)$. In particular, the random variables $n_i \max(1, (\log(n_i/R^2))^{1/2})$ possess a finite exponential moment. Applying Markov's exponential inequality then yields (3.14) and completes the proof of (3.9). □

Of course (3.7) and (3.9) also hold when replacing μ_N^+ by μ_N^-.

3.3.2. *Uniform integrability of the number of points.* The bound (3.9) implies that under $\mathbb{P}_N^\beta$, the random number of points $\mu_N^+\left(B(x, \frac{R}{\sqrt{N}})\right)$ is uniformly (as $N \to \infty$) integrable on Λ with overwhelming probablity. More precisely we have

Lemma 3.6. *For any $T, R > 0$ and any $\varepsilon > 0$ there exists $M' > 0$ (depending on T, ε and on β) such that for N large enough we have*

$$\int_\Lambda \mu_N^+\left(B(x, \frac{R}{\sqrt{N}})\right) \mathbf{1}_{\{\mu_N^+\left(B(x,\frac{R}{\sqrt{N}})\right) \geq M'\}} dx \leq \varepsilon$$

with probability $\geq 1 - \exp(-NT)$ under $\mathbb{P}_N^\beta$.

Proof. Indeed from (3.9) we control the $L^1(\Lambda)$ norm of the superlinear map ψ_R defined as

$$\psi_R(x) := \frac{x}{R^2} \max\left(1, \left(\log \frac{x}{R^2}\right)^{\frac{1}{2}}\right)$$

by M with $\mathbb{P}_N^\beta$-probability $\geq 1 - \exp(N f_\beta(M))$ with $\lim_{M\to\infty} f_\beta(M) = -\infty$. □

3.3.3. *Microscopic intensity versus macroscopic density.* We emphasize the following abuse of notation: in Lemma 3.7 and its proof, the quantities μ_N^+ and $\bar{P}_N$ are elements of a *deterministic* sequence.

Lemma 3.7. *Let $\{\vec{X}_N, \vec{Y}_N\}_N$ be a sequence of points in Λ^{2N}, let $\mu_N^+ := \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ and let $\bar{P}_N := i_N(\vec{X}_N, \vec{Y}_N)$. Assume that up to extraction the sequence $\{(\mu_N^+, \bar{P}_N)\}_N$ converges to $(\mu^+, \bar{P})$ where $\mu^+ \in \mathcal{P}(\Lambda)$ and $\bar{P} \in \mathcal{P}_{\mathrm{inv}}(\Lambda \times \mathcal{X})$. Then we have $\rho_{\bar{P}}^+ \leq \mu^+$ in the sense of nonnegative measures.*

Moreover under the assumption that μ^+ does not charge $\partial\Lambda$ and that for any $R > 1$, $x \mapsto \mu_N^+\left(B(x, \frac{R}{\sqrt{N}})\right)$ is uniformly integrable on Λ as $N \to \infty$ then $\rho_{\bar{P}}^+ = \mu^+$.

Of course the same results hold for the quantities associated to the negative charges as well.

Proof. Let χ be a non-negative test function in $C^0(\Lambda)$ and for any $R > 1$ let f_R be a smooth function satisfying

$$\frac{1_{C_{R-1}}}{|C_R|} \leq f_R \leq \frac{1_{C_R}}{|C_R|},$$

we also define $f_{R,N}$ as $f_{R,N}(x) := N f_R(\sqrt{N}x)$ for $x \in \Lambda$. Finally we define $\langle f_R, \mathcal{C}^+\rangle$ as

$$\langle f_R, \mathcal{C}^+\rangle := \int f_R d\mathcal{C}^+ \text{ for any } \mathcal{C}^+ \in \mathcal{X}^0(\mathbb{R}^2).$$

We compute (with $*$ being the convolution product)

$$\begin{aligned}(3.15)\quad \int \chi * f_{R,N} d\mu_N^+ &= \int \chi f_{R,N} * d\mu_N^+ = \int \chi(y) \int f_{R,N}(y-z) d\mu_N^+(z) dy \\ &= \int \chi(y) \sum_{i=1}^N f_R(\sqrt{N}(x_i - y)) dy \geq \int \chi(y) \langle f_R, \mathcal{C}^+\rangle d\bar{P}_N(y, \mathcal{C}),\end{aligned}$$

where the inequality is due to the possible loss of mass at the boundary. For any $M > 0$ we may write

$$\int \chi(y) \left\langle f_R, \mathcal{C}^+ \right\rangle d\bar{P}_N(y, \mathcal{C}) \geq \int \chi(y) \left(\left\langle f_R, \mathcal{C}^+ \right\rangle \wedge M \right) d\bar{P}_N(y, \mathcal{C})$$
$$\underset{N\to\infty}{\longrightarrow} \int \chi(y) \left(\left\langle f_R, \mathcal{C}^+ \right\rangle \wedge M \right) d\bar{P}(y, \mathcal{C})$$

and we have, by definition of f_R

$$\int \chi(y) \left(\left\langle f_R, \mathcal{C}^+ \right\rangle \wedge M \right) d\bar{P}(y, \mathcal{C}) \geq \frac{1}{R^2} \int \chi(y) \left(\mathcal{N}(0, R-1)(\mathcal{C}^+) \wedge M \right) d\bar{P}(y, \mathcal{C}).$$

By definition of the intensity it holds that

$$\lim_{R\to\infty} \lim_{M\to\infty} \frac{1}{R^2} \int \chi(y) \left(\mathcal{N}(0, R-1)(\mathcal{C}^+) \wedge M \right) d\bar{P}(y, \mathcal{C}) = \int \chi(y) \rho^+_{\bar{P}}(y).$$

Moreover, since μ_N^+ converges to μ^+ we have

$$\lim_{N\to\infty} \int \chi * f_{R,N} d\mu_N^+ = \int \chi d\mu^+ + o_R(1).$$

Finally, sending $R \to \infty, M \to \infty, N \to \infty$ we get

$$\int \chi d\mu^+ \geq \int \chi(y) \rho^+_{\bar{P}}(y)$$

for any non-negative continuous test function χ, which proves $\rho^+_{\bar{P}} \leq \mu^+$.

We next prove the equality under the additional assumption that μ^+ does not charge $\partial\Lambda$ and that that $x \mapsto \mu_N^+ \left(B(x, \frac{R}{\sqrt{N}}) \right)$ is uniformly integrable on Λ as $N \to \infty$. First, the difference between the last two terms in (3.15) is bounded as follows:

$$\int \chi(y) \sum_{i=1}^N f_R(\sqrt{N}(x_i - y)) dy$$
$$\leq \int \chi(y) \left\langle f_R, \mathcal{C}^+ \right\rangle d\bar{P}_N(y, \mathcal{C}) + ||\chi||_\infty \mu_N^+ \left(\{ x \in \Lambda, \operatorname{dist}(x, \partial\Lambda) \leq 2\frac{R}{\sqrt{N}} \} \right).$$

Since μ_N^+ converges to μ^+ which does not charge the boundary, the error term satisfies

$$||\chi||_\infty \mu^+ \left(\{ x \in \Lambda, \operatorname{dist}(x, \partial\Lambda) \leq 2\frac{R}{\sqrt{N}} \} \right) = o(1)$$

as $N \to \infty$, for any χ and R fixed.

Moreover, the uniform integrability assumption implies that $\mathcal{C} \mapsto \left\langle f_R, \mathcal{C}^+ \right\rangle$ is uniformly integrable against $d\bar{P}_N$ as $N \to \infty$ and we may for any $\delta > 0$ choose M large enough such that

$$\int \chi(y) \left\langle f_R, \mathcal{C}^+ \right\rangle d\bar{P}_N(y, \mathcal{C}) \leq \int \chi(y) \left(\left\langle f_R, \mathcal{C}^+ \right\rangle \wedge M \right) d\bar{P}_N(y, \mathcal{C}) + \delta$$

uniformly in N. Arguing as above we see that

$$\lim_{R\to\infty} \lim_{N\to\infty} \int \chi(y) \left(\left\langle f_R, \mathcal{C}^+ \right\rangle \wedge M \right) d\bar{P}_N(y, \mathcal{C}) \leq \int \chi(y) \rho^+_{\bar{P}}(y) \leq \int \chi(y) \rho^+_{\bar{P}}(y).$$

Eventually we get $\mu^+ \leq \rho^+_{\bar{P}} + \delta$ and we conclude by letting $\delta \to 0$. □

3.3.4. *Total intensity of the limit random point process.* From the previous lemmas we deduce that in the LDP we may restrict ourselves to random point processes with total intensity 1.

Lemma 3.8. *Let $\bar{P}$ be the law of a stationary tagged signed point process such that the intensity of positive charges satisfies $\int_\Lambda \rho^+_{\bar{P}} < 1$. Then we have*

$$\lim_{\varepsilon\to 0}\limsup_{N\to\infty}\frac{1}{N}\log\overline{\mathfrak{P}}^\beta_N(B(\bar{P},\varepsilon)) = -\infty. \tag{3.16}$$

Proof. Assume that (3.16) does not hold and that we have for some T

$$\lim_{\varepsilon\to 0}\limsup_{N\to\infty}\frac{1}{N}\log\overline{\mathfrak{P}}^\beta_N(B(\bar{P},\varepsilon)) \geq -T.$$

Using the relative compactness of $i_N(\Lambda^{2N})$ we may find a sequence $\{\vec{X}_N\}_N$ of points in Λ^{2N} such that $i_N(\vec{X}_N)$ converges to some $\bar{Q}\in B(\bar{P},\varepsilon)$. Up to extraction we may also assume that μ_N^+ converges to $\mu^+\in\mathcal{P}(\Lambda)$ and the point (3.7) of Lemma 3.5 ensures that we may assume that μ^+ has finite entropy, hence does not charge the boundary $\partial\Lambda$. Then, using Lemma 3.6 and Lemma 3.7 we obtain that $\rho^+_{\bar{Q}} = \mu^+$ and in particular $\rho^+_{\bar{Q}}$ has total mass 1. Thus in any ball $B(\bar{P},\varepsilon)$ we may find a random tagged point process $\bar{Q}_\varepsilon$ such that $\rho^+_{\bar{Q}_\varepsilon}$ has total mass 1. Moreover, again by Lemma 3.6, we may assume that the number of points in any disk is uniformly integrable under $\bar{Q}_\varepsilon$ as $\varepsilon\to 0$. Passing to the limit $\varepsilon\to 0$, it implies that $\rho^+_{\bar{P}}$ has total mass 1, which yields a contradiction. □

3.4. **Conclusion.** We now show how Theorems 1 and 2 follow once we have proven the following lower and upper bounds:

Proposition 3.9. *Let $\bar{P}\in\mathcal{P}_{\mathrm{inv},1}(\Lambda\times\mathcal{X})$. We have*

$$\lim_{\varepsilon\to 0}\liminf_{N\to\infty}\frac{1}{N}\log\int_{i_N^{-1}(B(P,\varepsilon))} e^{-\frac{\beta}{2}\left(\frac{1}{2\pi}\int_{\mathbb{R}^2}|\nabla V'_{N,r}|^2+\sum_{i=1}^N\log r(x'_i)+\log r(y'_i)\right)}d\vec{X}_N d\vec{Y}_N \geq -\overline{\mathcal{F}}_\beta(\bar{P}). \tag{3.17}$$

Proposition 3.10. *Let $\bar{P}\in\mathcal{P}_{\mathrm{inv},1}(\Lambda\times\mathcal{X})$. We have*

$$\lim_{\varepsilon\to 0}\limsup_{N\to\infty}\frac{1}{N}\log\int_{i_N^{-1}(B(P,\varepsilon))} e^{\left(-\frac{\beta}{2}\left(\frac{1}{2\pi}\int_{\mathbb{R}^2}|\nabla V'_{N,r}|^2+\sum_{i=1}^N\log r(x'_i)+\log r(y'_i)\right)\right)}d\vec{X}_N d\vec{Y}_N \leq -\overline{\mathcal{F}}_\beta(\bar{P}). \tag{3.18}$$

3.4.1. *Proof of Theorem 1.* Since we have exponential tightness, the proof of Theorem 1 reduces to proving a weak LDP. Thanks to Lemmas 2.14 and 3.8, the latter is easily deduced by combining Proposition 3.9 and Proposition 3.10. Indeed in view of (2.17) it only remains to show that $\log K_{N,\beta} = -\inf\overline{\mathcal{F}}_\beta + o(N)$, but combining the upper and lower bounds with the exponential tightness (and Lemma 3.8) we have

$$-\inf_{\mathcal{P}_{\mathrm{inv},1}}\overline{\mathcal{F}}_\beta \leq \liminf_{N\to\infty}\frac{1}{N}\log K_{N,\beta} \leq \limsup_{N\to\infty}\frac{1}{N}\log K_{N,\beta} \leq -\inf_{\mathcal{P}_{\mathrm{inv},1}}\overline{\mathcal{F}}_\beta. \tag{3.19}$$

Hence $\lim_{N\to\infty}\frac{1}{N}\log K_{N,\beta} = -\inf_{\mathcal{P}_{\mathrm{inv},1}}\overline{\mathcal{F}}_\beta$, which concludes the proof of Theorem 1.

We also get Corollary 1.3 from (3.19) and the fact that $\log K_{N,\beta} + \frac{\beta}{2}N\log N = \log Z_{N,\beta}$ as seen in Section 2.3.

3.4.2. *Proof of Theorem 2.* From Lemmas 2.12 and 2.14 we see that minimizers of $\overline{\mathcal{F}}_\beta$ are such that the intensity of both components of $\bar{P}^x$ are equal to 1 (for Lebesgue-a.e. $x \in \Lambda$). Thus the limit points of $\{\bar{P}_N\}_N$ have $\mathbb{P}_N^\beta$-a.s. both intensity measures equal to the uniform measure on Λ, and by Lemma 3.7 we see that any limit point of $\{\mu_N^+, \mu_N^-\}_N$ must be the uniform measure on Λ, almost surely under $\mathbb{P}_N^\beta$.

The rest of the paper is organized as follows: in Section 4 we give for the reader's convenience the proof of the main result of [GP77], in Section 5 we prove Proposition 3.9, and in Section 6 we prove Proposition 3.10.

4. The method of Gunson-Panta

In this section we recall the main steps of the analysis of Gunson-Panta as presented in [GP77] while keeping our notation when it is in conflict with that of [GP77]. In [GP77] the charges have absolute value $q > 0$, and for our concerns q should always be taken equal to 1.

Recall that the partition function is defined as

$$Z_{N,\beta} := \int_{\Lambda^{2N}} e^{-\frac{\beta}{2} w_N(\vec{X}_N, \vec{Y}_N)} d\vec{X}_N d\vec{Y}_N.$$

This is almost exactly what is denoted by Q_{2N}^* in [GP77, (2.2)], up to the fact that the domain of integration Λ is a square, in contrast to [GP77] where it is a disk. We have a factor $\beta/2$ but the definition of w_N counts each pairwise interaction twice, whereas in [GP77, (2.2)] the temperature factor is β but each pairwise interaction is counted only once.

In [GP77, (2.3)] an "electrostatic inequality" is used to bound below the interaction energy in terms of the quantity

$$\sum_{i=1}^N (\log r(x_i) + \log r(y_i)),$$

this is the same computation as in our Lemma 2.3. It yields the bound

$$Z_{N,\beta} \le \int_{\Lambda^{2N}} e^{-\frac{\beta}{2} \sum_{i=1}^N (\log r(x_i) + \log r(y_i))} d\vec{X}_N d\vec{Y}_N, \tag{4.1}$$

as expressed in [GP77, (2.4)] (up to notation, and the fact that in the latter, a minus sign is missing in the exponential).

Henceforth the signs of the charge will not play any role. For any M-tuple of points $\vec{S}_M = (S_1, \dots, S_M)$, let us define the map $F : \{1, \dots, M\} \to \{1, \dots, M\}$ such that

$$|S_i - S_{F(i)}| = \min_{j \in \{1, \dots, M\}} |S_i - S_j|.$$

With this notation we may rewrite (4.1) as

$$Z_{N,\beta} \le \int_{\Lambda^{2N}} e^{-\frac{\beta}{2} \sum_{i=1}^{2N} \log(\frac{1}{2}|S_i - S_{F_i}|)} d\vec{S}_{2N}. \tag{4.2}$$

To any M-tuple $\vec{S}_M$ we associate the (directed) graph $\hat{\gamma}(\vec{S}_M)$ of "nearest-neighbors", whose set of vertices is $\{1, \dots, M\}$ and such that there is a directed arrow from i to F_i for any $i \in \{1, \dots, M\}$. We observe the following

Lemma 4.1. *For any $\vec{S}_M$, the associated graph $\hat{\gamma}(\vec{S}_M)$ has between 1 and $M/2$ connected components. Each connected component is composed of a cycle of length 2, together with trees attached to the two vertices of the cycle.*

The graphs satisfying these properties are called "functional digraphs" (or "functional directed graphs") such that each connected component contains a cycle of order 2. For any even $M \geq 1$ and $1 \leq K \leq M/2$ let us denote by $\mathbf{D}_{M,K}$ the set of (isomorphism classes of) labeled functional digraphs with M vertices and K connected components, each possessing a cycle of order 2. A combinatorial computation (as the one leading to [GP77, (2.8)]) shows that

Lemma 4.2. *For any $M \geq 1$ and $1 \leq K \leq M/2$ the cardinality of $\mathbf{D}_{M,K}$ is bounded by*

$$|\mathbf{D}_{M,K}| \leq \frac{\Gamma(M+1)M^{M-2K}}{2^K\Gamma(K+1)\Gamma(M-2K+1)}. \tag{4.3}$$

If $\gamma \in \mathbf{D}_{M,K}$ is an isomorphism class, we denote by $\hat{\gamma}(\vec{S}_M) \equiv \gamma$ the event "$\hat{\gamma}(\vec{S}_M)$ is isomorphic to γ". We may then rewrite (4.2) by splitting the domain of integration according to the isomorphism class of $\hat{\gamma}(\vec{S}_{2N})$, this reads

$$Z_{N,\beta} \leq \sum_{K=1}^{N} \sum_{\gamma \in \mathbf{D}_{N,K}} \int_{\Lambda^{2N} \cap \{\hat{\gamma}(\vec{S}_{2N}) \equiv \gamma\}} e^{-\frac{\beta}{2}\sum_{i=1}^{2N} \log(\frac{1}{2}|S_i - S_{F_i}|)} d\vec{S}_{2N}. \tag{4.4}$$

Let $1 \leq K \leq N$ and $\gamma \in \mathbf{D}_{2N,K}$ be fixed, we turn to evaluating the quantity

$$\int_{\Lambda^{2N} \cap \{\hat{\gamma}(\vec{S}_{2N}) \equiv \gamma\}} e^{-\frac{\beta}{2}\sum_{i=1}^{2N} \log(\frac{1}{2}|S_i - S_{F_i}|)} d\vec{S}_{2N}. \tag{4.5}$$

Let $L_1, \dots, L_K$ be the K subsets of vertices associated to each connected component of the isomorphism class γ of graphs. For $k \in \{1, \dots, K\}$ we perform a change of variables on the variables S_i for $i \in L_k$. We denote by $c_k := \{i_k^a, i_k^b\}$ the two vertices on the cycle. We let for $i \in L_k$ such that $i \notin c_k$

$$u_i := \frac{1}{2}(S_i - S_{F_i}),$$

and we let

$$u_{i_k^a} := \frac{1}{2}(S_{i_k^a} - S_{i_k^b}), \quad u_{i_k^b} := \frac{1}{2}S_{i_k^b}.$$

With respect to the new variables, the integral in (4.5) is bounded by

$$4^{2N} \prod_{k=1}^{K} \int_{D_k} e^{-\frac{\beta}{2}\left(\sum_{i \in L_k \setminus c_k} \log u_i + 2\log u_{i_k^a}\right)} \prod_{i \notin L_k} du_i \, du_{i_k^a} \, du_{i_k^b}, \tag{4.6}$$

where D_k is a (suitably enlarged) domain of integration for the new variables. It may be observed that the new variables satisfy

$$\sum_{i \in L_k \setminus c_k} |u_i|^2 + |u_{i_k^a}|^2 + |u_{i_k^b}|^2 \leq C.$$

for a certain universal constant C, thus each integral term in (4.6) can be viewed as an integral over a simplex i.e. a multiple Dirichlet integral. Using classical results about such integrals following [GP77, Equation (2.9)] we have

Lemma 4.3. *For any integer $M \geq 1$ and $K \leq M/2$, $\gamma \in \mathbf{D}_{M,K}$, we have*

$$\prod_{k=1}^{K} \int_{D_k} e^{-\frac{\beta}{2}\left(\sum_{i \in L_k \setminus c_k} \log u_i + 2 \log u_{i_k^a}\right)} \prod_{i \notin L_k} du_i \, du_{i_k^a} \, du_{i_k^b} \\ \leq Diri_{M,K} := \frac{\left(Y(1-\frac{\beta}{4})\right)^{M-2K} \left(X(1-\frac{\beta}{2})\right)^{K}}{\Gamma\left((M-K) - \frac{M}{2}\frac{\beta}{2} + 1\right)}, \tag{4.7}$$

where X and Y are two functions independent of M and K and the bound (4.7) depends only on M, K, β and not on the isomorphism class inside $\mathbf{D}_{M,K}$.

With that lemma we deduce

$$Z_{N,\beta} \leq \sum_{K=1}^{N} |\mathbf{D}_{2N,K}| Diri_{2N,K} \\ \leq \sum_{K=1}^{N} \frac{\Gamma(2N+1)(2N)^{2N-2K}}{2^K \Gamma(K+1)\Gamma(2N-2K+1)} \frac{\left(Y(1-\frac{\beta}{4})\right)^{2N-2K} \left(X(1-\frac{\beta}{2})\right)^{K}}{\Gamma\left((2N-K) - N\frac{\beta}{2} + 1\right)}. \tag{4.8}$$

The last step is the evaluation of the right-hand side in (4.8). From [GP77, Equation (2.9)],

$$Z_{N,\beta} \leq (2N)^{\beta N/2} \sum_{K=1}^{N} \frac{\Gamma(2N+1)\exp(2N)}{\Gamma(K+1)\Gamma(2N-K+1)} \left(Y(1-\frac{\beta}{4})\right)^{2N-K} \left(\frac{X(1-\frac{\beta}{2})}{Y(1-\frac{\beta}{4})}\right)^{K}, \tag{4.9}$$

which gives in turn, using Newton's formula

$$Z_{N,\beta} \leq N^{\beta N/2} C_\beta^N, \tag{4.10}$$

where C_β depends only on β. The value of C_β is not important and it is thus enough to prove (4.9) up to a multiplicative constant of order C^N. In particular it yields an upper bound on the partition function

$$\log Z_{N,\beta} \leq \frac{\beta}{2} N \log N + C_\beta N. \tag{4.11}$$

Passing from (4.8) to (4.9) (up to a multiplicative constant of order C^N) is simple after observing that the summands in (4.8) and (4.9) differ by a factor

$$\frac{(2N)^{2N-2K}\Gamma(2N-K+1)}{\Gamma((2N-K) - N\frac{\beta}{2} + 1)\Gamma(2N-2K+1)}.$$

Using Stirling's estimate for the Gamma function we see that the logarithm of the previous expression is equal to

$$(2N-2K)\log N + (2N-K)\log N - (2N-K-N\frac{\beta}{2})\log N - (2N-2K)\log N + O(N).$$

After simplifying we see that the ratio of the two summands in (4.8) and (4.9) is bounded by $C_\beta^N N^{\frac{\beta N}{2}}$ for some constant C_β depending on β, whose precise value is not important here.

The thermodynamic limit for $\log Z_{N,\beta}$ (as expressed in our Proposition 1.4) is proved in [GP77, Sections 3 and 4] using an interesting "conjugation" trick. In this paper we only need to use an upper bound (as (4.11)) and more generally to follow the method of [GP77, Section

2] that we have just recalled. *A posteriori* our large deviation principle at scale N implies in particular that Proposition 1.1 holds.

5. Next order large deviations: lower bound

In this section, we use the blow-up coordinates as introduced in Section 2.3 and we prove the LDP lower bound announced in Proposition 3.9.

In the rest of this section $\bar{P}$ is a fixed stationary tagged signed point process in $\mathcal{P}_{\mathrm{inv},1}(\Lambda \times \mathcal{X})$ such that $\overline{\mathsf{ent}}[\bar{P}]$ is finite, otherwise there is nothing to prove.

5.1. Negative part of the energy.

First we observe that the negative part of the energy is semi-continuous in the suitable direction.

Lemma 5.1. *For any sequence $\{(\vec{X}_N, \vec{Y}_N)\}_N$ such that $i_N(\vec{X}_N, \vec{Y}_N) \in B(\bar{P}, \varepsilon)$, we have*

$$\liminf_{N\to\infty} -\frac{1}{N}\sum_{i=1}^{N}\left(\log r(x_i') + \log r(y_i')\right) \geq \overline{\mathbb{W}}^{\star}(\bar{P}) - o_\varepsilon(1) \quad \text{as } \varepsilon \to 0.$$

Proof. We fix a family $\{\chi_\tau\}_{\tau\in(0,1)}$ of non-negative bounded by 1 smooth functions such that $\chi_\tau \equiv 1$ on $C_{1-\tau}$, $\chi_\tau \equiv 0$ outside C_1, and such that for any $x \in \mathbb{R}^2$, $\chi_\tau(x)$ is nonincreasing with respect to τ. We also set $I_\tau := \int \chi_\tau$, and we have $I_\tau \to 1$ as $\tau \to 0$.

For any $M > 0$ we have

$$\begin{aligned}-\frac{1}{N}\sum_{i=1}^{N}\left(\log r(x_i') + \log r(y_i')\right) &\geq -\frac{1}{N}\sum_{i=1}^{N}\left(\log(r(x_i')\vee\tau) + \log(r(y_i')\vee\tau)\right)\\ &\geq -\int\left(\mathbb{W}_\tau^{\star}(\frac{1}{I_\tau}\chi_\tau, \mathcal{C}) \wedge M\right) d\bar{P}_N.\end{aligned}$$

The map $\mathcal{C} \mapsto \mathbb{W}_\tau^{\star}(\frac{1}{I_\tau}\chi_\tau, \mathcal{C}) \wedge M$ is continuous for the topology we use, hence

$$-\liminf_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\left(\log r(x_i') + \log r(y_i')\right) \geq \int\left(\mathbb{W}_\tau^{\star}(\frac{1}{I_\tau}\chi_\tau, \mathcal{C}) \wedge M\right) d\bar{P} - o_\varepsilon(1).$$

The monotone convergence theorem implies

$$\lim_{M\to\infty}\int\left(\mathbb{W}_\tau^{\star}(\frac{1}{I_\tau}\chi_\tau, \mathcal{C}) \wedge M\right) d\bar{P} = \int \mathbb{W}_\tau^{\star}(\frac{1}{I_\tau}\chi_\tau, \mathcal{C}) d\bar{P}.$$

Let us observe that $\mathbb{W}_\tau^{\star}(\cdot,\cdot)$ is linear in the first variable, in particular $\mathbb{W}_\tau^{\star}(\frac{1}{I_\tau}\chi_\tau, \mathcal{C}) = \frac{1}{I_\tau}\mathbb{W}^{\star}(\chi_\tau, \mathcal{C})$. The family of functions $\{\mathbb{W}_\tau^{\star}(\chi_\tau, \cdot)\}_{\tau\in(0,1)}$ is monotone in τ, and the monotone convergence theorem implies

$$\lim_{\tau\to 0}\int \frac{1}{I_\tau}\mathbb{W}_\tau^{\star}(\chi_\tau, \mathcal{C}) d\bar{P} = \int \mathbb{W}^{\star}(\mathbf{1}_{C_1}, \mathcal{C}) d\bar{P}.$$

By stationarity of $\bar{P}$ we have $\int \mathbb{W}^{\star}(\mathbf{1}_{C_1}, \mathcal{C}) d\bar{P} = \overline{\mathbb{W}}^{\star}(\bar{P})$. Chosing then τ small enough, M large enough and ε small enough depending on τ, M yields the result. □

To prove the LDP lower bound, it thus suffices to prove a lower bound for

$$\int_{\Lambda^{2N}\cap\{i_N(\vec{X}_N,\vec{Y}_N)\in B(\bar{P}^s,\varepsilon)\}} e^{-\frac{\beta}{2}\frac{1}{2\pi}\int_{\mathbb{R}^2}|\nabla V'_{N,r}|^2} d\vec{X}_N d\vec{Y}_N$$

(for notation see Section 2.3). This then becomes similar to the question treated in [LS15] and we follow the same strategy: we need to show that there is a large enough volume of configurations (in fact one which is logarithmically very close to the relative entropy of $\bar{P}^s$) on which the energy $\int_{\mathbb{R}^2} |\nabla V'_{N,r}|^2$ is not too large. For that we will split the box Λ into squares of microscopic size, and we will draw signed configurations independently at random in a Poissonian way in these squares, so that the total number of points in each square is the expected one. Sanov's theorem will guarantee that this generates a set of configurations with the right volume. Then we need to estimate the energy $\int_{\mathbb{R}^2} |\nabla V'_{N,r}|^2$ generated by each such configuration. We in fact need to make these energies restricted to each square depend only on the configuration in the square, which is a priori not the case. For that we use the idea of "screening" the electric field E_K generated in each square K, by modifying the configuration in a small neighborhood of the boundary of the square, to make the electric field energies independent and summable. This is accomplished by enforcing the boundary condition $E_K \cdot \vec{n} = 0$ on each boundary, which ensures that when pasting together these electric fields, the relation

$$-\operatorname{div} E = 2\pi\mathcal{C} \quad \text{in } \mathbb{R}^2 \tag{5.1}$$

holds globally. Indeed, a vector field which is discontinuous across an interface has a distributional divergence concentrated on the interface and equal to the jump of the normal derivative. With the choice (5.1), there is no extra divergence created across the interfaces between the squares. Even if the E_K's were gradients, the global E is in general no longer a gradient. This does not matter however, since the energy of the true electric field $\nabla V'_{N,r}$ generated by the configuration $\mathcal{C}$ will be shown to be smaller than that of E by Helmholtz projection. In the case of positive charges with a neutralizing background, this procedure was introduced in [SS12] and further refined in [SS15, RS15, PS15, LS15], to which we refer for more detail. We implement this program, with the appropriate adaptations needed for controlling dipoles, in the rest of this section.

5.2. **The screening lemma.**

Proposition 5.2. *Let $R > 0$ and let $\mathcal{C} = (\mathcal{C}^+, \mathcal{C}^-)$ be a simple signed configuration in C_R, and E an electric field satisfying*

$$-\operatorname{div} E = 2\pi(\mathcal{C}^+ - \mathcal{C}^-) \quad \text{in } C_R.$$

Let E_r be its truncation at nearest-neighbour half-distance as defined in (2.20).

Let $n^+ := \mathcal{C}^+(C_R)$, $n^- := \mathcal{C}^-(C_R)$, and $n := n^+ + n^-$, and for any $0 < \varepsilon < 1$ let

$$n^{+,\varepsilon}_{\text{int}} := \mathcal{C}^+(C_{R(1-\varepsilon)}), \quad n^{-,\varepsilon}_{\text{int}} := \mathcal{C}^-(C_{R(1-\varepsilon)}), \quad n^{\varepsilon}_{\text{int}} := n^{+,\varepsilon}_{\text{int}} + n^{-,\varepsilon}_{\text{int}},$$

and define

$$M := \frac{1}{R^2} \int_{C_R} |E_r|^2. \tag{5.2}$$

Then for any $0 < \varepsilon < 1$, if R is large enough, there exists a (measurable) family of signed configurations $\Phi^{\text{scr}}_{\varepsilon,R}(\mathcal{C}, E)$ such that for any $\mathcal{C}^{\text{scr}} \in \Phi^{\text{scr}}_{\varepsilon,R}(\mathcal{C}, E)$ we have

(1) $\mathcal{C}$ and $\mathcal{C}^{\text{scr}}$ coincide in $C_{R(1-\varepsilon)}$.

(2) There exists a vector field E^{scr} satisfying

(a) E^{scr} is compatible with $\mathcal{C}^{\mathrm{scr}}$ in C_R and is screened in the sense that

$$\begin{cases} -\mathrm{div}\, E^{\mathrm{scr}} = 2\pi \mathcal{C}^{\mathrm{scr}} & \text{in } C_R \\ E^{\mathrm{scr}} \cdot \vec{n} = 0 & \text{on } \partial C_R \end{cases} . \tag{5.3}$$

(b) The energy of E^{scr} (after truncation) is controlled by that of E

$$\int_{C_R} |E^{\mathrm{scr}}_r|^2 \leq \int_{C_R} |E_r|^2 + C\left(\frac{MR}{\varepsilon} + \varepsilon R^2 + \frac{n}{\varepsilon R}\left|\log \frac{n}{\varepsilon R}\right|\right). \tag{5.4}$$

(3) We have

$$n^{+,\varepsilon}_{\mathrm{int}}(\mathcal{C}) \leq n^+(\mathcal{C}^{\mathrm{scr}}) \leq n^+(\mathcal{C}) + C\left(\frac{MR}{\varepsilon} + 1 + \frac{n}{\varepsilon R}\left|\log \frac{n}{\varepsilon R}\right|\right) \tag{5.5}$$

and the same holds for $n^{-,\varepsilon}_{\mathrm{int}}, n^-$.

Moreover the map $\mathcal{C} \mapsto \Phi^{\mathrm{scr}}_{\varepsilon,R}(\mathcal{C}, E)$ is such that if A is a (measurable) subset of signed point configurations in $\mathcal{X}(C_R)$ such that the quantities n and $n^{\varepsilon}_{\mathrm{int}}$ are constant on A, then we have

$$\begin{aligned} \log \mathbf{Leb}^{\otimes 2R^2}\left(\bigcup_{\mathcal{C}\in A} \Phi^{\mathrm{scr}}_{\varepsilon,R}(\mathcal{C}, E)\right) &\geq \log \mathbf{Leb}^{\otimes n}(A) \\ &\quad - C\left((n - n^{\varepsilon}_{\mathrm{int}})\log R + R + \frac{MR}{\varepsilon} + \varepsilon R^2 + \frac{n}{\varepsilon R}\log\frac{n}{\varepsilon R}\right). \end{aligned} \tag{5.6}$$

Proof. Step 1: choice of a good "annulus". Consider the disjoint "annuli" $A_k = C_{R-8(k+1)} \backslash C_{R-8k}$ for k ranging from 1 to the integer part of $\frac{1}{8}\varepsilon R$. There are $[\frac{1}{8}\varepsilon R]$ such disjoint sets. The proportion of such k's such that

$$\int_{A_k} |E_r|^2 \leq \frac{20MR^2}{\varepsilon R}$$

is strictly larger than $\frac{1}{2}$. Similarly, the proportion of such k's such that

$$|\mathcal{C}|(A_k) \leq \frac{20n}{\varepsilon R}$$

is strictly larger than $\frac{1}{2}$. We deduce that there exists a $k_0 \in [1, [\varepsilon R]]$ such that

$$\int_{A_{k_0}} |E_r|^2 \leq \frac{20MR}{\varepsilon} \qquad |\mathcal{C}|(A_{k_0}) \leq \frac{20n}{\varepsilon R}. \tag{5.7}$$

For brevity, we set $A := A_{k_0}$ and $A_{\mathrm{int}} = \{x \in A, \mathrm{dist}(x, \partial A) \geq 2\}$.

Set next $\eta = \frac{\varepsilon R}{40n}$ and

$$E_{r,\eta} := E_r - \sum_{p \in \mathcal{C}^+ \cap A_{\mathrm{int}}} \nabla(f_{\eta \wedge r(p)} - f_{r(p)})(x-p) + \sum_{p \in \mathcal{C}^- \cap A_{\mathrm{int}}} \nabla(f_{\eta \wedge r(p)} - f_{r(p)})(x-p). \tag{5.8}$$

In other words, we replace $\delta_p^{(r(p))}$ by $\delta_p^{(\eta \wedge r(p))}$ for all the points p in A_{int}.

Computing as in [PS15, Lemma 2.3] we may write

$$\begin{aligned} \int_A |E_{r,\eta}|^2 &= \int_A |E_r|^2 + |E_{r,\eta} - E_r|^2 \\ &+ 2\int_A \sum_{p \in \mathcal{C}^+ \cap A_{\mathrm{int}}} \nabla(f_{\eta \wedge r(p)} - f_{r(p)})(x-p) \cdot E_r + \sum_{p \in \mathcal{C}^- \cap A_{\mathrm{int}}} \nabla(f_{\eta \wedge r(p)} - f_{r(p)})(x-p) \cdot E_r \end{aligned} \tag{5.9}$$

Integrating by parts, and using that $-\mathrm{div}\, E_r = 2\pi(\sum_{x\in\mathcal{C}^+}\delta_x^{(r(x))} - \sum_{x\in\mathcal{C}^-}\delta_x^{(r(x))})$, we have

$$\int_A \sum_{p\in\mathcal{C}^+\cap A_{\mathrm{int}}} \nabla(f_{\eta\wedge r(p)} - f_{r(p)})(x-p)\cdot E_r + \sum_{p\in\mathcal{C}^-\cap A_{\mathrm{int}}} \nabla(f_{\eta\wedge r(p)} - f_{r(p)})(x-p)\cdot E_r \tag{5.10}$$
$$= 2\pi \sum_{p\in\mathcal{C}^+\cap A_{\mathrm{int}}} \int_A (f_{\eta\wedge r(p)} - f_{r(p)})(x-p)\left(\sum_{x\in\mathcal{C}^+}\delta_x^{(r(x))} - \sum_{x\in\mathcal{C}^-}\delta_x^{(r(x))}\right)$$
$$+ 2\pi \sum_{p\in\mathcal{C}^-\cap A_{\mathrm{int}}} \int_A (f_{\eta\wedge r(p)} - f_{r(p)})(x-p)\left(\sum_{x\in\mathcal{C}^+}\delta_x^{(r(x))} - \sum_{x\in\mathcal{C}^-}\delta_x^{(r(x))}\right)$$

and there are no boundary terms since f_r and $f_{\eta\wedge r}$ vanish outside of $B(p, r(p))$ with $r(p) \le 1$. By the same argument, and since all the balls of radius $r(p)$ are disjoint, all the terms in the right-hand side of (5.10) vanish. We are thus left with

$$\int_A |E_{r,\eta}|^2 = \int_A |E_r|^2 + |E_{r,\eta} - E_r|^2 = \int_A |E_r|^2 + \sum_{p\in\mathcal{C}\cap A_{\mathrm{int}}} \int_{\mathbb{R}^2} |\nabla(f_{\eta\wedge r(p)} - f_{r(p)})|^2. \tag{5.11}$$

and using again the same integration by parts argument, we have

$$\int_{\mathbb{R}^2} |\nabla(f_{\eta\wedge r(p)} - f_{r(p)})|^2 = 2\pi(\log r(p) - \log(\eta\wedge r(p)) \le 2\pi|\log\eta|.$$

It follows with (5.7) that

$$\int_A |E_{r,\eta}|^2 \le \int_A |E_r|^2 + \frac{40\pi n}{\varepsilon R}\left|\log\frac{\varepsilon R}{40n}\right|. \tag{5.12}$$

Step 2: choice of a good boundary. Consider all the ∂C_t where t is chosen so that

$$\partial C_t \subset \{x \in A_{\mathrm{int}}, \mathrm{dist}(x, \partial A_{\mathrm{int}}) \ge 1\}.$$

By the bound $r(x) \le 1$, ∂K_t cannot intersect any $B(p, r(p))$ for $p \notin A_{\mathrm{int}}$. Moreover, by the choice of η and (5.7), we have $\eta|\mathcal{C}|(A) \le \frac{1}{2}$. Thus, the total perimeter of the balls $B(p, \eta\wedge r(p))$ with $p \in \mathcal{C}\cap A_{\mathrm{int}}$ is bounded by $\frac{1}{2}$. We deduce that there exists t such that $\partial C_t \subset A_{\mathrm{int}}$ and ∂C_t intersects none of the $B(p,\eta)$ for $p \in \mathcal{C}\cap A_{\mathrm{int}}$ and none of the $B(p, r(p))$ for $p \in \mathcal{C}\backslash A_{\mathrm{int}}$. Applying a mean value argument to the integrand in (5.12), and using (5.7), we may also assume that t is such that the restriction (or trace) of $E_{r,\eta}$ on ∂C_t is well defined as an L^2 function, and denoting

$$g := (E_{r,\eta}\cdot\vec{n})\lfloor_{\partial C_t} \tag{5.13}$$

where $\vec{n}$ denotes the inner unit normal, we have

$$\int_{\partial C_t} |g|^2 \le 10\left(\int_A |E_r|^2 + \frac{40\pi n}{\varepsilon R}\left|\log\frac{\varepsilon R}{40n}\right|\right) \le 10\left(\frac{20MR}{\varepsilon} + \frac{40\pi n}{\varepsilon R}\left|\log\frac{\varepsilon R}{40n}\right|\right). \tag{5.14}$$

Step 3: construction outside C_t

We take the C_t given by the previous step and keep the configuration in C_t unchanged, and discard the configuration in $C_R\backslash C_t$. This way the first item will be verified.

Consider next ∂C_t and partition each of its sides into segments I_l of length $\in [\frac{1}{2}, \frac{3}{2}]$. This yields a natural partition of $C_{t+1}\backslash C_t$ into disjoint rectangles $\mathcal{R}_l$, $l = 1, \cdots, L$. Each $\partial\mathcal{R}_l$ has four sides: one is I_l (or does not exist if $\mathcal{R}_l$ is a corner square), one (or two in case of a corner) belongs to ∂C_{t+1}, one is adjacent to $\partial\mathcal{R}_{l-1}$, one to $\mathcal{R}_{l+1}$.

For each l, we let g_l denote the restriction of g to I_l. We also define $n_0 = 0$ and for each $l \in [1, L]$,

$$n_l = \left[\sum_{k=1}^{l} \int g_k\right] - \sum_{k=0}^{l-1} n_k, \qquad c_l = \sum_{k=1}^{l} \int g_k - \left[\sum_{k=1}^{l} \int g_k\right] \tag{5.15}$$

where $[\cdot]$ is the integer part. We observe that

$$|c_l| \le 1, \qquad \sum_{k=1}^{l} n_k = \left[\sum_{k=1}^{l} \int g_k\right] \tag{5.16}$$

and

$$n_l = c_l - c_{l-1} - \int g_l. \tag{5.17}$$

In each $\mathcal{R}_l$ we let Λ_l be a set of $|n_l|$ points of sign equal to that of n_l, and which are a perturbation of a fixed regular square lattice Λ_l^0 of sidelength $1/\sqrt{|n_l|}$. More formally, to each $z_i \in \Lambda_l^0$ which is at distance $\ge 1/\sqrt{|n_l|}$ from $\partial\mathcal{R}_l$, we associate a point x_i satisfying $|x_i - z_i| \le 1/(4\sqrt{|n_l|})$, and set $\Lambda_l = \{x_i\}_{i=1}^{|n_l|}$. We let h_l be the mean zero solution to

$$\begin{cases} -\Delta h_l = 2\pi sgn(n_l) \sum_{p\in\Lambda_l} \delta_p & \text{in } \mathcal{R}_l \\ \nabla h_l \cdot \vec{n} = g_l & \text{on } \partial\mathcal{R}_l \cap I_l \\ \nabla h_l \cdot \vec{n} = -c_{l-1} & \text{on } \partial\mathcal{R}_l \cap \partial\mathcal{R}_{l-1} \\ \nabla h_l \cdot \vec{n} = c_l & \text{on } \partial\mathcal{R}_l \cap \partial\mathcal{R}_{l+1} \\ \nabla h_l \cdot \vec{n} = 0 & \text{on } \partial\mathcal{R}_l \cap \partial C_{t+1}. \end{cases}$$

One may check that this equation is solvable, and has a unique solution with mean zero, in view of (5.17). We may also write $h_l = u_l + v_l$ where

$$\begin{cases} -\Delta u_l = 2\pi \frac{n_l}{|\mathcal{R}_l|} & \text{in } \mathcal{R}_l \\ \nabla u_l \cdot \vec{n} = g_l & \text{on } \partial\mathcal{R}_l \cap I_l \\ \nabla u_l \cdot \vec{n} = -c_{l-1} & \text{on } \partial\mathcal{R}_l \cap \partial\mathcal{R}_{l-1} \\ \nabla u_l \cdot \vec{n} = c_l & \text{on } \partial\mathcal{R}_l \cap \partial\mathcal{R}_{l+1} \\ \nabla u_l \cdot \vec{n} = 0 & \text{on } \partial\mathcal{R}_l \cap \partial C_{t+1}. \end{cases}$$

and

$$\begin{cases} -\Delta v_l = 2\pi sgn(n_l) \sum_{p\in\Lambda_l} \delta_p - 2\pi \frac{n_l}{|\mathcal{R}_l|} & \text{in } \mathcal{R}_l \\ \nabla v_l \cdot \vec{n} = 0 & \text{on } \partial\mathcal{R}_l. \end{cases}$$

Both equations have a unique solution with zero average. Since the points in Λ_l are well-separated from the boundary, using the same notation as in (2.20), we have

$$\begin{cases} -\mathrm{div}\,(\nabla v_l)_r = 2\pi sgn(n_l) \sum_{p\in\Lambda_l} \delta_p^{(r(p))} - 2\pi \frac{n_l}{|\mathcal{R}_l|} & \text{in } \mathcal{R}_l \\ (\nabla v_l)_r \cdot \vec{n} = 0 & \text{on } \partial\mathcal{R}_l, \end{cases}$$

and we may write $(\nabla v_l)_r = \sum_{p\in\Lambda_l} \nabla G_p$ with

$$\begin{cases} -\Delta G_p = 2\pi sgn(n_l) \left(\delta_p^{(r(p))} - \frac{1}{|\mathcal{R}_l|}\right) & \text{in } \mathcal{R}_l \\ (\nabla G_p) \cdot \vec{n} = 0 & \text{on } \partial\mathcal{R}_l. \end{cases}$$

We have that for each $p \in \Lambda_l$,

$$\int_{\mathcal{R}_l} |\nabla G_p|^2 \le C(|\log r(p)| + 1).$$

This can be proved by comparing G_p to $f_{r(p)}(x-p)$ (defined in (2.6)), for example as in [PS15, proof of (6.23)]. Using that the separation of the points is at least of order $1/\sqrt{|n_l|}$, we may then write

$$\int_{\mathcal{R}_l} |(\nabla v_l)_r|^2 \le C|n_l| \log |n_l| + \sum_{p\neq p' \in \Lambda_l} \nabla G_p \cdot \nabla G_{p'},$$

and by arrangement of the points near a lattice, the second sum is seen to be comparable to $-\sum_{p\neq p'\in\Lambda_l} \log|p-p'|$. This term is itself known to be asymptotic as $|n_l| \to \infty$ to $-(n_l)^2 \int_{\mathcal{R}_l\times\mathcal{R}_l} \log|x-y|\, d\mu(x)\, d\mu(y)$ where μ is the limit of $\frac{1}{|n_l|}\sum_{p\in\Lambda_l}\delta_p$ (here the uniform measure on $\mathcal{R}_l$). We thus conclude that whether $|n_l| \to \infty$ or not, we have

$$\int_{\mathcal{R}_l} |(\nabla v_l)_r|^2 \le C(n_l)^2. \tag{5.18}$$

On the other hand, by elliptic estimates (for example [RS15, Lemma 5.8]), we have

$$\int_{\mathcal{R}_l} |\nabla u_l|^2 \le C\left(\int g_l^2 + c_{l-1}^2 + c_l^2\right) \tag{5.19}$$

with C universal. Since $|c_l| \le 1$ for every l, in view of (5.17) we have $(n_l)^2 \le 3 + 2\int g_l^2$ and thus, combining (5.18) and (5.19), we find

$$\int_{\mathcal{R}_l} |(\nabla h_l)_r|^2 \le C\left(1 + \int g_l^2\right). \tag{5.20}$$

We now define E^{scr} to be E in C_t and to be $\sum_l \mathbf{1}_{\mathcal{R}_l}\nabla h_l$ in $C_{t+1}\backslash C_t$, and $\mathcal{C}^{\mathrm{scr}}$ to be

$$\mathcal{C}^{\mathrm{scr}} := (\mathcal{C}\cap C_t) \cup (\cup_l \Lambda_l)$$

(with sign). We see that the normal components of E^{scr} agree on each interface of $\partial\mathcal{R}_l$, and thus

$$\begin{cases} -\mathrm{div}\, E^{\mathrm{scr}} = 2\pi\mathcal{C}^{\mathrm{scr}} & \text{in } \mathcal{R}_l \\ E^{\mathrm{scr}}\cdot \vec{n} = 0 & \text{on } \partial C_{t+1}. \end{cases} \tag{5.21}$$

Also, in view of (5.20) we have

$$\int_{C_t\backslash C_{t-1}} |E_r^{\mathrm{scr}}|^2 \le C\left(1 + \int g^2\right).$$

We conclude with (5.14) that

$$\int_{C_t} |E_r^{\mathrm{scr}}|^2 \le \int_{C_R} |E_r|^2 + C\left(\frac{MR}{\varepsilon} + 1 + \frac{n}{\varepsilon R}\left|\log\frac{\varepsilon R}{n}\right|\right).$$

Next, we extend (if needed) the configuration to $C_R\backslash C_{t+1}$ by just adding squares with dipoles. More precisely, we partition $C_R\backslash C_{t+1}$ into rectangles $\mathcal{R}$ of sidelengths in $[1,2]$. In each rectangle we place a positive charge p_+ and a negative charge p_- separated from each other and from the boundary of the rectangle by at least $1/4$. We then solve for

$$\begin{cases} -\Delta u = 2\pi(\delta_{p_+} - \delta_{p_-}) & \text{in } \mathcal{R} \\ \nabla u \cdot \vec{n} = 0 & \text{on } \partial\mathcal{R}. \end{cases}$$

We check as above that $\int_{\mathcal{R}} |(\nabla u)_r|^2 \le C$ for each such rectangle, and pasting together the electric fields $(\nabla u)_r$ thus constructed and the one constructed in C_{t+1}, we find a vector field and a family of configurations satisfying all the desired conditions, and we see that this extension has added an energy at most proportional to the volume of $C_R \backslash C_{t+1}$, i.e. $C\varepsilon R^2$.

Step 4: control on the number of points

By construction, the point configuration has not been changed in $C_{R(1-\varepsilon)}$, hence the left-hand inequality in (5.5) holds. The number of points that have been added is given by $\sum_{l=1}^{L} n_l$. In view of (5.17) and (5.14) we obtain

$$\sum_{l=1}^{L} n_l \le \sum_{l=1}^{L} n_l^2 \le C\left(R + \frac{MR}{\varepsilon} + \frac{n}{\varepsilon R}\left|\log \frac{\varepsilon R}{n}\right|\right),$$

which yields the right-hand inequality in (5.5).

Step 5: volume estimate

We now turn to the proof of (5.6). Since we have discarded the point configuration in $C_R \backslash C_t$, in which there were at most $n - n_{\text{int}}^{\varepsilon}$ points, we have lost a logarithmic volume bounded (in absolute value) by

$$(n - n_{\text{int}}^{\varepsilon}) \log |C_R \backslash K_t|. \tag{5.22}$$

On the other hand, the points that are constructed in each rectangle $\mathcal{R}_l$ were allowed to move independently in a small perturbation of the lattice of sidelength $1/\sqrt{|n_l|}$, e.g. they may be chosen arbitrarily in a disk of radius $\frac{1}{4\sqrt{|n_l|}}$ up to a multiplicative constant in the estimates. This allows us to create a volume of configurations of order

$$\left(\frac{1}{4\sqrt{|n_l|}}\right)^{2|n_l|}$$

in each rectangle $\mathcal{R}_l$. Summing over l, we see that the (absolute value of the) logarithmic volume contribution of the points that are created is bounded by

$$\sum_{l=1}^{L} C|n_l| \log |n_l| \le C \sum_{l=1}^{L} n_l^2 \le C\left(L + \int g^2\right)$$

in view of (5.17). We have $L \le R$ and $\int g^2$ is bounded as in (5.14), which allows us to bound the previous expression by

$$C\left(R + \frac{MR}{\varepsilon} + \frac{n}{\varepsilon R} \log \frac{n}{\varepsilon R}\right). \tag{5.23}$$

Combining (5.22) and (5.23) yields (5.6). □

5.3. **Screening the best electric field.** For any $R > 0$, for any $\mathcal{C} \in \mathcal{X}(C_R)$, we let $\mathcal{O}_R(\mathcal{C})$ be the set of electric fields which are compatible with $\mathcal{C}$ in C_R, i.e. such that $-\operatorname{div} E = 2\pi\mathcal{C}$ in C_R.

We may then define $F_R(\mathcal{C})$ to be the "best energy" associated to $\mathcal{C}$ in C_R, i.e.

$$F_R(\mathcal{C}) := \min\left\{\frac{1}{R^2}\int_{C_R} |E_r|^2, E \in \mathcal{O}_R(\mathcal{C})\right\}. \tag{5.24}$$

Since we may always consider the local electric field associated to $\mathcal{C}$, the set $\mathcal{O}_R(\mathcal{C})$ is non empty. If $\{E^{(k)}\}_k$ is a sequence in $\mathcal{O}_R(\mathcal{C})$ such that $\frac{1}{R^2}\int_{C_R}|E_r|^2$ is bounded, then using Lemma 2.4 we see that $\{E^{(k)}\}_k$ converges weakly (up to a subsequencen extraction) to some E in $L^p_{\mathrm{loc}}(C_R,\mathbb{R}^2)$. Moreover we have $E\in\mathcal{O}_R(\mathcal{C})$ and the sequence $\{E^{(k)}_r\}_k$ converges weakly to E_r in L^2. By the weak lower semi-continuity of the L^2 norm we obtain

$$\int_{C_R}|E_r|^2 \le \liminf_{k\to\infty}\int_{C_R}|E_r^{(k)}|^2.$$

This ensures that the minimum in (5.24) exists.

We next let $\Gamma_R : \mathcal{X}(C_R)\to\mathcal{A}$ be a measurable choice of an optimal electric field, i.e. be such that $\Gamma_R(\mathcal{C})\in\mathcal{O}_R(\mathcal{C})$ and

$$\frac{1}{R^2}\int_{C_R}|\Gamma_{R,r}|^2 = F_R(\mathcal{C}).$$

Let us also make the following observation: if $\bar{P}$ is a stationary tagged signed point process, we have, for any $R>0$

$$\mathbf{E}_{\bar{P}}\left[F_R(\mathcal{C})\right] \le \overline{\mathbb{W}}^o(\bar{P}). \tag{5.25}$$

Indeed from Lemma 2.7 we know that we may find a stationary electric field process $\bar{P}^{\mathrm{elec}}$ which is compatible with $\bar{P}$ and such that

$$\mathbf{E}_{\bar{P}^{\mathrm{elec}}}\left[\left(\frac{1}{R^2}\int_{C_R}|E_r|^2\right)\right] = \overline{\mathbb{W}}^o(\bar{P}),$$

and the left-hand side is by definition $\ge \int F_R(\mathcal{C})d\bar{P}$.

Lemma 5.3. *The map F_R is upper semi-continuous on $\mathcal{X}(C_R)$.*

Proof. The proof goes as in [LS15, Lemma 5.8]. First, if $\mathcal{C}_1$ is a signed point configuration in C_R and $\mathcal{C}_2$ is close to $\mathcal{C}_1$, then they have the same number of points (for both components). We may then evaluate the energy of the "Neumann" electric field $\tilde{E}$ associated to $\mathcal{C}_1-\mathcal{C}_2$ (i.e. the solution to $-\mathrm{div}\,\tilde{E} = 2\pi(\mathcal{C}_1-\mathcal{C}_2)$ with zero mean and vanishing Neumann boundary conditions and see that it can be made arbitrarily small if $\mathcal{C}_2$ is close enough to $\mathcal{C}_1$. By adding $\tilde{E}$ to a properly an electric field in $\mathcal{O}_R(\mathcal{C}_1)$ of almost minimal energy we may construct an element of $\mathcal{O}_R(\mathcal{C}_2)$ whose energy is bounded above by $F_R(\mathcal{C}_1)+o(1)$ as $d_{\mathcal{X}}(\mathcal{C}_2,\mathcal{C}_1)$ goes to zero. □

Henceforth for any $R>0$ and any $\varepsilon\in(0,1)$, if $\mathcal{C}$ is a signed point configuration in C_R we let $\Phi^{\mathrm{scr}}_{R,\varepsilon}(\mathcal{C})$ be the set of point configurations obtained by applying Proposition 5.2 to $\mathcal{C}$ and $\Gamma_R(\mathcal{C})$, in other words we let (with a slight abuse of notation)

$$\Phi^{\mathrm{scr}}_{R,\varepsilon}(\mathcal{C}) := \Phi^{\mathrm{scr}}_{R,\varepsilon}(\mathcal{C},\Gamma_R(\mathcal{C})).$$

To any configuration in $\Phi^{\mathrm{scr}}_{R,\varepsilon}(\mathcal{C})$ is associated a compatible electric field E^{scr} whose energy is bounded in terms of $F_R(\mathcal{C})$ according to the conclusions of Proposition 5.2

$$\int_{C_R}|E^{\mathrm{scr}}_r|^2 \le F_R(\mathcal{C}) + C\left(\frac{R}{\varepsilon}F_R(\mathcal{C}) + \varepsilon R^2 + \frac{n}{\varepsilon R}\log\frac{n}{\varepsilon R}\right). \tag{5.26}$$

5.4. **Construction of configurations.** We now turn to the construction of configurations by cutting the domain into microscopic squares as announced.

For any $N\ge 1$ we let $\Lambda' := \sqrt{N}\Lambda$.

Tiling the domain. For any integer $N \geq 1$ and any $R > 0$, we let

$$\bar{R} := R(1 + (\log R)^{-1/10}). \tag{5.27}$$

If R is such that $\sqrt{N}/\bar{R}$ is an integer, we let for convenience $m_{N,R} := N/\bar{R}^2$. We let $\mathcal{K}_{N,R} := \{\bar{K}_i\}_{i=1,\dots,m_{N,R}}$ be a collection of closed squares with disjoint interior which tile Λ' by translated copies of $C_{\bar{R}}$. For any i we denote by z_i the center of $\bar{K}_i$, and we let K_i be the square of center z_i and sidelength R (and whose sides are parallel to those of $\bar{K}_i$). Finally for any $\varepsilon \in (0,1)$ we let $K_{i,\varepsilon}$ be the square of center z_i and sidelength $R(1-\varepsilon)$. In particular we have $K_{i,\varepsilon} \subset K_i \subset \bar{K}_i$.

5.4.1. *Generating approximating microstates.* In the following lemma we show how to generate configurations with enough phase-space volume and ressembling any given $\bar{P}$.

Lemma 5.4. *Let* $\left((\mathbf{C}_1^+, \mathbf{C}_1^-), \dots, (\mathbf{C}_{m_{N,R}}^+, \mathbf{C}_{m_{N,R}}^-)\right)$ *be* $m_{N,R}$ *independent random variables such that* $(\mathbf{C}_i^+, \mathbf{C}_i^-)$ *is distributed as* $\mathbf{\Pi}^s_{|K_i}$*, in other words* $\mathbf{C}_i^+$ *and* $\mathbf{C}_i^-$ *are the restriction to* K_i *of a couple of independent Poisson point processes of intensity* 1*. We condition this* $m_{N,R}$*-tuple of random variables so that the total number of points of each sign is about* $N\left(\frac{R}{\bar{R}}\right)^2$*, more precisely*

$$\sum_{i=1}^{m_{N,R}} \mathbf{C}_i^+ = \sum_{i=1}^{m_{N,R}} \mathbf{C}_i^- = \lceil N\left(\frac{R}{\bar{R}}\right)^2 \rceil. \tag{5.28}$$

We define $\dot{\mathfrak{M}}_{N,R}$ *as the law of the following random variable in* $\Lambda \times \mathcal{X}$*:*

$$\frac{1}{m_{N,R}} \sum_{i=1}^{m_{N,R}} \delta_{(N^{-1/2}z_i, \theta_{z_i}\cdot(\mathbf{C}_i^+, \mathbf{C}_i^-))}. \tag{5.29}$$

Moreover let $\mathbf{C}$ *be the signed point process obtained as the union of the signed point processes* $(\mathbf{C}_i^+, \mathbf{C}_i^-)$ *i.e.*

$$\mathbf{C} := \left(\sum_{i=1}^{m_{N,R}} \mathbf{C}_i^+, \sum_{i=1}^{m_{N,R}} \mathbf{C}_i^-\right)$$

and define $\widehat{\mathfrak{M}}_{N,R}$ *as the law of the random variable in* $\Lambda \times \mathcal{X}^0$

$$\frac{1}{N}\int_{\Lambda'} \delta_{(N^{-1/2}z, \theta_z\cdot\mathbf{C})} dz.$$

Then for any $\bar{P} \in \mathcal{P}_{s,1}(\Lambda \times \mathcal{X})$ *the following inequality holds*

$$\liminf_{R\to\infty} \lim_{\nu\to 0} \liminf_{N\to\infty} \frac{1}{m_{N,R}} \log \dot{\mathfrak{M}}_{N,R}\left(B(\bar{P}, \nu)\right) \geq -\overline{\mathsf{ent}}[\bar{P}], \tag{5.30}$$

moreover for any $\delta > 0$ *we have*

$$\liminf_{R\to\infty} \lim_{\nu\to 0} \liminf_{N\to\infty} \frac{1}{m_{N,R}} \log \left(\dot{\mathfrak{M}}_{N,R}, \widehat{\mathfrak{M}}_{N,R}\right)\left(B(\bar{P}, \nu) \times B(\bar{P}, \delta)\right) \geq -\overline{\mathsf{ent}}[\bar{P}], \tag{5.31}$$

where $\left(\dot{\mathfrak{M}}_{N,R}, \widehat{\mathfrak{M}}_{N,R}\right)$ *denotes the joint law of* $\dot{\mathfrak{M}}_{N,R}$ *and* $\widehat{\mathfrak{M}}_{N,R}$ *with the natural coupling.*

Proof. First let us forget about the condition on the number of points (i.e. we consider independent Poisson point processes) and about the tags (i.e. let us replace $\bar{P}$ by a signed point process P in $\mathcal{P}_{\mathrm{inv}}(\mathcal{X})$). Then for any fixed R there holds a LDP for $\dot{\mathfrak{M}}_{N,R}$ at speed $m_{N,R}$ with rate function $\mathrm{Ent}[\cdot|\mathbf{\Pi}^s_R]$. This is a consequence of the classical Sanov theorem (see [DZ10, Section 6.2]) because the random variables $\theta_{z_i} \cdot (\mathbf{C}_i^+, \mathbf{C}_i^-)$ are i.i.d. Taking the limit $R \to \infty$ yields

$$\lim_{R\to\infty} \lim_{\nu\to 0} \liminf_{N\to\infty} \frac{1}{m_{N,R}} \log \dot{\mathfrak{M}}_{N,R}\left(B(P,\nu)\right) \geq -\mathsf{ent}[P].$$

We may extend this LDP to the context of tagged (signed) point processes by arguing as in [LS15, Section 7], where it is also shown that the condition on the number of points does not alter the LDP. This leads to (5.30). The lower bound (5.31) follows from (5.30) by elementary manipulations as sketched in [LS15, Section 7]. □

Further conditioning on the points. Let $\varepsilon = (\log R)^{-3}$. For any $i \in \{1, \dots, m_{N,R}\}$ we let n_i be the number of points in K_i and $n_{i,\mathrm{int}}$ be the number of points in $K_{i,\varepsilon}$ (we add a $+$ or $-$ superscript in order to restrict ourselves to points with a positive or negative charge).

Lemma 5.5. *The conclusions of Lemma 5.4 hold after conditioning the random variables* $\left((\mathbf{C}_1^+, \mathbf{C}_1^-), \dots, (\mathbf{C}^+_{m_{N,R}}, \mathbf{C}^-_{m_{N,R}})\right)$ *to satisfy the following additional conditions:*

$$\sum_{i=1}^{m_{N,R}} \frac{n_i}{\varepsilon R}\left|\log \frac{n_i}{\varepsilon R}\right| \leq (\varepsilon R)^{-1/4} N, \tag{5.32}$$

$$\sum_{i=1}^{m_{N,R}} (n_i^{\pm} - n^{\pm}_{i,\mathrm{int}}) \log R \leq (\log R)^{-1/2} N. \tag{5.33}$$

Proof. It is enough to show that both events occur with probability $1 - \exp(-NT)$ with T tending to ∞ as $R \to \infty$, when throwing points as a signed Poisson point process of intensity 1 in $\bigcup_{i=1}^{m_{N,R}} K_i$. The result then follows from standard large deviations estimates. Indeed, to prove that we may assume (5.32) without changing the volume of microstates, we may observe that the exponential moments of

$$\mathcal{C} \mapsto \sqrt{\varepsilon R} \frac{1}{\varepsilon R} n \left|\log \frac{n}{\varepsilon R}\right|$$

under the law of a Poisson point process in C_R are bounded by $O(R^2)$ as $R \to \infty$. In particular,

$$\begin{aligned}
\log \mathbf{\Pi}^s \left(\sum_{i=1}^{m_{N,R}} \frac{n_i}{\varepsilon R}\left|\log \frac{n_i}{\varepsilon R}\right| > (\varepsilon R)^{-1/4} N \right) &= \log \mathbf{\Pi}^s \left(\sum_{i=1}^{m_{N,R}} \sqrt{\varepsilon R} \frac{n_i}{\varepsilon R}\left|\log \frac{n_i}{\varepsilon R}\right| > (\varepsilon R)^{1/4} N \right) \\
&\leq -(\varepsilon R)^{1/4} N + \log \mathbf{E}_{\mathbf{\Pi}^s} \exp\left(\sum_{i=1}^{m_{N,R}} \sqrt{\varepsilon R} \frac{n_i}{\varepsilon R}\left|\log \frac{n_i}{\varepsilon R}\right| \right) \\
&\leq -(\varepsilon R)^{1/4} N + \frac{N}{R^2} \log \mathbf{E}_{\mathbf{\Pi}^s_{|C_R}} \exp \sqrt{\varepsilon R} \frac{1}{\varepsilon R} n \left|\log \frac{n}{\varepsilon R}\right| \leq -(\varepsilon R)^{1/4} N + O(N).
\end{aligned}$$

In particular (5.32) indeed occurs with probability of order $1 - \exp(-(\varepsilon R)^{1/4} N)$.

To prove that we may assume (5.33) we may argue similarly, by first observing that the exponential moments of

$$\mathcal{C} \mapsto (n_i - n_{i,\mathrm{int}})(\log R)^2$$

under a standard Poisson point process (of intensity 1) in C_R are bounded by $O(R^2)$ as $R \to \infty$. Indeed the quantity $n_i - n_{i,\mathrm{int}}$ is nothing but the number of points in the thin layer $C_R \backslash C_{R(1-\varepsilon)}$ which has an area of order $R^2\varepsilon = R^2(\log R)^{-3}$. We then deduce that

$$\log \mathbf{\Pi}\left(\sum_{i=1}^{m_{N,R}} (n_i^{\pm} - n_{i,\mathrm{int}}^{\pm}) \log R > (\log R)^{-1/2} N\right) \leq -C(\log R)^{1/4} N,$$

which implies that (5.33) indeed occurs with large enough probability. □

5.4.2. *Screening microstates.*

Lemma 5.6. *Let $\bar{P} \in \mathcal{P}_{s,1}(\Lambda \times \mathcal{X})$ and $\delta > 0$ be fixed. Let $N, R, \bar{R}$ be as above. For any $\nu > 0$ there exists a set A^{mod} of signed point configurations in Λ' of the form $\mathcal{C}^{\mathrm{mod}} = \sum_{i=1}^{m_{N,R}} (\mathcal{C}_i^{\mathrm{mod},+}, \mathcal{C}_i^{\mathrm{mod},-})$ where $(\mathcal{C}_i^{\mathrm{mod},+}, \mathcal{C}_i^{\mathrm{mod},-})$ is a signed point configuration in K_i, satisfying*

$$\sum_{i=1}^{m_{N,R}} \mathcal{C}_i^{\mathrm{mod},+}(\Lambda') = \sum_{i=1}^{m_{N,R}} \mathcal{C}_i^{\mathrm{mod},-}(\Lambda') = N \tag{5.34}$$

and such that the following holds

(1) If R is large enough, ν small enough and N large enough we have for any $\mathcal{C}^{\mathrm{mod}} \in A^{\mathrm{mod}}$

$$\frac{1}{N} \int_{\Lambda'} \delta_{(N^{-1/2} z, \theta_z \cdot \mathcal{C}^{\mathrm{mod}})} dz \in B(\bar{P}, \frac{3\delta}{4}). \tag{5.35}$$

(2) For any $\mathcal{C}^{\mathrm{mod}}$ in A^{mod}, there exists an electric field E^{mod} satisfying

(a) E^{mod} is compatible with $\mathcal{C}^{\mathrm{mod}}$

$$\begin{cases} \mathrm{div}\,(E^{\mathrm{mod}}) = 2\pi \mathcal{C}^{\mathrm{mod}} & \textit{in } \Lambda' \\ E^{\mathrm{mod}} \cdot \vec{n} = 0 & \textit{on } \partial\Lambda' \end{cases}. \tag{5.36}$$

(b) The energy of E^{mod} is bounded by

$$\limsup_{R \to \infty, \nu \to 0, N \to \infty} \frac{1}{2\pi} \int_{\mathbb{R}^2} |E_r^{\mathrm{mod}}|^2 \leq \overline{\mathbb{W}}^o(\bar{P}) + \delta, \tag{5.37}$$

uniformly on $\mathcal{C}^{\mathrm{mod}} \in A^{\mathrm{mod}}$.

(3) There is a good volume of such microstates

$$\liminf_{R \to \infty, \nu \to 0, N \to \infty} \frac{1}{N} \log \frac{\mathbf{Leb}^{2N}}{|\Lambda'|^{2N}} \left(A^{\mathrm{mod}}\right) \geq -\overline{\mathsf{ent}}[\bar{P}]. \tag{5.38}$$

Proof. For any $\nu > 0$, let us write the conditions for a signed point configuration $\mathcal{C} := \sum_{i=1}^{m_{N,R}} \mathcal{C}_i$

$$\frac{1}{\Lambda'} \int_{\Lambda'} \delta_{(N^{-1/2} z, \theta_z \cdot \mathcal{C})} dx \in B(\bar{P}, \delta) \tag{5.39}$$

$$\frac{1}{m_{N,R}} \sum_{i=1}^{m_{N,R}} \delta_{(N^{-1/2} z_i, \theta_{z_i} \cdot \mathcal{C}_i)} \in B(\bar{P}, \nu). \tag{5.40}$$

By Lemma 5.4 we know that given $\delta > 0$, for any N, R, ν (such that $N/R \in \mathbb{N}$) there exists a set A^{abs} ("abs" as "abstract" because we generate them abstractly - and not by hand - using Sanov's theorem as explained in the previous section) of configurations $\mathcal{C}^{\mathrm{abs,s}} = \sum_{i=1}^{m_{N,R}} \mathcal{C}_i^{\mathrm{abs,s}}$ with N points, where $\mathcal{C}_i^{\mathrm{abs,s}}$ is a point configuration in the square K_i, such that for any $\mathcal{C}^{\mathrm{abs,s}} \in A^{\mathrm{abs}}$ (5.39) and (5.40) hold and satisfying

$$\liminf_{R\to\infty,\nu\to 0,N\to\infty} \frac{1}{2N} \log \frac{\mathbf{Leb}^{2N}}{|\Lambda_{N,R}|^{2N}}(A^{\mathrm{abs}}) \geq -\int_\Lambda \mathsf{ent}[\bar{P}^x]dx. \tag{5.41}$$

To see how Lemma 5.4 yields (5.41) it suffices to note that the law of the $2N$-points signed point process $\mathbf{C}$ of Lemma 5.4 coincides with the law of the point process induced by the $2N$-th product of the normalized Lebesgue measure on Λ', and with this observation (5.31) gives (5.41).

We let A^{mod} be the set of configurations obtained after applying the screening procedure described in Section 5.2 with the parameter ε chosen as

$$\varepsilon = \frac{1}{\log^3 R}. \tag{5.42}$$

More precisely, for each $\mathcal{C}^{\mathrm{abs,s}}$ in A^{abs} we decompose $\mathcal{C}^{\mathrm{abs,s}}$ as $\sum_{i=1}^{m_{N,R}} \mathcal{C}_i^{\mathrm{abs,s}}$ where $\mathcal{C}_i^{\mathrm{abs,s}}$ is a signed point configuration in K_i, and for any $i = 1, \dots, m_{N,R}$ we let $\Phi_i^{\mathrm{scr}}(\mathcal{C}^{\mathrm{abs,s}})$ be the set of signed point configurations obtained after screening the configuration $\mathcal{C}_i^{\mathrm{abs,s}}$ in K_i. Combining (5.5) with (5.28) and (5.32), (5.33) we see that any signed point configuration $\mathcal{C}^{\mathrm{mod}}$ has a total number of points with positive charge between $N - o(N)$ and N. We may then complete the point configurations by just adding squares with dipoles in the remaining layers $\bigcup_{i=1,\dots,m_{N,R}} (\bar{K}_i \backslash K_i)$, in such a way that (5.34) is satisfied.

We then let $\Phi^{\mathrm{mod}}(\mathcal{C}^{\mathrm{abs,s}})$ be the set of signed point configurations in Λ' obtained as the cartesian product of the $\Phi_i^{\mathrm{scr}}(\mathcal{C}^{\mathrm{abs,s}})$

$$\Phi^{\mathrm{mod}}(\mathcal{C}^{\mathrm{abs,s}}) := \prod_{i=1}^{m_{N,R}} \Phi_i^{\mathrm{scr}}(\mathcal{C}^{\mathrm{abs,s}})$$

and A^{mod} ("mod" as "modified") is defined as the image of A^{abs} by Φ^{mod}. Since $\bar{P}$ and δ are given, the set A^{mod} depends on the parameters N, R, ε, ν.

Let us now check that A^{mod} satisfies the conclusions of Lemma 5.6.

Distance to $\bar{P}$. To prove the first item we claim that the screening procedure preserves the closeness of the continuous average to $\bar{P}$ as expressed in (5.39) (however in general it does not preserve that of the discrete average (5.40)). The proof of such a claim was already given (in a slightly different setting) in [LS15, Section 6.3.2].

Since the topology on $\mathcal{X}^0$ is *local*, when comparing two random signed point processes we can localize the configurations to a square of fixed size R_0, up to a uniform error which goes to 0 as $R_0 \to \infty$. Now, the main point is that the screening procedure only modifies the configurations in a thin layer of size εR, (where ε has been chosen in (5.42)) in each square K_i. In particular, when R is large (hence ε is small), the vast majority of the translates of a given square R_0 by $z \in \Lambda'$ does not intersect any such thin layer, so that the configurations in them have not been modified when passing from $\mathcal{C}^{\mathrm{abs,s}}$ to $\mathcal{C}^{\mathrm{mod}}$. Fixing R_0 large enough and sending $R \to \infty$ we may thus bound the distance between the continuous average for $\mathcal{C}^{\mathrm{abs,s}}$ and the one for $\mathcal{C}^{\mathrm{mod}}$ by at most $\delta/4$.

Energy. First we associate to any $\mathcal{C}^{\mathrm{mod}} \in A^{\mathrm{mod}}$ a screened electric field E^{mod}. We know by definition of the map Φ^{scr} that for $\mathcal{C}^{\mathrm{mod}} \in A^{\mathrm{mod}}$, for any $i = 1 \dots m_{N,R}$ there exists an electric field E_i^{mod} such that

$$\begin{cases} \mathrm{div}\,(E_i^{\mathrm{mod}}) = 2\pi \mathcal{C}_i^{\mathrm{mod}} & \text{in } K_i \\ E_i^{\mathrm{mod}} \cdot \vec{n} = 0 & \text{on } \partial K_i \end{cases}$$

An electric field $\bar{E}_i^{\mathrm{mod}}$ satisfying the analogous relation on $\bar{K}_i \backslash K_i$ can also easily be constructed, and its energy can be bounded be the number of dipoles added in that layer. Setting $E^{\mathrm{mod}} := \sum_i E_i^{\mathrm{mod}} \mathbf{1}_{K_i} + \bar{E}_i^{\mathrm{mod}} \mathbf{1}_{\bar{K}_i \backslash K_i}$ provides an electric field satisfying (5.36). We now turn to bounding its energy. Let $\mathcal{C}^{\mathrm{abs,s}} \in A^{\mathrm{abs}}$ be such that $\mathcal{C}^{\mathrm{mod}}$ is obtained from $\mathcal{C}^{\mathrm{abs,s}}$ after screening. For any $i = 1, \dots, m_{N,R}$, the energy of E_i^{mod} is bounded as in (5.26) in terms of the "best energy" associated to $\mathcal{C}_i^{\mathrm{abs,s}}$. We thus have, by summing the contributions of each square K_i and $\bar{K}_i \backslash K_i$

$$\begin{aligned} (5.43) \quad \int_{\Lambda'} |E_r^{\mathrm{mod}}|^2 = \sum_{i=1}^{m_{N,R}} \int_{K_i} |E_{i,r}^{\mathrm{mod}}|^2 + \int_{\bar{K}_i \backslash K_i} |\bar{E}_{i,r}^{\mathrm{mod}}|^2 \leq \sum_{i=1}^{m_{N,R}} F_R(\mathcal{C}_i^{\mathrm{abs,s}}) \\ + C \sum_{i=1}^{m_{N,R}} \frac{R}{\varepsilon} F_R(\mathcal{C}_i^{\mathrm{abs,s}}) + \sum_{i=1}^{m_{N,R}} \frac{n_i}{\varepsilon R} \left| \log \frac{n_i}{\varepsilon R} \right| + N\varepsilon + o(N) \end{aligned}$$

where n_i is the number of points $|\mathcal{C}_i^{\mathrm{abs,s}}|(K_i)$.

We now use the fact that the discrete average of the configurations in the square K_i is close to $\bar{P}$ and that F_R is upper semi-continuous (see Lemma 5.3). We thus have

$$(5.44) \qquad \limsup_{R\to\infty, \nu\to 0} \frac{1}{m_{N,R}} \sum_{i=1}^{m_{N,R}} F_R(\mathcal{C}_i^{\mathrm{abs,s}}) \leq \int F_R(\mathcal{C}) d\bar{P} \leq \overline{\mathbb{W}}^o(\bar{P})$$

where the last inequality follows from (5.25). Combining (5.43), (5.44) and (5.32) proves (5.37) .

Volume. We now wish to bound the volume loss between the set A^{abs} of microstates generated "abstractly" and the set A^{mod} of configurations obtained after modification by the screening procedure. From the conclusions Proposition 5.2 we may estimate the cost (in logarithmic volume) of screening the signed point configurations. According to (5.6) the loss can be controlled by

$$\int_{\mathcal{C}^{\mathrm{abs,s}} \in A^{\mathrm{abs}}} C \left(\sum_{i=1}^{m_{N,R}} ((n_i - n_{i,\mathrm{int}}) \log R + R) + \frac{R}{\varepsilon} \sum_{i=1}^{m_{N,R}} \left(F_R(\mathcal{C}_i^{\mathrm{abs,s}}) + \varepsilon R^2 \right) + \sum_{i=1}^{m_{N,R}} \frac{n_i}{\varepsilon R} \left| \log \frac{n_i}{\varepsilon R} \right| \right).$$

We have $m_{N,R} R = o(N)$, and $\lim_{R,N\to\infty} \frac{1}{N} m_{N,R} \varepsilon R^2 = 0$ due to the choice (5.42).

We also have, from (5.32)

$$\lim_{R\to\infty} \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{m_{N,R}} \frac{n_i}{\varepsilon R} \left| \log \frac{n_i}{\varepsilon R} \right| = 0.$$

Since (5.44) holds we obtain

$$\lim_{R\to\infty} \lim_{\nu\to 0} \lim_{N\to\infty} \frac{1}{N} \frac{R}{\varepsilon} \sum_{i=1}^{m_{N,R}} F_R(\mathcal{C}_i^{\mathrm{abs,s}}) = 0.$$

Finally from (5.33) we get that

$$\sum_{i=1}^{m_{N,R}} (n_i - n_{i,\mathrm{int}}) \log R = o(N)$$

where $n_i = |\mathcal{C}_i^{\mathrm{abs,s}}|(C_R)$ and $n_{i,\mathrm{int}} = |\mathcal{C}_i^{\mathrm{abs,s}}|(C_{R(1-\varepsilon)})$, which concludes the proof of (5.38). □

Combining Lemmas 5.1 and 5.6 yields Proposition 3.9.

6. Next order large deviations: upper bound

In this section we prove the Large Deviations upper bound stated in Proposition 3.10.

6.1. Positive part of the energy.

First we observe that the positive part of the energy is semi-continuous in the suitable direction.

Lemma 6.1. *Let $\bar{P} \in \mathcal{P}_{\mathrm{inv},1}(\Lambda \times \mathcal{X})$. For any sequence $\{(\vec{X}_N, \vec{Y}_N)\}_N$ such that $i_N(\vec{X}_N, \vec{Y}_N) \in B(\bar{P}, \varepsilon)$, we have*

$$\liminf_{N\to\infty} \frac{1}{2\pi N} \int_{\mathbb{R}^2} |\nabla V'_{N,r}|^2 \geq \overline{\mathbb{W}}^o(\bar{P}) - o_\varepsilon(1).$$

Proof. Define $\bar{P}^{\mathrm{elec}}_{(\vec{X}_N,\vec{Y}_N)}$ as

$$\bar{P}^{\mathrm{elec}}_{(\vec{X}_N,\vec{Y}_N)} := \int_\Lambda \delta_{(x, V'_N(\sqrt{N}x + \cdot))} dx, \tag{6.1}$$

i.e. $\bar{P}^{\mathrm{elec}}_{(\vec{X}_N,\vec{Y}_N)}$ is the push-forward of the Lebesgue measure on Λ by $x \mapsto V'_N(\sqrt{N}x + \cdot)$, where V'_N was defined in (2.16). Assume that $\int_{\mathbb{R}^2} |\nabla V'_{N,r}|^2 \leq CN$ (otherwise there is nothing to prove). For any $m > 0$ we have

$$\int_\Lambda \frac{1}{|C_m|} \int_{C_m} |E_r|^2 d\bar{P}^{\mathrm{elec}}_{(\vec{X}_N,\vec{Y}_N)}(x, E) \leq \frac{1}{N} \int_{\mathbb{R}^2} |\nabla V'_{N,r}|^2. \tag{6.2}$$

The bound (6.2) implies that the push-forward of $\bar{P}^{\mathrm{elec}}_{(\vec{X}_N,\vec{Y}_N)}$ by $(z, E) \mapsto (z, E_r)$ is tight in $\mathcal{P}(\Lambda \times L^2_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2))$. Using Lemma 2.4 we also get the tightness of $\bar{P}^{\mathrm{elec}}_{(\vec{X}_N,\vec{Y}_N)}$ in $\mathcal{P}(\Lambda \times L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2))$. The associated random tagged signed point process $\bar{P}_{(\vec{X}_N,\vec{Y}_N)}$ (i.e. the push-forward of $\bar{P}^{\mathrm{elec}}_{(\vec{X}_N,\vec{Y}_N)}$ by $(z, E) \mapsto (z, \mathrm{Conf}(E))$ is also tight in $\mathcal{P}(\Lambda \times \mathcal{X})$, arguing as in Lemma 2.11. Up to a subsequence extraction, we may thus find $\bar{P}^{\mathrm{elec}}_0 \in \mathcal{P}(\Lambda \times L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2))$ and $P_0 \in \mathcal{P}(\Lambda \times \mathcal{X})$ such that

(1) $\bar{P}^{\mathrm{elec}}_{(\vec{X}_N,\vec{Y}_N)}$ converges to $\bar{P}^{\mathrm{elec}}_0$ as $N \to \infty$
(2) $\bar{P}_{(\vec{X}_N,\vec{Y}_N)}$ converges to $\bar{P}_0$ as $N \to \infty$.

It is not hard to check that the first marginal of $\bar{P}_0$ is the Lebesgue measure on Λ, that its second marginal is stationary, and that $\bar{P}_0 \in B(\bar{P}, 2\varepsilon)$.

Using Lemma 2.6, we obtain from (6.2) that for any $m > 0$

$$\int_\Lambda \frac{1}{|C_m|} \int_{C_m} |E_r|^2 d\bar{P}^{\mathrm{elec}}_0(x, E) \leq \liminf_{N\to\infty} \frac{1}{N} \int_{\mathbb{R}^2} |\nabla V'_{N,r}|^2.$$

In particular, letting $m \to \infty$ and using the definition of $\overline{\mathbb{W}}^o$ we obtain

$$\overline{\mathbb{W}}^o(\bar{P}_0) \le \liminf_{N\to\infty} \frac{1}{N}\int_{\mathbb{R}^2} |\nabla V'_{N,r}|^2.$$

We conclude by letting $\varepsilon \to 0$ and using the lower semi-continuity of $\overline{\mathbb{W}}^o$ among random stationary tagged processes, as stated in Lemma 2.8. □

6.2. **Bound on the nearest neighbor contributions.** We are now left to bound from above

$$\int_{\Lambda^{2N}\cap i_N^{-1}(B(\bar{P},\varepsilon))} e^{-\frac{\beta}{2}\sum_{i=1}^N \log r(x'_i)+\log r(y'_i)} d\vec{X}_N d\vec{Y}_N.$$

For $0<\tau<1$ we will distinguish between the points whose nearest neighbor is at distance $\geq \tau$ and those with a very close neighbor, at distance $\le \tau$. We thus write

$$\sum_{i=1}^N \log r(x'_i) + \log r(y'_i) = \sum_{i=1}^N \log(r(x'_i)\vee\tau) + \log(r(y'_i)\vee\tau) + \sum_{i=1}^N \log(\frac{r(x'_i)}{\tau}\wedge 1) + \log(\frac{r(y'_i)}{\tau}\wedge 1). \tag{6.3}$$

Points at distance $\geq \tau$. The contributions due to the interactions of points at distance $\geq \tau$ is continuous, as expressed by the following

Lemma 6.2. *Let $\bar{P}\in\mathcal{P}_{\mathrm{inv},1}(\Lambda\times\mathcal{X})$. For any sequence $\{\vec{X}_N,\vec{Y}_N\}_N$ such that $i_N(\vec{X}_N,\vec{Y}_N)\in B(\bar{P},\varepsilon)$ and such that $\mathcal{N}(\mathcal{C},C_1)$ is uniformly integrable against $d\bar{P}_N$ as $N\to\infty$, we have*

$$\liminf_{N\to\infty} -\sum_{i=1}^N \log(r(x'_i)\vee\tau) + \log(r(y'_i)\vee\tau) \geq -\mathbb{W}^\star_\tau(\bar{P}) + o_\varepsilon(1).$$

Proof. For any $t>0$ let χ_t be a smooth nonnegative function such that $\chi_t\equiv 1$ in C_{1-t} and $\chi_t\equiv 0$ outside C_{1+t} and such that $\int\chi_t = 1$. Let Λ_t be the square $\{x\in\Lambda, d(x,\partial\Lambda)\geq t\}$. For N large enough we have

$$-\frac{1}{N}\sum_{i=1}^N \log(r(x'_i)\vee\tau) + \log(r(y'_i)\vee\tau) \geq -\int \mathbf{1}_{\Lambda_t}(x)\left(\mathbb{W}^\star_\tau(\chi_t,\mathcal{C})\right) d\bar{P}_N.$$

The map $\mathcal{C}\mapsto -\mathbb{W}^\star_\tau(\chi_t,\mathcal{C})$ is continuous and bounded by $(-\log(\tau))\mathcal{N}(\mathcal{C},C_{1+1})$, and since $\mathcal{N}(\mathcal{C},C_{1+t})$ is uniformly integrable against $d\bar{P}_N$ we have

$$\lim_{N\to\infty} -\int \mathbf{1}_{\Lambda_t}(x)\left(\mathbb{W}^\star_\tau(\chi_t,\mathcal{C})\right) d\bar{P}_N = -\int \mathbf{1}_{\Lambda_t}(x)\left(\mathbb{W}^\star_\tau(\chi_t,\mathcal{C})\right) d\bar{P} + o_\varepsilon(1).$$

Letting $t\to 0$ yields the result. □

Contribution of close dipoles. We now turn to bounding the contributions to the Boltzmann factor $e^{-\beta w_N}$ due to pairs of points which are at distance $\le\tau$ from each other, and see that this quantity is negligible when $\tau\to 0$. Using (6.3) we see that we are left with bounding

$$\int_{i_N^{-1}(B(\bar{P},\varepsilon))} e^{-\frac{\beta}{2}\sum_{i=1}^N \log\frac{r(x'_i)}{\tau}\wedge 1 + \log\frac{r(y'_i)}{\tau}\wedge 1}\, d\vec{X}_N\, d\vec{Y}_N.$$

We prove

Lemma 6.3. *We have*

$$\limsup_{\tau\to 0,\varepsilon\to 0,N\to\infty} \frac{1}{N}\log\int_{i_N^{-1}(B(\bar{P},\varepsilon))} e^{-\frac{\beta}{2}\sum_{i=1}^N \log\frac{r(x_i')}{\tau}\wedge 1+\log\frac{r(y_i')}{\tau}\wedge 1}\, d\vec{X}_N\, d\vec{Y}_N \le -\overline{\mathsf{ent}}(\bar{P}). \tag{6.4}$$

This will rely on the method of [GP77] described in Section 4.

Proof. For each configuration, we denote by n the number of points of any sign for which $r(z_i') \le \tau$, and separate over the value of n. Without loss of generality, we may assume that these points are the first n_x ones for the x's and n_y ones for the y's, with $n_x + n_y = n$. We may write

$$\begin{aligned}\int_{i_N^{-1}(B(\bar{P},\varepsilon))} & e^{-\frac{\beta}{2}\sum_{i=1}^{2N}\log\frac{r(x_i')}{\tau}\wedge 1+\log\frac{r(y_i')}{\tau}\wedge 1}\, d\vec{X}_N\, d\vec{Y}_N \\ &\le \sum_{n=0}^{2N}\sum_{n_x+n_y=n}\binom{N}{n_x}\binom{N}{n_y}\int_{\Lambda^n} e^{-\frac{\beta}{2}\sum_{i=1}^n \log\frac{r(z_i')}{\tau}\wedge 1}\, dz_1\dots dz_n \\ &\qquad \times\int_{i_N^{-1}(B(\bar{P},\varepsilon))} dx_{n_x+1}\dots dx_N\, dy_{n_y+1}\dots dy_{2N}.\end{aligned} \tag{6.5}$$

For any $n \ge 0$, $1 > \tau > 0$ and $N \ge 1$, define $\Lambda^n_{N,\tau}$ as the set of n-tuples of points in Λ such that all the nearest-neighbor distances at blown-up (by $\sqrt{N}$) scale are smaller than τ. We define

$$Z(n,\tau,\beta,N) = \int_{\Lambda^n_{N,\tau}} e^{-\frac{\beta}{2}\sum_{i=1}^n \log\frac{r(z_i')}{\tau}}\, dz_1\dots dz_n, \tag{6.6}$$

with $z_i' = z_i\sqrt{N}$.

Fixing the parameter $\delta = 1/\sqrt{|\log\tau|}$, we may rewrite (6.5) as

$$\begin{aligned}\int_{i_N^{-1}(B(\bar{P},\varepsilon))} e^{-\frac{\beta}{2}\sum_{i=1}^{2N}\log\frac{r(x_i')}{\tau}\wedge 1+\log\frac{r(y_i')}{\tau}\wedge 1}\, d\vec{X}_N\, d\vec{Y}_N \le \sum_{n=\lfloor\delta N\rfloor+1}^{2N} Z(n,\tau,\beta,N)|\Lambda|^{2N-n} \\ +\sum_{n=0}^{\lfloor\delta N\rfloor}\sum_{n_x+n_y=n}\binom{N}{n_x}\binom{N}{n_y} Z(n,\tau,\beta,N)\int_{\Lambda^{2N-n}\cap i_N^{-1}(B(\bar{P},\varepsilon))} dx_{n_x+1}\dots dx_N\, dy_{n_y+1}\dots dy_{2N}.\end{aligned}$$

Next, we claim that if $n \le N$ (which is the case here), we have

$$Z(n,\tau,\beta,N) \le \tau^n C^n. \tag{6.7}$$

Assuming this claim is true, and observing that $i_N(\vec{X}_N,\vec{Y}_N) \in B(\bar{P},\varepsilon)$ implies that

$$i_{N-n}(x_n,\dots,x_N,y_n,\dots,y_N) \in B(\bar{P},2\varepsilon)$$

with obvious notation (for δ small enough depending on ε), so we may then write

$$\begin{aligned}(6.8)\quad &\int_{i_N^{-1}(B(\bar{P},\varepsilon))} e^{-\frac{\beta}{2}\sum_{i=1}^{2N}\log\frac{r(x_i')}{\tau}\wedge 1+\log\frac{r(y_i')}{\tau}\wedge 1}\,d\vec{X}_N\,d\vec{Y}_N \\ &\le \sum_{n=\lfloor\delta N\rfloor+1}^{2N}\sum_{n_x+n_y=k}\binom{N}{n_x}\binom{N}{n_y}Z(n,\tau,\beta,N) \\ &+\sum_{n=0}^{\lfloor\delta N\rfloor}\sum_{n_x+n_y=n}\binom{N}{n_x}\binom{N}{n_y}Z(n,\tau,\beta,N)\int_{i_N^{-1}(B(\bar{P},\varepsilon))} dx_{n_x+1}\dots dx_N\,dy_{n_y+1}\dots dy_{2N}.\end{aligned}$$

The first term in the right-hand side of (6.8) is easily bounded above by $C^N\tau^{\delta N}$ in view of (6.7). For the second one, we note that by direct computations one may show that for $n\le\delta N$,

$$\sum_{n_x+n_y=n}\binom{N}{n_x}\binom{N}{n_y}\le C_\delta^N$$

with $\lim_{\delta\to 0}C_\delta=1$. Combining this with (6.7) and inserting into (6.8) we thus are led to

$$\begin{aligned}(6.9)\quad &\int_{i_N^{-1}(B(\bar{P},\varepsilon))} e^{-\frac{\beta}{2}\sum_{i=1}^{2N}\log\frac{r(x_i')}{\tau}\wedge 1+\log\frac{r(y_i')}{\tau}\wedge 1}\,d\vec{X}_N\,d\vec{Y}_N \\ &\le C^N\tau^{\delta N}+\delta N C_\delta^N C^{\delta N}\int_{\Lambda^{2N-2\delta N}\cap i_N^{-1}(B(\bar{P},\varepsilon))} dx_{\delta N+1}\dots dx_N\,dy_{\delta N+1}\dots dy_{2N}.\end{aligned}$$

By the choice of $\delta=1/\sqrt{|\log\tau|}$ we find that the first term is logarithmically negligible and thus

$$\begin{aligned}(6.10)\quad &\limsup_{N\to\infty}\frac{1}{N}\log\int_{\Lambda^{2N}\cap i_N^{-1}(B(\bar{P},\varepsilon))} e^{-\frac{\beta}{2}\sum_{i=1}^{2N}\log\frac{r(x_i')}{\tau}\wedge 1+\log\frac{r(y_i')}{\tau}\wedge 1}\,d\vec{X}_N\,d\vec{Y}_N \\ &\le \log C_\delta + C\delta+\limsup_{N\to\infty}\frac{1}{N}\log\int_{\Lambda^{2N-2\delta N}\cap i_N^{-1}(B(\bar{P},\varepsilon))} dx_{\delta N+1}\dots dx_N\,dy_{\delta N+1}\dots dy_{2N}.\end{aligned}$$

But we have, from Proposition 2.10

$$\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\frac{1}{N}\log\int_{\Lambda^{2N-2\delta N}\cap i_N^{-1}(B(\bar{P},\varepsilon))} dx_{\delta N+1}\dots dx_N\,dy_{\delta N+1}\dots dy_{2N} \le -N(1-\delta)\overline{\mathsf{ent}}(\bar{P})$$

so letting $\varepsilon\to 0$ and $\tau\to 0$ (hence $\delta\to 0$) in (6.10), we obtain the conclusion.

We now check (6.7), following Section 4. We will establish the more general bound

$$(6.11)\qquad Z(n,\tau,\beta,N)\le\left(\frac{C\tau n}{N}\right)^n,$$

which implies (6.7) when $n\le N$. We may assume that $n\ge 1$, otherwise (6.11) is obvious (both sides are equal to 1). First, we rewrite $Z(n,\tau,\beta,N)$ as

$$Z(n,\tau,\beta,N)=\int_{\Lambda^n_{N,\tau}} e^{-\frac{\beta}{2}\sum_{i=1}^n\log\frac{r(S_i)\sqrt{N}}{\tau}}\,d\vec{S}_n.$$

As in Section 4, for each configuration $(S_1, \dots, S_n)$, we form the nearest-neighbor function $i \mapsto F_i$, and associate to the configuration a directed graph with $K \in [1, n]$ connected components, each component comprising a cycle of length 2 together with trees attached to the two vertices of the cycle. We know moreover that the distances to the nearest neighbor $|S_i - S_{F_i}|$ are bounded by $\tau/\sqrt{N}$.

Splitting between isomorphisms classes of graphs and following the computations and using the notation of Section 4, we now have to bound

$$\int_{\Lambda^n_{N,\tau} \cap \{\hat{\gamma}(\vec{S_n}) \equiv \gamma\}} e^{-\frac{\beta}{2}\sum_{i=1}^n \log\left(\frac{\frac{1}{2}|S_i - S_{F_i}|\sqrt{N}}{\tau}\right)} d\vec{S_n}. \tag{6.12}$$

Let $L_1, \dots, L_K$ be the K subsets of indices associated to γ. We make the change of variables: for $i \in L_k$ such that $i \notin c_k$

$$u_i := \sqrt{\frac{N}{n}} \frac{1}{2\tau}(S_i - S_{F_i}),$$

and

$$u_{i_k^a} := \sqrt{\frac{N}{n}} \frac{1}{2\tau}(S_{i_k^a} - S_{i_k^b}), \quad u_{i_k^b} := \sqrt{\frac{N}{n}} \frac{1}{2} S_{i_k^b}.$$

With respect to the new variables, the integral in (6.12) is bounded by

$$\left(\sqrt{\frac{n}{N}}\right)^{2n} e^{-\frac{\beta}{2} n \log \sqrt{\frac{n}{N}}} 4^{p_k} \tau^{2(p_k-1)} \left(\frac{1}{\sqrt{N}}\right)^{\frac{1}{2}\beta p_k} \int_{D'_k} e^{-\frac{\beta}{2}\left(\sum_{i \in L_k \setminus c_k} \log u_i + 2 \log u_{i_k^a}\right)} \prod_{i \notin L_k} du_i \, du_{i_k^a} \, du_{i_k^b},$$

where D'_k is a (suitably enlarged) domain of integration for the new variables where they satisfy

$$\sum_{i \in L_k \setminus c_k} |u_i|^2 \le 1, \qquad |u_{i_k^a}| \le \frac{1}{2\sqrt{n}}, \qquad |u_{i_k^b}| \le 1$$

and p_k denotes the cardinality of L_k. Clearly D'_k is included in the set that was denoted D_k in Section 4, so we deduce using Lemma 4.3 that

$$\begin{aligned}\int_{\Lambda^n_{N,\tau} \cap \{\hat{\gamma}(\vec{S_n}) \equiv \gamma\}} e^{-\frac{\beta}{2}\sum_{i=1}^n \log\left(\frac{\frac{1}{2}|S_i - S_{F_i}|\sqrt{N}}{\tau}\right)} d\vec{S_n} &\le \left(\frac{n}{N}\right)^{n(1-\frac{\beta}{4})} \frac{\tau^{2\sum_{k=1}^K (p_k - 1)}}{N^{\beta/4 \sum_{k=1}^K p_k}} Diri_{n,K} \\ &\le \left(\frac{n}{N}\right)^{n(1-\frac{\beta}{4})} \frac{\tau^{2(n-K)}}{N^{n\beta/4}} Diri_{n,K}.\end{aligned}$$

Summing over all isomorphism classes of γ, we deduce that

$$\begin{aligned}Z(n, \tau, \beta, N) &\le \left(\frac{n}{N}\right)^{n(1-\frac{\beta}{4})} \sum_{K=1}^{n/2} |\mathbf{D}_{n,K}| Diri_{n,K} \frac{\tau^{2(n-K)}}{N^{n\beta/4}} \\ &\le \left(\frac{n}{N}\right)^{n(1-\frac{\beta}{4})} \frac{\tau^n}{N^{n\beta/4}} \sum_{K=1}^{n/2} |\mathbf{D}_{n,K}| Diri_{n,K} \le \left(\frac{n}{N}\right)^{n(1-\frac{\beta}{4})} \frac{\tau^n}{N^{n\beta/4}} \left(\frac{n}{2}\right)^{\beta n/4} C_\beta^{n/2},\end{aligned} \tag{6.13}$$

which yields (6.11). □

7. Leading order large deviations

In this section, we prove a large deviations upper bound at the leading order (N^2) the joint law of the empirical measures μ_N^+ and μ_N^-, with rate function given by (7.1).

The limiting energy. For any two probability measures μ^+, μ^- in $\mathcal{P}(\Lambda)$ we let

$$H(\mu^+, \mu^-) := \min\left\{\int_{\mathbb{R}^2} |E|^2, E \in L^2(\mathbb{R}^2, \mathbb{R}^2) \text{ s.t. } -\operatorname{div} E = 2\pi(\mu^+ - \mu^-)\right\}, \tag{7.1}$$

which represents the electrostatic interaction energy between μ^+ and μ^-. The infimum in (7.1) is achieved because the L^2-norm is coercive and lower semi-continuous for the weak-L^2 topology. In fact it is not difficult to check that if $H(\mu^+, \mu^-)$ is finite, it is equal to $\int_{\mathbb{R}^2} |E^{\mathrm{loc}}|^2$ where E^{loc} is the "local electric field" defined as $E^{\mathrm{loc}}(x) := \int_{\mathbb{R}^2} -\nabla \log|x-t|(d\mu^+(t) - d\mu^-(t))$.

The functional H takes value in $[0, +\infty]$ and we have

$$H(\mu^+, \mu^-) = 0 \iff \mu^+ = \mu^-. \tag{7.2}$$

Large deviations upper bound. We give a large deviation upper bound at speed N^2 for the joint law of the empirical measures μ_N^+ and μ_N^-.

Proposition 7.1. *For any $\mu^+, \mu^- \in \mathcal{P}(\Lambda)$ we have*

$$\limsup_{\varepsilon\to 0} \limsup_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}_N^\beta\left(\{(\mu_N^+, \mu_N^-) \in B((\mu^+, \mu^-), \varepsilon)\}\right) \le -\frac{\beta}{2} H(\mu^+, \mu^-). \tag{7.3}$$

Together with (7.2) this essentially says that we must have $\mu_N^+ \approx \mu_N^-$ for N large, except with very small (of order e^{-N^2}) probability.

Proof. Let μ^+, μ^- be in $\mathcal{P}(\Lambda)$ and $\varepsilon > 0$. Using Lemma 2.3 we may write

$$\begin{aligned} &\mathbb{P}_N^\beta\left(\{(\mu_N^+, \mu_N^-) \in B((\mu^+, \mu^-), \varepsilon)\}\right) \\ &\le \frac{1}{Z_{N,\beta}} \int_{\Lambda^{2N} \cap \{(\mu_N^+, \mu_N^-) \in B((\mu^+, \mu^-), \varepsilon)\}} d\vec{X}_N d\vec{Y}_N \exp\left(-\frac{\beta}{2}\left(\int |\nabla V_{N,r}|^2 + \sum_{i=1}^N \log(r(x_i)) + \sum_{i=1}^N \log(r(y_i))\right)\right). \end{aligned} \tag{7.4}$$

We claim that $\int |\nabla V_{N,r}|^2$ is lower-semi continuous in the following sense: if $\mu_N^+ \to \mu^+$ and $\mu_N^- \to \mu^-$ then

$$\liminf_{N\to\infty} \frac{1}{N^2} \int_{\mathbb{R}^2} |\nabla V_{N,r}|^2 \ge H(\mu^+, \mu^-). \tag{7.5}$$

Indeed, a uniform bound on the L^2-norm implies that $\{\frac{1}{N}\nabla V_{N,r}\}_N$ is tight in $L^2(\mathbb{R}^2, \mathbb{R}^2)$, let us denote by E a limit point. For any N we have by definition

$$-\operatorname{div}(\nabla V_{N,r}) = 2\pi\left(\sum_{i=1}^N \delta_{x_i}^{r(x_i)} - \sum_{i=1}^N \delta_{y_i}^{r(y_i)}\right)$$

and it easily implies that

$$-\operatorname{div} E = 2\pi(\mu^+ - \mu^-).$$

On the other hand, by lower semi-continuity of the L^2-norm with respect to weak convergence we have

$$\liminf_{N\to\infty} \frac{1}{N^2} \int_{\mathbb{R}^2} |\nabla V_{N,r}|^2 \ge \int_{\mathbb{R}^2} |E|^2,$$

thus (7.5) holds, and together with (7.4) it yields

$$\begin{aligned}(7.6)\quad \limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\frac{1}{N^2}\log \mathbb{P}_N^{\beta}\left(\{(\mu_N^+,\mu_N^-)\in B((\mu^+,\mu^-),\varepsilon)\}\right)\\ \le -\frac{\beta}{2}H(\mu^+,\mu^-) - \liminf_{N\to\infty}\frac{1}{N^2}\log Z_{N,\beta}\\ +\limsup_{N\to\infty}\frac{1}{N^2}\log\int_{\Lambda^{2N}}\exp\left(\frac{\beta}{2}\left(\sum_{i=1}^N \log(r(x_i)) + \sum_{i=1}^N \log(r(y_i))\right)\right)d\vec{X}_N d\vec{Y}_N.\end{aligned}$$

Combining (7.6) with the control (1.4) on the partition function and with (3.1) we get

$$\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\frac{1}{N^2}\log \mathbb{P}_N^{\beta}\left(\{(\mu_N^+,\mu_N^-)\in B((\mu^+,\mu^-),\varepsilon)\}\right)\le -\frac{\beta}{2}H(\mu^+,\mu^-),$$

which concludes the proof of the proposition. □

Large deviations lower bound (sketch). We remark (without providing details) that a complementary large deviations lower bound, with the same rate function, can be derived by adapting the approximation constructions in [AZ98, Ser15]. Indeed, to construct an approximate configuration of points with enough volume (in the exponential in N^2 scale), one may proceed as follows. First, one cancels the common parts of μ_+ and μ_- by positioning pairs of positive and negative charges (henceforth referred to as dipoles) with intra-dipole separation of N^{-10} say and inter-dipole separation of at least N^{-1}. Then, one may use the construction of e.g. [Ser15, Theorem 2.3] and construct a sequence of "well-separated" configurations (separation at least $\eta N^{-1/2}$ between points with small enough η) so that $\limsup_{N\to\infty} N^{-2} w_N(\vec{X}_N, Y_N) \le H(\mu_+,\mu_-) + C(\eta)$ where $C(\eta)\to_{\eta\to 0} 0$; the argument in [AZ98] shows that the volume of such configurations is large enough.

Appendix A. Appendix: tail estimates for the complex Gaussian multiplicative chaos, by Wei Wu

In this appendix we apply Theorem 1 and Corollary 1.3 to obtain the tail asymptotics of subcritical complex Gaussian multiplicative chaos on $\mathbb{R}^2$. Let h be an instance of the Gaussian free field (GFF) on $\mathbb{R}^2$, which we define below. Let $D\subset\mathbb{R}^2$ be a bounded domain, we are interested in the distribution of $\left|\int_D e^{i\beta h(x)}dx\right|$, for $\beta\in(0,\sqrt{2})$. This object appears in different contexts, such as the partition function of complex random energy type models, the scaling limit of the compactified height function in two dimensional dimer model [Dub11], and the electric vertex operator in conformal field theory [Gaw97]. Another motivation comes from the conjecture that this object describes the scaling limit of the magnetization of XY model in the plasma phase [FS81]. The Lee-Yang Theorem was proved for the XY model (see [LS81]), therefore one may further conjecture that the characteristic function of $\int_D e^{i\beta h(x)}dx$ has pure imaginary zeros. Here we focus on another perspective, which is the tail behavior of $\int_D e^{i\beta h(x)}dx$, and our approach is based on an identity relating the moments of $\left|\int_D e^{i\beta h(x)}dx\right|$ to the partition function of a two component Coulomb gas (Lemma A.1 below). For simplicity we set $D=\Lambda$, the unit cube.

Formally, let $H_0(\mathbb{R}^2)=\left\{\varphi:\varphi\in C_0^\infty(\mathbb{R}^2),\ \text{s.t.}\int\varphi(x)\,dx=0\right\}$, and denote by $H(\mathbb{R}^2)$ the completion of $H_0(\mathbb{R}^2)$ in $L^2(\mathbb{R}^2)$. The GFF is defined as a random distribution in $(H(\mathbb{R}^2))'$, such that for any $\rho_1, \rho_2 \in H(\mathbb{R}^2)$, the covariances are given by

$$\mathrm{Cov}\left(\langle h, \rho_1\rangle, \langle h, \rho_2\rangle\right) = -\int\int \rho_1(x)\,\rho_2(y)\log|x-y|\,dxdy,$$

where $\langle h, \rho\rangle \doteq \int h(x)\,\rho(x)\,dx$. Therefore, to give a mathematical definition of $e^{i\beta h}$, one immediately runs into the issue that h is only defined up to an additive constant, so that one can only hope to define $e^{i\beta h}$ up to a multiplicative constant. However, as we explain below, $\left|\int_D e^{i\beta h(x)}dx\right|^2$ can be uniquely defined, and we then simply set $\left|\int_D e^{i\beta h(x)}dx\right| = \sqrt{\left|\int_D e^{i\beta h(x)}dx\right|^2}$.

Let h_r denote the GFF on a disk $D_r = \{z : |z| \le r\}$ with zero boundary condition, and let $h^{(m)}$ be the massive GFF on $\mathbb{R}^2$ with mass m. h_r, $h^{(m)}$ are Gaussian processes with covariances given by

$$\begin{aligned}\mathrm{Cov}\left(h_r(x), h_r(y)\right) &\doteq g_r(x,y)\\ &= -\log|x/r - y/r| + \log\left|1 - x\bar{y}/r^2\right|,\end{aligned}$$

and $\mathrm{Cov}\big(h^{(m)}(x), h^{(m)}(y)\big) = \left(-\Delta + m^2\right)^{-1}(x,y)$, respectively. Then, as was explained in Section 6 of [LRV15], both $e^{i\beta h_r}$ and $e^{i\beta h^{(m)}}$ are well defined as random distributions in $(C_0^\infty(\mathbb{R}^2))'$ for $\beta \in (0, \sqrt{2})$. Therefore we can define the random variable $\left|\int_D e^{i\beta h(x)}dx\right|^2$ as the distributional weak limit of $\left|\int_D e^{i\beta h_r(x)}dx\right|^2$ as $r \to \infty$, or the distributional weak limit of $\left|\int_D e^{i\beta h^{(m)}(x)}dx\right|^2$ as $m \to 0$. Indeed, given $\rho_1, \rho_2 \in H(\mathbb{R}^2)$, since

$$\lim_{r\to\infty}\mathrm{Cov}\left(\langle h_r, \rho_1\rangle, \langle h_r, \rho_1\rangle\right) = \lim_{m\to 0}\mathrm{Cov}(\langle h^{(m)}, \rho_1\rangle, \langle h^{(m)}, \rho_1\rangle) = \mathrm{Cov}\left(\langle h, \rho_1\rangle, \langle h, \rho_1\rangle\right),$$

it is possible to prove that the two constructions lead to the same limiting object. We will not give a proof of the equivalence in this appendix, but instead construct the limiting object using the first approach.

As was explained in Section 6.2 of [LRV15], one needs a regularization procedure to define $e^{i\beta h_r}$. For $\varepsilon > 0$ and $x \in D_r$, let $h_r^\varepsilon(x)$ denote the average of h on the circle of radius ε centered at x (we assume h to be identically zero outside D_r). h_r^ε is a Gaussian process with covariances given by

$$\begin{aligned}\mathrm{Cov}\left(h_r^\varepsilon(x), h_r^\varepsilon(y)\right) &\doteq g_r^\varepsilon(x,y)\\ &= \frac{1}{(2\pi\varepsilon)^2}\int_{|z-x|=\varepsilon}\int_{|w-y|=\varepsilon} g_r(z,w)\,dzdw.\end{aligned}$$

It was shown in [LRV15] that as $\varepsilon \to 0$,

$$\varepsilon^{-\beta^2/2}\int_D e^{i\beta h_r^\varepsilon(x)}dx \to \int_D e^{i\beta h_r(x)}dx \text{ in } L_p, \text{ for } p > 1.$$

Some properties of the circle averaged field h_r^ε were summarized in Section 3.1 of [DS11]. In particular,

$$\mathrm{Var} h_r^\varepsilon(x) = \log C(x; D_r) - \log\varepsilon, \tag{A.1}$$

where $C(x; D_r) = r(1 - |x|^2/r^2)$ is the conformal radius of D_r from x. Also notice that when $|x-y| > 2\varepsilon$, $g_r^\varepsilon(x,y) = g_r(x,y)$, and $g_r^\varepsilon(x,y) \to 0$ if either x or y tends to ∂D_r. Moreover, for all $r > 2$ (which we will assume for the rest of the argument) and $\varepsilon < 1$,

$$\sup_{x,y\in D_r} |g_r^\varepsilon(x,y)| \le -\log\varepsilon + C_1, \text{ for some } C_1 < \infty.$$

We may therefore compute, for $k \in \mathbb{N}$,

$$\begin{aligned}
&\text{(A.2)} \quad \mathbb{E}\left|\int_D e^{i\beta h_r(x)}dx\right|^{2k} \\
&= \lim_{\varepsilon\to 0}\mathbb{E}\left|\varepsilon^{-\beta^2/2}\int_D e^{i\beta h_r^\varepsilon(x)}dx\right|^{2k} \\
&= \lim_{\varepsilon\to 0}\varepsilon^{-\beta^2 k}\mathbb{E}\int_{D^{\otimes 2k}}\prod_{i=1}^k e^{i\beta h_r^\varepsilon(x_i)}\prod_{j=1}^k e^{-i\beta h_r^\varepsilon(y_j)}d\vec{x}d\vec{y} \\
&= \lim_{\varepsilon\to 0}\varepsilon^{-\beta^2 k}\int_{D^{\otimes 2k}} e^{-\frac{\beta^2}{2}\mathrm{Var}\left(\sum_{i=1}^k h_r^\varepsilon(x_i)-\sum_{i=1}^k h_r^\varepsilon(y_i)\right)}d\vec{x}d\vec{y} \\
&= \lim_{\varepsilon\to 0}\int_{D^{\otimes 2k}}\prod_{i=1}^k\left(\frac{1}{C(x_i;D_r)\,C(y_i;D_r)}\right)^{\beta^2/2}\left(\frac{\prod_{1<i<j<k}e^{-g_r^\varepsilon(x_i,x_j)}e^{-g_r^\varepsilon(y_i,y_j)}}{\prod_{i,j}e^{-g_r^\varepsilon(x_i,y_j)}}\right)^{\beta^2} d\vec{x}d\vec{y},
\end{aligned}$$

where we applied (A.1) to obtain the last equation. We now show that the limit above equals

$$\begin{aligned}
&\text{(A.3)} \quad \int_{D^{\otimes 2k}}\prod_{i=1}^k\left(\frac{1}{C(x_i;D_r)\,C(y_i;D_r)}\right)^{\beta^2/2}\left(\frac{\prod_{1<i<j<k}e^{-g_r(x_i,x_j)}e^{-g_r(y_i,y_j)}}{\prod_{i,j}e^{-g_r(x_i,y_j)}}\right)^{\beta^2} d\vec{x}d\vec{y} \\
&= \int_{D^{\otimes 2k}}\left(\frac{\prod_{1<i<j<k}|x_i-x_j|\,|y_i-y_j|}{\prod_{i,j}|x_i-y_j|}\right)^{\beta^2} F_r(\vec{x},\vec{y},\beta)\,d\vec{x}d\vec{y},
\end{aligned}$$

where

$$F_r(\vec{x},\vec{y},\beta) = \prod_{i=1}^k\left(1-\frac{|x_i|^2}{r^2}\right)^{-\frac{\beta^2}{2}}\left(1-\frac{|y_i|^2}{r^2}\right)^{-\frac{\beta^2}{2}}\left[\frac{\prod_{i,j}|1-x_i\bar{y}_j/r^2|}{\prod_{1<i<j<k}|1-x_i\bar{x}_j/r^2|\,|1-y_i\bar{y}_j/r^2|}\right]^{\beta^2}.$$

To show the equality of (A.2) and (A.3), given $\varepsilon > 0$, set

$$D_\varepsilon = \left\{(\vec{x},\vec{y})\in D^{\otimes 2k} : \min_{i,j}|x_i-x_j|\ge 2\varepsilon, \min_{i,j}|y_i-y_j|\ge 2\varepsilon, \min_{i,j}|x_i-y_j|\ge 2\varepsilon\right\}.$$

For $(\vec{x},\vec{y}) \in D_\varepsilon$, the integrand in (A.2) and (A.3) coincide. Since the integrand in (A.3) has integrable singularities when $\beta \in (0,\sqrt{2})$, it suffices to prove

$$\lim_{\varepsilon\to 0}\int_{D^{\otimes 2k}\setminus D_\varepsilon}\prod_{i=1}^k\left(\frac{1}{C(x_i;D_r)\,C(y_i;D_r)}\right)^{\beta^2/2}\left(\frac{\prod_{1<i<j<k}e^{-g_r^\varepsilon(x_i,x_j)}e^{-g_r^\varepsilon(y_i,y_j)}}{\prod_{i,j}e^{-g_r^\varepsilon(x_i,y_j)}}\right)^{\beta^2} d\vec{x}d\vec{y} = 0.$$

Given $(x_1,...,x_k),(y_1,...,y_k)\in\mathbb{R}^k$, one can define the Gale-Shapley matching $\sigma(\vec{x},\vec{y})\in S_k$ of $\vec{x}$ with $\vec{y}$, by

(1) Find x_i and y_j, such that

$$\forall i', j',\ |x_i-y_j| < |x_i-y_{j'}| \text{ and } |x_i-y_j| < |x_{i'}-y_j|,$$

and set $\sigma(i) = j$.

(2) Delete the points that have been matched in Step 1.

(3) Iterate the procedure until all the points have been matched.

Set

$$B_\varepsilon = (D^{\otimes 2k} \backslash D_\varepsilon) \cap \{(\vec{x}, \vec{y}) : \sigma(\vec{x}, \vec{y}) = \mathrm{Id}\}.$$

And by symmetry, it suffices to prove the integration over B_ε vanishes as $\varepsilon \to 0$.

Using the properties of $g_r^\varepsilon(\cdot, \cdot)$, we have

$$\sup_{\varepsilon < e^{-C_1}} \sup_{\substack{x,y,z \in D \\ |x-y| \leq 2|x-z|}} \frac{e^{-g_r^\varepsilon(x,y)}}{e^{-g_r^\varepsilon(x,z)}} < \infty.$$

Therefore, by the same argument as [LRV15, Lemma A.2], for $(\vec{x}, \vec{y}) \in B_\varepsilon$ we have the upper bound

$$\left(\frac{\prod_{1<i<j<k} e^{-g_r^\varepsilon(x_i,x_j)} e^{-g_r^\varepsilon(y_i,y_j)}}{\prod_{i,j} e^{-g_r^\varepsilon(x_i,y_j)}} \right)^{\beta^2} \leq C(k, \beta) \prod_{i=1}^k e^{\beta^2 g_r^\varepsilon(x_i,y_i)},$$

thus

$$\int_{B_\varepsilon} \prod_{i=1}^k \left(\frac{1}{C(x_i; D_r) C(y_i; D_r)} \right)^{\beta^2/2} \left(\frac{\prod_{1<i<j<k} e^{-g_r^\varepsilon(x_i,x_j)} e^{-g_r^\varepsilon(y_i,y_j)}}{\prod_{i,j} e^{-g_r^\varepsilon(x_i,y_j)}} \right)^{\beta^2} d\vec{x} d\vec{y}$$

$$\leq C(k, \beta) \prod_{i=1}^k \int_{B_\varepsilon} \left(\frac{1}{C(x_i; D_r) C(y_i; D_r)} \right)^{\beta^2/2} e^{\beta^2 g_r^\varepsilon(x_i,y_i)} dx_i dy_i. \tag{A.4}$$

Now, for $(\vec{x}, \vec{y}) \in B_\varepsilon$, if for some i, $|x_i - y_i| < 2\varepsilon$, then because $g_r^\varepsilon(x_i, y_i) \leq -\log \varepsilon + O(1)$, the integrand in (A.4) is bounded by $O(\varepsilon^{-\beta^2})$. The volume of the point configuration is at most $O(\varepsilon^2)$, thus (A.4) has integrable singularities. Since the volume of B_ε goes to zero as $\varepsilon \to 0$, we conclude that (A.2) equals (A.3).

From the explicit expression of F_r, we see that $F_r(\vec{x}, \vec{y}, \beta) \to 1$ uniformly for all $(\vec{x}, \vec{y}) \in D^{\otimes 2k}$, as $r \to \infty$. Finally, noting Corollary 1.3, we can send $r \to \infty$ and apply the dominated convergence theorem to obtain

Lemma A.1. *For $\beta \in (0, \sqrt{2})$, $k \in \mathbb{N}$, and $\left|\int_D e^{i\beta h(x)} dx\right|$ defined in the sense above, we have*

$$\mathbb{E} \left| \int_D e^{i\beta h(x)} dx \right|^{2k} = Z_{k,2\beta^2},$$

where $Z_{k,2\beta^2}$ is defined as (1.2).

Corollary A.2. *For $\beta \in (0, \sqrt{2})$,*

$$\mathbb{P}\left(\left| \int_D e^{i\beta h(x)} dx \right| > x \right) = \exp\left(-c^*(\beta) x^{\frac{2}{\beta^2}} + o(x^{\frac{2}{\beta^2}}) \right), \quad \text{as } x \to \infty,$$

and

$$c^*(\beta) = \beta^2 \exp\left(-1 + \beta^{-2} \inf_{\mathcal{P}_{inv,1}(\Lambda, \mathcal{X}^s)} \bar{\mathcal{F}}_\beta \right).$$

(See (1.9) for the definition of $\bar{\mathcal{F}}_\beta$.)

Proof. By Chebyshev's inequality and Lemma A.1,

$$\log \mathbb{P}\left(\left| \int_D e^{i\beta h(x)} dx \right| > x \right) \leq \log \frac{\mathbb{E} \left| \int_D e^{i\beta h(x)} dx \right|^{2k}}{x^{2k}} = \log Z_{k,2\beta^2} - 2k \log x.$$

Apply Corollary 1.3, choose k^* to optimize the right hand side while neglecting the $o(k)$ term in $\log Z_{k,2\beta^2}$, we obtain $k^* = \lfloor \beta^{-2} c^*(\beta) x^{\frac{2}{\beta^2}} \rfloor$, and

$$\mathbb{P}\left(\left|\int_D e^{i\beta h(x)} dx\right| > x\right) \leq \exp\left(-c^*(\beta) x^{\frac{2}{\beta^2}} + o(x^{\frac{2}{\beta^2}})\right).$$

For the lower bound, fix constants C_1, d_1, such that $C_1 d_1 + \int_{C_1}^{\infty} \exp\left(-c^*(\beta) x^{\frac{2}{\beta^2}}\right) dx = 1$. Let Y be the non-negative random variable whose p.d.f is given by

$$f(x) = \begin{cases} d_1 & \text{if } 0 \leq x \leq C_1 \\ \exp\left(-c^*(\beta) x^{\frac{2}{\beta^2}}\right) & \text{if } x > C_1 \end{cases}.$$

An explicit computation gives $\log \mathbb{E} Y^{2k} = \log \mathbb{E} \left|\int_D e^{i\beta h(x)} dx\right|^{2k} + o(k)$. Given $\delta > 0$, denote $Y_\delta = (1-\delta) Y^2$. Then there exists $k_0(\delta)$, such that for all $k > k_0(\delta)$, $\mathbb{E} Y_\delta^k \leq \mathbb{E}\left|\int_D e^{i\beta h(x)} dx\right|^{2k}$. Take $C_2 = C_2(\delta) < \infty$, such that for all $k \in \mathbb{N}$, $\mathbb{E}(Y_\delta - C_2)^k \leq \mathbb{E}\left|\int_D e^{i\beta h(x)} dx\right|^{2k}$. Therefore the tail of $\left|\int_D e^{i\beta h(x)} dx\right|$ dominates the tail of $Y_\delta - C_2$. Then for $x > \sqrt{C_1}$,

$$\begin{aligned} \mathbb{P}\left(\left|\int_D e^{i\beta h(x)} dx\right|^2 > x^2\right) &\geq \mathbb{P}\left(Y_\delta - C_2 > x^2\right) \\ &= \mathbb{P}\left(Y > \sqrt{\frac{x^2 + C_2}{1-\delta}}\right) \\ &= \exp\left(-c^*(\beta)\left(\frac{x^2 + C_2}{1-\delta}\right)^{1/\beta^2}\right). \end{aligned}$$

Taking x sufficiently large yields the lower bound. □

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(Thomas Leblé) Sorbonne Universités, UPMC Univ. Paris 06, CNRS, UMR 7598, Laboratoire Jacques-Louis Lions, 4, place Jussieu 75005, Paris, France.
E-mail address: `leble@ann.jussieu.fr`

(Sylvia Serfaty) Sorbonne Universités, UPMC Univ. Paris 06, CNRS, UMR 7598, Laboratoire Jacques-Louis Lions, 4, place Jussieu 75005, Paris, France.
& Institut Universitaire de France
& Courant Institute, New York University, 251 Mercer st, New York, NY 10012, USA.
E-mail address: `serfaty@ann.jussieu.fr`

(Ofer Zeitouni) Department of Mathematics, Weizmann Institute of Science, POB 26, Rehovot 76100, Israel
& Courant Institute, New York University, 251 Mercer Street, New York, NY 10012, USA.
E-mail address: `ofer.zeitouni@weizmann.ac.il`

(Wei Wu) Courant Institute, New York University, 251 Mercer st, New York, NY 10012, USA.
E-mail address: `weiwu@cims.nyu.edu`