Filling invariants for lattices in symmetric spaces

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(joint work with Enrico Leuzinger)

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Conjecture (Thurston, Gromov, Leuzinger-Pittet, Bestvina-Eskin-Wortman)

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Let X be an $(n-1)$-connected simplicial complex or manifold and let $\alpha \in C_{n-1}(X)$ be a cycle. Define

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- $\mathrm{FV}^n_{\mathbb{R}^k}(r^{n-1}) = C_n r^n$ for $k \geq n$ (i.e., $\mathrm{FV}^n_{\mathbb{R}^k}(V) = C_n V^{\frac{n}{n-1}}$)

Filling invariants as geometric group invariants

If X and Y are bilipschitz equivalent, then there is a $C > 0$ such that

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In particular, if G is a group acting geometrically on an n–connected space X, we can define $\mathrm{FV}^n_G = \mathrm{FV}^n_X$ (up to constants).

Examples: negative curvature

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- If X has pinched negative curvature, then we can fill curves using geodesics. These discs have area linear in the length of their boundary, so $\mathrm{FV}^2(n) \sim n$.
- In fact, G is a group with sub-quadratic Dehn function ($\mathrm{FV}^2 \precnsim n^2$) if and only if G is δ-hyperbolic (Gromov).

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- If X has nonpositive curvature, we can fill curves with geodesics, but the discs may have quadratically large area.
- But the class of groups with quadratic Dehn functions is extremely rich; it includes Thompson's group (Guba), many solvable groups (Leuzinger-Pittet, de Cornulier-Tessera), some nilpotent groups (Gromov, Sapir-Ol'shanskii, others), lattices in symmetric spaces (Druţu, Y., Cohen, others), and many more.

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- But subsets of nonpositively curved spaces can have stranger behavior!

Sol_3 and Sol_5

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$$\begin{aligned}\mathrm{Sol}_3 &\subset \left\{ \begin{pmatrix} e^a & 0 & x \\ 0 & e^b & y \\ 0 & 0 & 1 \end{pmatrix} \right\} \\ &\cong \left\{ \begin{pmatrix} e^a & x \\ 0 & 1 \end{pmatrix} \right\} \times \left\{ \begin{pmatrix} e^b & y \\ 0 & 1 \end{pmatrix} \right\} = \mathbb{H}^2 \times \mathbb{H}^2\end{aligned}$$

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But Sol_5 has spheres which are exponentially difficult to fill!

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- So Sol_{2k-1} contains $(k-1)$-spheres (intersections with flats) with exponentially large filling area (Gromov)
- But there are plenty of lower-dimensional surfaces to fill lower-dimensional spheres, so $\mathrm{FV}^n(V) \approx V^{\frac{n}{n-1}}$ when $n < k$ (Y.)

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Theorem (Leuzinger-Y.)

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- A *nonuniform* lattice is a lattice that acts with noncompact quotient

Lattices act on subsets of X

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If Γ is a nonuniform lattice, the quotient $\Gamma\backslash X$ has cusps. Cutting out the cusps corresponds to cutting out horoballs in X.

Lemma

If Γ is a nonuniform lattice, then there is an r_0 such that for $r \geq r_0$, Γ acts geometrically on a set $X(r) \subset X$ such that $X(r)$ is contractible and approximates the r–neighborhood of Γ.
We can write $X(r) = X \setminus \bigcup_i H_i$, where the H_i are a collection of horoballs in X.

Low dimensions

Dimension 1:

- (Lubotzky-Mozes-Raghunathan) If X has rank ≥ 2, then $d_\Gamma(x, y) \approx d_{X(r_0)}(x, y) \approx d_G(x, y)$ for all $x, y \in \Gamma$.

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- (Cohen) $\mathrm{FV}^2_{\mathrm{SP}_p(\mathbb{Z})}(n) \lesssim n^2$ when $p \geq 5$ (i.e., rank ≥ 5).

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Dimension > 2:

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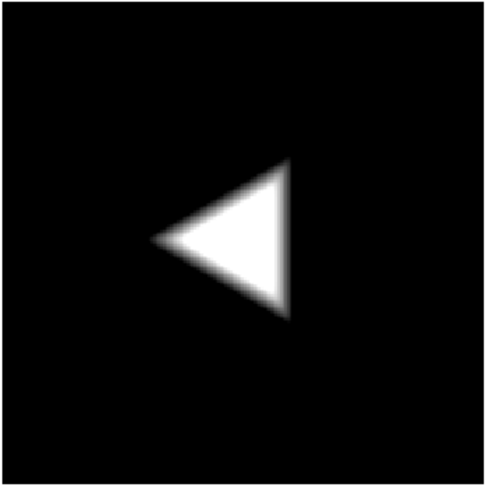
A flat in $SL_3(\mathbb{R})$



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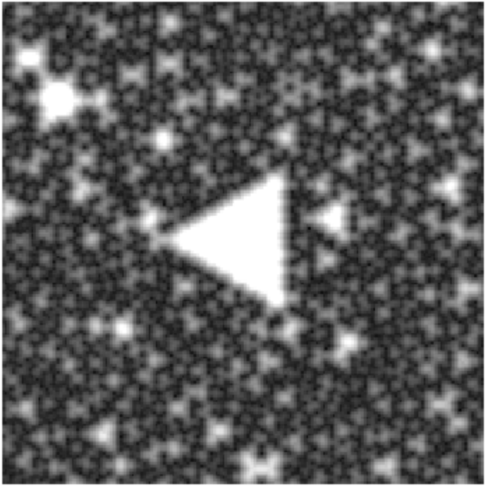

Lower bounds using random flats

Results of Kleinbock and Margulis imply:

Lemma (see Kleinbock-Margulis)

There is a $c > 1$ such that if $x \in X$ and $\rho = d(x, \Gamma)$, then there is a flat E passing through x such that the sphere $S_E(c\rho) \subset E$ of radius $c\rho$ satisfies

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Corollary

$$\mathrm{FV}^k_\Gamma(\rho^{k-1+c}) \approx e^{\rho}.$$

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How do we construct random simplices?

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Corollary

If $\alpha : S^{n-1} \to X(r_0)$ and $V = \operatorname{mass} \alpha$,

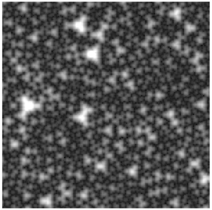
$$\mathrm{FV}^n_{X(c + c\log V)}(\alpha) \lesssim V^{\frac{n}{n-1}}.$$

Using the retraction $X(c + c\log V) \to X(r_0)$,

$$\mathrm{FV}^n_\Gamma(V) \approx V^{c + \frac{n}{n-1}}.$$

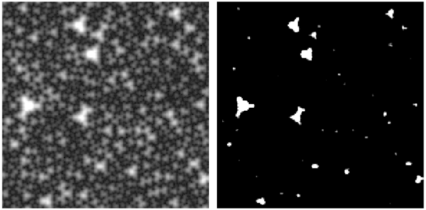
Bootstrapping

A filling is made of random flats:



Bootstrapping

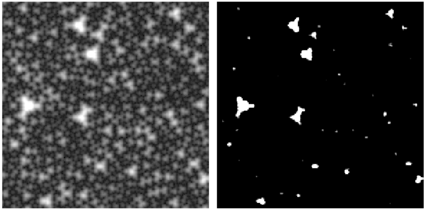
A filling is made of random flats:



- Cut out the parts of E that lie in a thin part.

Bootstrapping

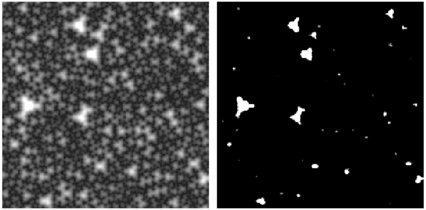
A filling is made of random flats:



- Cut out the parts of E that lie in a thin part.
- Replace them with a disc of polynomial area.

Bootstrapping

A filling is made of random flats:



- Cut out the parts of E that lie in a thin part.
- Replace them with a disc of polynomial area.
- The result is a filling that lies in $X(r_0)$ and has volume $\approx V^{\frac{n}{n-1}}$.